ELSEVIER
Journal of Pure and Applied Algebra 133 (1998) 197–202
JOURNAL OF PURE AND APPLIED ALGEBRA

Modules whose small submodules have Krull dimension

Christian Lomp
Mathematisches Institut, Heinrich-Heine Universität Düsseldorf, 40225 Düsseldorf, Germany

Abstract

The main aim of this paper is to show that an $AB5^*$ module whose small submodules have Krull dimension has a radical having Krull dimension. The proof uses the notion of dual Goldie dimension. © 1998 Elsevier Science B.V. All rights reserved.

1991 Math. Subj. Class.: 16P20; 16D10; 16A55; 16A49; 16S90

1. Introduction

Let R be an associative ring with unit and let M be a left unital R-module. Denote the socle of M, the intersection of all essential submodules of M, by $Soc(M)$. A well-known theorem by Goodearl (see [5, Proposition 3.6] or [1, Proposition 4]) asserts that $M/Soc(M)$ is noetherian if and only if every factor module M/N with N essential in M is noetherian. This can easily be extended to show that $M/Soc(M)$ has Krull dimension if and only if M/N has Krull dimension for every essential submodule N of M (see [9, Proposition 2]). Denote the radical of M, the sum of all small submodules of M, by $Rad(M)$. Dual to Goodearl's result, Al-Khazzi and Smith proved that $Rad(M)$ is artinian if and only if every small submodule of M is artinian (see [1, Theorem 5]). They asked in [1]: If every small submodule has finite uniform dimension (Goldie dimension). Does $Rad(M)$ have finite uniform dimension?

Puczyłowski answered this question in the negative and showed that there exists a $\mathbb{Z}$-module M such that every small submodule is noetherian and hence has Krull dimension but $Rad(M)$ does not have Krull dimension (see [9, Example]).

Since we wish to dualize Goodearl's Theorem it is natural to ask if the Al-Khazzi–Smith Theorem can be extended for arbitrary Krull dimension to modules which satisfy property $AB5^*$.

Theorem 5 is our main theorem and shows that for a module having $AB5^*$ the following implication holds. If every small submodules has finite hollow dimension

PII: S0022-4049(97)00194-1

(dual Goldie dimension) then every submodule of $Rad(M)$ has finite hollow dimension. This can be seen as dual to [3, Lemma 5.14]: If M/N has finite uniform dimension for every essential submodule N of M, then every factor module of $M/Soc(M)$ has finite uniform dimension.

2. Definitions

For the definition of Krull dimension we refer to [3, Ch. 6]. A module M is said to be *uniform* if $M \neq 0$ and every non-zero submodule is essential in M. M is said to have *finite uniform dimension* (or *finite Goldie dimension*) if there is a monomorphism from a finite direct sum of proper uniform submodules of M to M such that the image is essential in M. It is well-known that this is equivalent to the property that M has no infinite independent family of non-zero submodules and that there is a maximal finite independent family of uniform submodules (see [3, Theorem 5.9]). We denote the cardinality of this family by $udim(M)$ and call $udim(M)$ the *uniform dimension* of M.

A module M is said to be *hollow* if $M \neq 0$ and every proper submodule is small in M. Hollow modules were introduced by Fleury [4]. M is said to have *finite hollow dimension* if there is an epimorphism with a small kernel from M to a finite direct sum of non-zero hollow factor modules. It can be shown that in this case there is a number n such that M does not allow an epimorphism to a direct sum with more than n summands. We denote this by $hdim(M) = n$ and call $hdim(M)$ the *hollow dimension* of M. For any submodule N of M we have $hdim(M/N) \leq hdim(M)$.

Definition. Let M be an R-module and $\{N_\lambda\}_\Lambda$ a family of proper submodules of M. $\{N_\lambda\}_\Lambda$ is called *coindependent* (see [11]) if for every $\lambda \in \Lambda$ and finite subset $J \subseteq \Lambda \backslash \{\lambda\}$,

$$N_\lambda + \bigcap_{j \in J} N_j = M$$

holds (convention: if J is empty, then set $\bigcap_J N_j := M$).

It can be shown, that a module M has finite hollow dimension if and only if every coindependent family of submodules is finite (see [6, Corollary 13]). For more information on dual Goldie dimension we refer to [6, 10–12].

Definition. An R-module M has property $AB5^*$ if for every submodule N and inverse systems $\{M_i\}_{i \in I}$ of submodules of M the following holds:

$$N + \bigcap_{i \in I} M_i = \bigcap_{i \in I} (N + M_i)$$

Examples of modules with $AB5^*$ are artinian or linearly compact modules. Herbera and Shamsuddin proved the following result:

Lemma 1 (Herbera and Shamsuddin [7, Lemma 6] or Brodskii [2]). *For a module M with property $AB5^*$ the following statements are equivalent*:

(a) *Every factor module of M has finite uniform dimension.*

(b) *Every submodule of M has finite hollow dimension.*

It is easy to see that implication (b)$\Rightarrow$(a) always holds (see [13, Proposition 12]) and that (a)$\Rightarrow$(b) is false in general (for example $M =_{\mathbb{Z}} \mathbb{Z}$).

Definition. Let M be an R-module and $\{N_\lambda\}_\Lambda$ a family of proper submodules. Then $\{N_\lambda\}_\Lambda$ is called *completely coindependent* if for every $\lambda \in \Lambda$,

$$N_\lambda + \bigcap_{\mu \neq \lambda} N_\mu = M$$

holds.

A completely coindependent family is coindependent, but the converse is not true in general (for example, $\{p\mathbb{Z}\}$ in ${}_{\mathbb{Z}}\mathbb{Z}$ where p runs through all prime numbers). Considering Herbera and Shamsuddin's proof of Lemma 1 we get:

Lemma 2. *Every coindependent family of submodules of a module with property $AB5^*$ is completely coindependent.*

3. Modules whose small submodules have Krull dimension

In this section we will prove our main theorem. First we prove:

Lemma 3. *Let M be an R-module, $\{N_\lambda\}_\Lambda$ a completely coindependent family of proper submodules of M and $|\Lambda| \geq 2$. Assume that for every $\lambda \in \Lambda$ there exists a submodule L_λ such that $N_\lambda \subsetneq L_\lambda$. Let $L := \bigcap_{\lambda \in \Lambda} L_\lambda$ and $N := \bigcap_{\lambda \in \Lambda} N_\lambda$. Then $\{(N_\lambda \cap L)/N\}_\Lambda$ forms a completely coindependent family of proper submodules of L/N.*

Proof. Let $\lambda \in \Lambda$. Then $N_\lambda + L = L_\lambda \cap (N_\lambda + \bigcap_{\mu \neq \lambda} L_\mu) = L_\lambda \cap M = L_\lambda$. Since $N_\lambda \neq L_\lambda$ we have $N_\lambda \cap L \subsetneq L$. Moreover, $(N_\lambda \cap L) + \bigcap_{\mu \neq \lambda}(N_\mu \cap L) = L$ is straightforward. Thus, $\{N_\lambda \cap L\}_\Lambda$ forms a completely coindependent family of proper submodules of L. Hence, $N \subsetneq N_\lambda \cap L$ for all $\lambda \in \Lambda$ and $\{(N_\lambda \cap L)/N\}_\Lambda$ is a completely coindependent family of proper submodules of L/N. $\square$

The next definition dualizes the notion of an essential extension.

Definition. Let $N \subseteq L \subseteq M$ be submodules of M. We say that L *lies above* N (in M) if $L/N \ll M/N$. Note that L lies above N if and only if $N + K = M$ holds whenever $L + K = M$ holds for a submodule K of M.

Lemma 4. *Let M be an R-module with $AB5^*$, $\{L_\lambda\}_\Lambda$ a coindependent family of submodules such that for each $\lambda \in \Lambda$ there exists a submodule $N_\lambda \subseteq L_\lambda$ such that L_λ lies above N_λ in M. Then $\bigcap_\Lambda L_\lambda$ lies above $\bigcap_\Lambda N_\lambda$ in M.*

Proof. Let Ω denote the set of all finite subsets of Λ. Define for all $J \in \Omega$ $A_J := \bigcap_{j\in J} L_j$ and $B_J := \bigcap_{j\in J} N_J$. By induction on the cardinality of J it is easy to show that A_J lies above B_J for all $J \in \Omega$ (see [11, Proposition 1.6]). Since $\{A_J\}_{J\in\Omega}$ and $\{B_J\}_{J\in\Omega}$ are inverse systems, we get for $K \subset M$,

$$M = K + \bigcap_{\lambda\in\Lambda} L_\lambda = K + \bigcap_{J\in\Omega} A_J = \bigcap_{J\in\Omega} (K + A_J) = \bigcap_{J\in\Omega} (K + B_J)$$

$$= K + \bigcap_{J\in\Omega} B_J = K + \bigcap_{\lambda\in\Lambda} N_\lambda. \qquad \square$$

Definition. Let M be an R-module and N, L submodules of M. Then N is called a *supplement* of L in M if N is minimal with respect to $N + L = M$. Note that N is a supplement of L in M if and only if $N + L = M$ and $N \cap L \ll N$ (see [14, Ch. 41]). A module is called *amply supplemented* if whenever $N + L = M$ holds for two submodules of M, then N contains a supplement of L in M. Any module with $AB5^*$ is amply supplemented (see [14, 47.9]). As a generalization of a supplement, we say that N is a *weak supplement* of a module L in M if $N + L = M$ and $N \cap L \ll M$ holds.

We will now state our main result:

Theorem 5. *Let M be an R-module having $AB5^*$ such that every small submodule of M has finite hollow dimension. Then every submodule of $Rad(M)$ has finite hollow dimension.*

Proof. Let G be a submodule of $Rad(M)$ with $G \not\ll M$ and assume $\{N_\lambda\}_\Lambda$ to be a coindependent family of proper submodules of G that can be assumed to be completely coindependent by Lemma 2. Moreover, we assume that $|\Lambda| \geq 2$. For all $\lambda \in \Lambda$ there exist elements $x_\lambda \in Rad(M) \setminus N_\lambda$ since the N_λ's are proper submodules of G. Hence $Rx_\lambda \ll M$ and $L_\lambda := N_\lambda + Rx_\lambda \neq N_\lambda$. Let $N := \bigcap_\Lambda N_\lambda$ and $L := \bigcap_\Lambda L_\lambda$. Applying Lemma 3, we get that $\{(N_\lambda \cap L)/N\}_\Lambda$ is a completely coindependent family of proper submodules of L/N. Next we will show that L/N has finite hollow dimension so that Λ has to be finite. Since L_λ lies above N_λ for all $\lambda \in \Lambda$, we get by applying Lemma 4 that L lies above N in M. Let K be a weak supplement of L in M. Then $L/N \simeq (L \cap K)/(N \cap K)$ yields $hdim(L/N) \leq hdim(L \cap K)$. By assumption, $L \cap K \ll M$ has finite hollow dimension. Thus, L/N has finite hollow dimension and Λ must be finite. This shows that every coindependent family of submodules of G must be finite. Hence every submodule of $Rad(M)$ has finite hollow dimension. $\square$

Let us recall a result by Lemonnier to prove the next theorem.

Proposition 6 (Lemonnier [8, Proposition 1.3]). *Let M be an R-module such that every non-zero factor module of M has finite uniform dimension and contains a non-zero submodule having Krull dimension. Then M has Krull dimension.*

Theorem 7. *Let M be an R-module having $AB5^*$ such that every small submodule of M has Krull dimension. Then $Rad(M)$ has Krull dimension.*

Proof. It is well-known that a module having Krull dimension has finite uniform dimension (see [3, 6.2]). Hence, every factor module of a small submodule N of M has finite uniform dimension. Since N has $AB5^*$ every submodule of N has finite hollow dimension, by Lemma 1. Hence, by Theorem 5, every submodule of $Rad(M)$ has finite hollow dimension. By Lemma 1 every factor module of $Rad(M)$ has finite uniform dimension. In order to apply Lemonnier's Proposition, we need to show that every non-zero factor module of $Rad(M)$ contains a non-zero submodule having Krull dimension. Let $L \subset Rad(M)$ and $x \in Rad(M)\backslash L$; then $Rx \ll M$ so that Rx has Krull dimension and hence $(Rx+L)/L \subseteq Rad(M)/L$ has Krull dimension. Applying Proposition 6, $Rad(M)$ has Krull dimension. □

Corollary 8. *Let M be an R-module such that $Rad(M)$ has $AB5^*$ and every small submodule of M has Krull dimension. Then every submodule of $Rad(M)$ that has a weak supplement in M has Krull dimension.*

Proof. By Theorem 7, the radical of every submodule contained in $Rad(M)$ has Krull dimension. Since $Rad(N) = N \cap Rad(M)$ holds for every supplement N in M (see [14, 41.1]), every supplement in M that is a submodule of $Rad(M)$ has Krull dimension. Let $L \subseteq Rad(M)$ and $K \subseteq M$ a weak supplement of L in M. Then $Rad(M) = L + (Rad(M) \cap K)$. Since $Rad(M)$ has $AB5^*$ it is amply supplemented. Thus, there exists a supplement $N \subseteq L$ of $K \cap Rad(M)$ in $Rad(M)$ such that $Rad(M) = N + (Rad(M) \cap K)$ and $N \cap Rad(M) \cap K = N \cap K \ll N$ holds. Moreover $L = N + (L \cap K)$ and $M = N + K$ holds. Thus, N is a supplement of K in M, implying that N has Krull dimension. Because $L/N \simeq (L \cap K)/(N \cap K)$ with $L \cap K \ll M$, L/N has Krull dimension and hence has L also. □

Acknowledgements

The research for this paper was done at the University of Glasgow. The material in this paper will form part of the author's MSc thesis at the University of Glasgow and part of the author's Diplomarbeit for the Heinrich–Heine Universität Düsseldorf. The author would like to express his gratitude to his supervisors Professor P.F. Smith and Professor R. Wisbauer, for all their help, advice and encouragement. He would like to thank especially P.F. Smith for bringing his attention to this problem and for helpful discussions about it.

References

[1] I. Al-Khazzi, P.F. Smith, Modules with chain conditions on superfluous submodules, Comm. Algebra 19 (1991) 2331–2351.
[2] G.M. Brodskii, Modules lattice isomorphic to linearly compact modules, Math. Notes 59 (1996) 123–127.
[3] N.V. Dung, D. van Huynh, P.F. Smith, R. Wisbauer, Extending Modules, Pitman Research Notes in Mathematics Series, vol. 313, Longman, New York, 1994.
[4] P. Fleury, A note on dualizing Goldie dimension, Canad. Math. Bull. 17 (1974) 511–517.
[5] K. Goodearl, Singular torsion and the splitting properties, Am. Math. Soc. Memoirs 124 (1972).
[6] P. Grzeszczuk, E.R. Puczyłowski, On Goldie and dual Goldie dimension, J. Pure Appl. Algebra 31 (1984) 47–54.
[7] D. Herbera, A. Shamsuddin, Modules with semi-local endomorphism ring, Proc. Amer. Math. Soc. 123 (1995) 3593–3600.
[8] B. Lemonnier, Dimension de Krull et codeviation. Application au theoreme d'Eakin, Comm. Algebra 6 (1978) 1647–1665.
[9] E.R. Puczyłowski, On the uniform dimension of the radical of a module, Comm. Algebra 23 (1995) 771–776.
[10] E. Reiter, A dual to the Goldie ascending chain condition on direct sums of submodules, Bull. Cal. Math. Soc. 73 (1981) 55–63.
[11] T. Takeuchi, On cofinite-dimensional modules, Hokkaido Math. J. 5 (1976) 1–43.
[12] K. Varadarajan, Dual Goldie dimension, Comm. Algebra 7 (1979) 565–610.
[13] K. Varadarajan, Properties of endomorphism rings, Acta Math. Hung. 74 (1997) 83–92.
[14] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, Reading, 1991.