Semigroups Generated by Nilpotent Transformations

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In a seminal paper published in 1966, John Howie characterised the elements of $T_X$, the semigroup (under composition) of all total transformations of a set $X$ into itself, which can be written as a product of idempotents in $T_X$. We now initiate the study of the subsemigroup of $P_X$, the semigroup of all partial transformations of $X$, which is generated by the nilpotents of $P_X$.

1. INTRODUCTION

In [8], Howie investigated the subsemigroup of $T_X$ generated by all the idempotents of $T_X$. His work was later extended to $P_X$ by Evseev and Podran [5, 6] (and independently for finite $X$ by Sullivan [17]). Howie's result was generalised in a different direction by Kim [11], and it has also been considered in both a topological and a totally ordered setting (see [13, 15] for brief summaries of this latter work). The analogous idea for endomorphisms of a Boolean ring has been studied by Magill [12], and that for linear transformations of a vector space by both Erdos [4] and Dawlings [2] (in the finite-dimensional case) and by Reynolds and Sullivan [14] (in the infinite-dimensional case). The notion has also been extended to bounded linear operators on a separable Hilbert space [3]. Besides all this work, Howie and others have explored various ramifications of the original result (see [9], for example).

Since $P_X$ contains a zero, it contains nilpotents, and so it is natural to ask for a description of the subsemigroup of $P_X$ generated by all the nilpotents of $P_X$; this idea is related to a problem raised by Schwarz in [16]. In Section 2, we consider the situation when $X$ is finite: rather surprisingly this happens to be just as complex as the infinite case (considered in Section 3) and indeed the answer depends on whether $X$ contains an even or an odd number of elements; Theorems 1, 2, 3, and 4 supply the main results of the paper.

Naturally, we shall investigate each of the settings alluded to in the opening paragraph in a series of further papers. However, even more than
this, our final results (Theorems 5 and 6) suggest that it may be interesting to study the role of nilpotents in arbitrary semigroups containing a zero; this has already been done for completely 0-simple semigroups in [7].

2. THE FINITE CASE

Terminology will be that of [1, 18]; we note in particular the now standard method of displaying non-zero $\alpha \in \mathcal{P}_X$, the semigroup of all partial transformations of $X$, as

$$\alpha = \left( \begin{array}{c} A_i \\ x_i \end{array} \right) \quad \text{or} \quad \alpha = \left( \begin{array}{c} A_1 \cdots A_r \\ x_1 \cdots x_r \end{array} \right).$$

the former being used when $X$ is infinite and the latter when $X$ is finite, it being understood in both cases that $\text{ran } \alpha = \{ x_i : i \in I \}$, $\text{dom } \alpha = \bigcup \{ A_i : i \in I \}$, and $A_i = x_i \alpha^{-1}$ for each $i \in I$, where $I$ is an appropriate (finite or infinite) index set.

Any $\lambda \in \mathcal{P}_X$ for which there exists $m \geq 1$ such that $\lambda^m = \emptyset$ is called a nilpotent of $\mathcal{P}_X$; if $\lambda^m = \emptyset$, where $\lambda \neq \emptyset$ and $\lambda^{m-1} \neq \emptyset$, we say $\lambda$ has index $m$. Following [1, Vol. 1], we call $|X \setminus \text{ran } \alpha|$ the defect of $\alpha \in \mathcal{P}_X$; in addition, for each $\alpha \in \mathcal{P}_X$, we call $X \setminus \text{dom } \alpha$ (or its cardinal) the gap in $\alpha$. Finally, $\mathcal{L}_X$ shall denote the subsemigroup of $\mathcal{P}_X$ generated by all the nilpotents of $\mathcal{P}_X$.

Clearly, if $\lambda$ is nilpotent then $\lambda$ has non-zero gap and non-zero defect: since $\text{dom } \lambda \beta \subseteq \text{dom } \alpha$ and $\text{ran } \lambda \beta \subseteq \text{ran } \beta$ for all $\alpha, \beta \in \mathcal{P}_X$, any finite product of nilpotents must also have non-zero gap and non-zero defect. In this section we shall begin by showing that the converse is true when $X$ is finite and contains an even number of elements; that is, for such $X$, any $\alpha \in \mathcal{P}_X$ with non-zero gap belongs to $\mathcal{L}_X$. However, to achieve this goal we need some additional terminology from [19].

We say $\alpha \in \mathcal{P}_X$ is a $k$-chain (or a chain with length $k$) if $\text{dom } \alpha = \{ \alpha_1, \ldots, \alpha_k \}$ and

$$\alpha = \left( \begin{array}{c} a_1 a_2 \cdots a_k \\ a_2 a_3 \cdots a_{k+1} \end{array} \right),$$

where $a_1, \ldots, a_{k+1}$ are all distinct, and that $\alpha$ is a $k$-cycle (or a cycle with rank $k$) if $\alpha$ can be displayed as above where $a_1 = a_{k+1}$ and $a_1, \ldots, a_k$ are all distinct. Chains will be written as $[a_1, \ldots, a_{k+1}]$ and cycles as $(a_1, \ldots, a_k)$: when no loss of generality will occur, we shall abbreviate this notation to $[1, \ldots, k+1]$ and $(1, \ldots, k)$, respectively. If $\alpha, \beta \in \mathcal{I}_X$, the semigroup of all 1–1 partial transformations of $X$, we say $\alpha, \beta$ are disjoint when

$$(\text{dom } \alpha \cup \text{ran } \alpha) \cap (\text{dom } \beta \cup \text{ran } \beta) = \emptyset.$$
Note that when $\alpha, \beta \in \mathcal{I}_\mathcal{X}$ are disjoint, $\alpha \cup \beta$ (regarding $\alpha$ and $\beta$ as subsets of $X \times X$) is also an element of $\mathcal{I}_\mathcal{X}$.

If $|X| = 1$, $\mathcal{L}_\mathcal{X} = \{ \varnothing \}$, and when $|X| = 2$, $\mathcal{L}_\mathcal{X}$ equals the set of all 1-1 constants in $\mathcal{I}_\mathcal{X}$ with $\varnothing$ adjoined. Throughout the rest of this section, $X$ will be finite with $|X| = n \geq 3$. To arrive at our first Theorem, we require a series of Lemmas.

**Lemma 1.** Any idempotent $\alpha \in \mathcal{I}_\mathcal{X}$ with rank $r \leq n - 1$ can be written as a product of two chains each with length $r$.

**Proof.** Note that if $b \notin \text{dom } \alpha = \{ a_1, \ldots, a_r \}$, then

$$
\begin{pmatrix}
(a_1 \cdots a_r) \\
(a_1 \cdots a_r)
\end{pmatrix} = 
\begin{pmatrix}
(a_1, a_2 \cdots a_r) \\
(a_2, a_3 \cdots a_r)
\end{pmatrix} 
\begin{pmatrix}
b \\
a_r, a_{r-1} \cdots a_1
\end{pmatrix}.
$$

To simplify the statement of the next Lemma, we now say $\alpha \in \mathcal{I}_\mathcal{X}$ is an extended $k$-chain ($k \geq 1$) if $\alpha$ is the disjoint union of a $k$-chain and an idempotent (the latter possibly being empty); the concept of an extended 1-chain will play a fundamental role in our characterisation of $\alpha \in \mathcal{L}_\mathcal{X}$ when $|X|$ is odd.

**Lemma 2.** Every $\alpha \in \mathcal{I}_\mathcal{X}$ with rank $r \leq n - 1$ can be written as a product of extended 1-chains each of which has rank $r$.

**Proof.** We first suppose $\alpha$ is a chain with length $r \leq n - 1$ and put $\alpha = [1, \ldots, r + 1]$. The result is immediate when $r = 1$; when $r = 2$, note that

$$
\begin{pmatrix}
1 & 2 \\
2 & 3
\end{pmatrix} = 
\begin{pmatrix}
1 & 2 \\
1 & 3
\end{pmatrix} 
\begin{pmatrix}
1 & 3 \\
2 & 3
\end{pmatrix},
$$

and hence every extended 2-chain of rank $r$ can be written as a product of precisely two extended 1-chains each of which has rank $r$. Now, if $r = 2k$, where $k \geq 2$, then for each $i = 0, 1, \ldots, k - 1$, we let $\lambda_i = [r - 2i - 1, r - 2i, r - 2i + 1] \cup \{ Y(i) \}$, where

$$
Y(i) = \left( \text{dom } \alpha \cup \text{ran } \alpha \right) \setminus \{ r - 2i - 1, r - 2i, r - 2i + 1 \},
$$

and observe that $\alpha = \lambda_0 \cdots \lambda_{k - 1}$. If $r = 2k + 1$, where $k \geq 1$, then

$$
\alpha = 
\begin{pmatrix}
1 & 2 & 3 & \cdots & r \\
1 & 3 & 4 & \cdots & r + 1
\end{pmatrix} 
\begin{pmatrix}
1 & 3 & 4 & \cdots & r + 1 \\
2 & 3 & 4 & \cdots & r + 1
\end{pmatrix},
$$

where $[2, 3, \ldots, r + 1]$ is a chain with even rank. From the case just considered, it is easy to see that $\alpha$ can be expressed in the desired format, in fact, using only transformations whose domain and range are contained in
Moreover, we may conclude that any chain with even (odd) rank can be written as a product of an even (odd) number of extended 1-chains (this fact will be useful in subsequent work).

We now assume $\alpha = (a_1, \ldots, a_s) \cup l_V$, where $V$ is some subset of $X$ disjoint with $\{a_1, \ldots, a_s\}$. If $s = 1$, we apply Lemma 1 and the conclusion of the preceding paragraph to obtain the desired result. If $s \geq 2$ then $\alpha = \pi_2 \cdots \pi_s$, where each $\pi_i = (a_1, a_i) \cup l_{Z(i)}$ and $Z(i) = V \cup \{a_2, \ldots, a_s\ \setminus \ a_i\}$ for $i = 2, \ldots, s$.

Now choose $x \notin \text{dom} \ \alpha$ and observe that

$$
\begin{pmatrix}
a_1 & a_i \\
a_i & a_1
\end{pmatrix} = \begin{pmatrix}
a_1 & a_i \\
a_i & x
\end{pmatrix} \circ \begin{pmatrix}
a_i & x \\
a_i & a_1
\end{pmatrix}.
$$

The result follows in this case after suitably redefining the $Z[i]$ wherever necessary.

To complete the proof we now consider arbitrary $\alpha \in \mathcal{A}_X$ with rank $\alpha = r \leq n - 1$ and, using [19, Theorem 1], write $\alpha$ as a disjoint union of cycles $\gamma_i$, $(1 \leq i \leq s)$ and chains $\gamma_j$, $(s + 1 \leq j \leq t)$:

$$
\alpha = \gamma_1 \cup \cdots \cup \gamma_s \cup \gamma_{s+1} \cup \cdots \cup \gamma_t,
$$

where in this representation some $\gamma_i$, $(1 \leq i \leq s)$ may be idempotent (that is, equal to a cycle of rank 1). For each $k = 1, \ldots, s$, let $Z(k) = \text{dom} \ \alpha \setminus \text{dom} \ \gamma_k$ and for each $k = s + 1, \ldots, t$, let

$$
Z(k) = \text{dom} \ \pi \cup \text{ran}(\gamma_{k+1} \cup \cdots \cup \gamma_{k-1}) \cup \text{dom}(\gamma_k \cup \cdots \cup \gamma_t),
$$

where $\pi = \gamma_1 \cup \cdots \cup \gamma_s$. Put $\beta_k = \gamma_k \cup l_{Z(k)}$ for $k = 1, \ldots, t$ and note that each $\beta_k$ has rank $r$. We now have $\alpha = \beta_1 \circ \cdots \circ \beta_t$, where, by Lemma 1 and the discussion in the two previous paragraphs, each $\beta_k$ can be expressed as a product of transformations of the desired form.

**Lemma 3.** If $\alpha \in \mathcal{A}_X$ has non-zero gap and rank $\alpha \leq n - 2$ then $\alpha \in \mathcal{L}_X$.

**Proof.** We first note that any idempotent with non-zero gap belongs to $\mathcal{L}_X$. If $a_i \in A_j$ for $i = 1, \ldots, r$, and $b \notin \bigcup \{A_i : i = 1, \ldots, r\}$, where $A_1, \ldots, A_r$ are pairwise disjoint subsets of $X$, then

$$
\begin{pmatrix}
A_1 \cdots A_r \\
a_1 \cdots a_r
\end{pmatrix} = \begin{pmatrix}
A_1 & A_2 \cdots A_{r-1} & A_r \\
a_2 & a_3 \cdots a_r & b
\end{pmatrix} \circ \begin{pmatrix}
a_1 & a_2 \cdots a_r \\
a_1 & a_2 \cdots a_{r-1} & a_r
\end{pmatrix}.
$$

Now suppose $\alpha \in \mathcal{A}_X$ and write

$$
\alpha = \begin{pmatrix}
A_1 \cdots A_r \\
x_1 \cdots x_r
\end{pmatrix} = \begin{pmatrix}
A_1 \cdots A_r \\
a_1 \cdots a_r
\end{pmatrix} \circ \begin{pmatrix}
a_1 \cdots a_r \\
x_1 \cdots x_r
\end{pmatrix},
$$
where \( a_i \in A \) for \( i = 1, \ldots, r \). The result now follows from Lemma 2 and the following general equation. All letters appearing are assumed to be distinct; note that this is the only place where \( r \leq n - 2 \) is used in the proof.

\[
\begin{pmatrix}
a & y_1 & \cdots & y_{r-1} \\
b & y_1 & \cdots & y_{r-1}
\end{pmatrix} = \begin{pmatrix}
a & y_1 & y_2 & \cdots & y_{r-1} \\
b & y_1 & y_2 & \cdots & y_{r-1}
\end{pmatrix} \cdot \begin{pmatrix}
y_1 & y_2 & y_3 & \cdots & c \\
1 & 1 & 1 & \cdots & 1
\end{pmatrix}.
\]

**Lemma 4.** If \(|X| - n \geq 4\) is even then any \( \alpha \in \mathcal{A}_X \) with rank \( n - 1 \) that is the disjoint union of a 1-chain and an idempotent can be written as the product of three chains each with length \( n - 1 \).

**Proof.** We suppose \( n = 2k \) and consider the equation

\[
\begin{pmatrix}
1 & 3 & 4 & \cdots & n \\
2 & 3 & 4 & \cdots & n
\end{pmatrix} = \begin{pmatrix}
1 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & n \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots & 2
\end{pmatrix} \cdot \begin{pmatrix}
3 & 4 & 5 & 2 \\
2 & 3 & 4 & \cdots & n
\end{pmatrix},
\]

where in the above \( n + 1 = 2 \) when \( n = 4, 6, 8, \ldots \). We now observe that the \((n - 1)\)-cycle appearing in the above equation equals the product

\[
\begin{pmatrix}
3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots & 2 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & n - 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & n \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & n
\end{pmatrix}.
\]

We have already observed that in general if \( \alpha \in \mathcal{A}_X \) then \( \alpha \) has non-zero gap. On the other hand, if \( \alpha \in \mathcal{A}_X \) has non-zero gap and \( X \) is finite then rank \( \alpha \leq n - 1 \) (where \( n \) denotes \(|X|\)); hence, Lemma 2, 3, and 4 combine to produce:

**Theorem 1.** If \( X \) is finite and contains an even number of elements then \( \alpha \in \mathcal{A}_X \) if and only if \( \alpha \) has non-zero gap.

Before proceeding with the case when \( n \) is odd, we present three results of independent interest. The Corollary will be indirectly used in proving Theorem 2, but all three results are stated in greater generality than is necessary for the remaining task of this section (nonetheless they bear comparison with Theorem 3 in Section 3). If \( \alpha = [a_1, \ldots, a_{r+1}] \) is a chain of length \( r \), we call \( a_1 \) the initial point of \( \alpha \) and \( a_{r+1} \) the terminal point of \( \alpha \); in what follows, when a chain \( \alpha \) is displayed in some two-row configuration, we may write its initial point \( z \) as \( \hat{z} \) to highlight where one should start reading \( \alpha \) as a chain. If \( \alpha, \beta \in \mathcal{A}_X \) are disjoint, we say \( \alpha \) can be welded to \( \beta \) if there exists chains \( \lambda, \mu \) such that \( \alpha \cup \beta = \lambda \cdot \mu \).

**Lemma 5.** If \( \alpha \in \mathcal{A}_X \) is a disjoint union of even cycles and has rank \( \leq n - 1 \), then \( \alpha \) can be written as a product of precisely two chains with the same rank as \( \alpha \).
Proof. Suppose \((a_1, \ldots, a_r)\) and \((b_1, \ldots, b_s)\) are disjoint even cycles with \(r + s \leq n - 1\), so that \(r = 3, 5, \ldots\) and \(s = 3, 5, \ldots\), and choose \(z \notin \{a_1, \ldots, a_r, b_1, \ldots, b_s\}\). Now consider the two mappings

\[
\lambda = \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & \cdots & a_r & b_1 & b_2 & b_3 & \cdots & b_s \\
z & a_1 & a_2 & a_3 & \cdots & a_{r-1} & a_r & b_1 & b_2 & \cdots & b_{s-1} \end{pmatrix}
\]

\[
\mu = \begin{pmatrix}
z & a_1 & a_2 & a_3 & \cdots & a_{r-1} & a_r & b_1 & b_2 & \cdots & b_{s-1} \\
a_2 & a_3 & a_4 & \cdots & a_r & b_1 & b_2 & b_3 & \cdots & b_s \end{pmatrix},
\]

where \(\tilde{a}_{r+1} = a_1\) for \(r = 3, 5, \ldots\) and \(\tilde{b}_{s+1} = b_1\) for \(s = 3, 5, \ldots\). Clearly, \(\lambda\) and \(\mu\) are chains with initial points \(b\) and \(z\) and terminal points \(z\) and \(b\), respectively, such that \(\lambda \circ \mu = (a_1, \ldots, a_r) \cup (b_1, \ldots, b_s)\). More generally, if \(\alpha\) is any disjoint union of even cycles \(\pi_i\) for \(i = 1, \ldots, s\) with rank \(\alpha \leq n - 1\), choose \(z \notin \text{dom } \alpha\) and then start a welding process by first welding \(\pi_1\) to \(\pi_2\), and then \(\pi_1 \cup \pi_2\) to \(\pi_3, \ldots\) and so on, at each step using the method indicated above to split the next cycle into two chains whose terminal (initial) point can be identified with the initial (terminal) point of the two chains already constructed.

Lemma 6. If \(\alpha \in \mathcal{F}_X\) is a disjoint union of two odd cycles whose orders differ by at most 2 and if rank \(\alpha \leq n - 1\), then \(\alpha\) can be written as a product of two chains with the same rank as \(\alpha\).

Proof. Suppose \(\alpha = (a_1, \ldots, a_r) \cup (b_1, \ldots, b_s)\), where \(r \leq s \leq r + 2\) and \(r = 2, 4, \ldots\) and \(s = 2, 4, \ldots\). When \(r = s\), we consider

\[
\lambda_1 = \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & \cdots & a_r & b_1 & b_2 & b_3 & b_4 & \cdots & b_r \\
z & b_r & b_{r-1} & b_{r-2} & \cdots & b_1 & a_r & a_{r-1} & a_{r-2} & \cdots & a_1 \end{pmatrix}
\]

\[
\mu_1 = \begin{pmatrix}
z & b_r & b_{r-1} & b_{r-2} & \cdots & b_2 & a_r & a_{r-1} & a_{r-2} & \cdots & a_1 \\
a_2 & a_3 & a_4 & \cdots & a_{r+1} & b_2 & b_3 & b_4 & \cdots & b_s \end{pmatrix},
\]

where \(\tilde{a}_{r+2} = a_1\) and \(\tilde{b}_{s+1} = b_1\) equal \(a_1\) and \(b_1\), respectively, when \(r = 2, 4, \ldots\). If \(s = r + 2\), we consider

\[
\lambda_2 = \begin{pmatrix}
a_1 & a_2 & a_3 & \cdots & a_{r+1} & b_1 & b_2 & b_3 & b_4 & \cdots & b_{s-1} & b_s \\
z & b_s & b_{s-1} & \cdots & a_r & a_{r-1} & a_r & a_{r-2} & \cdots & a_2 & a_1 \end{pmatrix}
\]

\[
\mu_2 = \begin{pmatrix}
z & b_s & b_{s-1} & \cdots & b_2 & a_r & a_{r-1} & a_r & a_{r-2} & \cdots & a_2 & a_1 & b_1 & b_2 \\
a_2 & a_3 & a_4 & \cdots & a_{r+1} & b_2 & b_3 & b_4 & \cdots & b_{s+1} & b_s \end{pmatrix},
\]

where now \(\tilde{a}_{r+3} = a_1\) and \(\tilde{b}_{s+1} = b_1\) equal \(a_1\) and \(b_1\), respectively, when \(r - 2, 4, \ldots\). From Lemmas 1, 5, and 6 we readily obtain:
Corollary 1. Any even permutation $\alpha$ with rank $\leq n-1$ can be written as a product of precisely two chains with the same rank as $\alpha$ if, whenever its decomposition into disjoint cycles contains an odd cycle, then all such cycles can be arranged so that no two consecutive ones have order differing by more than 2.

We now aim to consider the case when $n$ is odd: the next sequence of Lemmas will eventually show that in this situation $\lambda_X$ consists of all $\alpha \in \mathcal{P}_X$ with non-zero gap and rank $\leq n-2$ together with all "even transformations" in $\mathcal{Y}_X$ with rank $n-1$. If $\beta \in \mathcal{Y}_X$, we call $\beta$ an even transformation if it is an even permutation of its domain, or a chain with even rank, or a disjoint union of an odd permutation and a chain with odd rank, or a disjoint union of any of the previous three types of transformation. It is important to note that for the purpose of this definition non-empty idempotents in $\mathcal{Y}_X$ are to be regarded as even permutations (as is customary when discussing $\text{Alt}(X)$, for example).

Lemma 7. If $\alpha \in \mathcal{Y}_X$ is an even permutation with rank $r \leq n-1$ then $\alpha$ can be written as a product of an even number of chains each with rank $r$.

Proof. In view of the above Corollary we need only consider the situation when $\alpha$ contains at least two arbitrary disjoint odd cycles, $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_s)$ say, where $r = 2p$ and $s = 2q$ for some integers $p, q$. However, note that

$$(a_1, \ldots, a_r) = (a_1, a_2, a_3) \circ (a_1, a_4, a_5) \circ \cdots \circ (a_1, a_{r-2}, a_{r-1}) \circ (a_1, a_r)$$

and

$$(b_1, \ldots, b_s) = (b_1, b_2) \circ (b_1, b_3, b_4) \circ \cdots \circ (b_1, b_{s-1}, b_s)$$

and so the result follows by a straightforward application of Corollary 1.

Lemma 8. If $\alpha \in \mathcal{Y}_X$ is a chain with even rank $r$ then $\alpha$ can be written as a product of precisely two chains each with rank $r$.

Proof. If $r = 2k$ for some $k \geq 2$, we observe that the product of the chains $\lambda$ and $\mu$, where

$$\lambda = \begin{pmatrix} 1 & 2 & 3 & \cdots & r \\ r+1 & 1 & 2 & \cdots & r-1 \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} r+1 & \hat{1} & 2 & \cdots & r-1 \\ 2 & 3 & 4 & \cdots & r+1 \end{pmatrix}$$

equals the chain $[1, \ldots, r+1]$.

Lemma 9. If $\alpha \in \mathcal{Y}_X$ is a disjoint union of an odd permutation and a chain with odd rank, then $\alpha$ can be written as a product of an even number of chains each with the same rank as $\alpha$. 
Proof. Suppose \( \alpha = \pi \cup \lambda \), where \( \pi \) is an odd permutation and \( \lambda \) is a chain with odd rank. Then \( \pi = \sigma \circ (1, 2) \), where \( \sigma \) is some even permutation (to be regarded as the identity on \( \{1, 2\} \) when \( \pi \) equals \((1, 2)\) alone). Thus, \( \alpha = (\sigma \cup 1_\gamma) \circ \mu \), where \( Y = \text{dom} \lambda \) and \( \mu = (1, 2) \cup \lambda \cup 1_Z \) for \( Z = \text{dom} \sigma \setminus \{1, 2\} \). In view of Lemmas 1 and 7, it will suffice to show that the disjoint union of a 2-cycle and a chain with odd rank can be written as a product of just two chains. However, this is apparent after forming the product \( \lambda \cdot \mu \) of the two chains (we assume \( r \geq 4 \) is even)

\[
\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & r - 1 \\ r & r - 1 & 1 & 2 & \cdots & r - 3 \end{pmatrix}, \quad \mu = \begin{pmatrix} r & r - 1 & 1 & 2 & \cdots & r - 3 \\ 2 & 1 & 4 & 5 & \cdots & r \end{pmatrix}.
\]

By combining Lemmas 7, 8, and 9 it is not too hard to see:

**Corollary 2.** If \( \alpha \in \mathcal{J}_X \) is an even transformation with rank \( r \leq n - 1 \), then \( \alpha \) is a product of an even number of chains each with rank \( r \).

If \( \alpha = \lambda_1 \cdots \lambda_s \), where each \( \lambda_i \) is nilpotent and rank \( \alpha = n - 1 \), then each \( \lambda_i \in \mathcal{J}_X \) and has rank \( n - 1 \); thus, each \( \lambda_i \) is a chain and, if \( n \) is odd, each \( \lambda_i \) has even rank. Hence, the task is to now prove a partial converse of Corollary 2, and for this we need:

**Lemma 10.** If \( \alpha \in \mathcal{J}_X \) has rank \( n - 1 \), then \( \alpha \) is even if and only if it is a product of an even number of extended 1-chains.

Proof. By Corollary 2, every even transformation with rank \( n - 1 \) is a product of an even number of chains each with rank \( n - 1 \). If \( n \) is even, each chain has odd rank and (by a remark in the first paragraph of the proof of Lemma 2) is the product of an odd number of extended 1-chains; thus, \( \alpha \) is a product of an even number of extended 1-chains as required. A similar argument when \( n \) is odd produces the same result.

Now suppose \( \alpha \in \mathcal{J}_X \) has rank \( n - 1 \) and that \( \alpha = \lambda_1 \lambda_2 \cdots \lambda_{2s-1} \lambda_2 \) is a product of an even number of extended 1-chains \( \lambda_i \) (\( 1 \leq i \leq 2s \)). Since \( \lambda_1 \lambda_2 \) is either an idempotent or an extended 2-chain (and hence, in either case, equals an even transformation) we can establish an inductive procedure to show that \( \alpha \) is even. In fact, it will suffice to assume that \( \beta = \lambda_1 \cdots \lambda_{2s-1} \) is even and \( \lambda_{2s-1} \lambda_{2s} \) is an extended 2-chain, \( \mu = [a, b, c] \cup 1_Y \) say, and then show under these circumstances that \( \alpha \) is even. However, before proceeding to do this, note that since \( \alpha = \beta \circ \mu \) has rank \( n - 1 \), ran \( \beta \) must equal dom \( \mu \), and so \( c \notin \text{ran} \beta \).

Clearly, if \( \beta \) fixes both \( a \) and \( b \), the result holds. As an abbreviation we now let \( \mathcal{A} \) denote the decomposition of \( \beta \) into disjoint cycles and chains; we commence our analysis with the situation in which \( \mathcal{A} \) contains a cycle \( \pi = (x_1, \ldots, x_r) \) whose domain \( D \) intersects \( \{a, b\} \). If \( a = x_1 \) and \( b = x_r \), then
\(\pi \circ \mu = [x_1, x_2, \ldots, x_{r-1}, c] \cup (x_r);\) if \(r\) is odd then \(\overline{\beta} = \beta \mid (\text{dom } \beta \setminus D)\) is even and so \(\beta \circ \mu\) is also even, whereas if \(r\) is even then \(\beta\) must contain an odd cycle or an odd chain (since \(\beta\) is even) and so \(\beta \circ \mu\) is again even. We now suppose \(r \geq 3\) and \([a, b] \subseteq D\) but \(a \neq x_1\) and \(b \neq x_r\). Then \(\pi \circ \mu\) is the disjoint union of a \(p\)-cycle and a \(q\)-chain, where \(p + q = r\). If \(r\) is odd then one of \(p, q\) must be odd and the other even; whatever happens, \(\pi \circ \mu\) is even (by definition) and so, as before, \(\beta \circ \mu\) is even. If \(r\) is even then both \(p, q\) are even or both are odd: in either event, \(\beta\) must be odd and so \(\beta \circ \mu\) remains even. When \(a = x_q\) say and \(\beta\) fixes \(b\), \(\pi \circ \mu\) always equals a chain with length \(r + 1\); namely,

\[
\pi \circ \mu = [x_q, x_{q + 1}, \ldots, x_r, x_1, x_2, \ldots, x_{q - 1}, b, c].
\]

If \(r\) is odd, \(\overline{\beta} = \beta \mid \text{dom } \beta \setminus (D \cup b)\) is even and so \(\beta \circ \mu\) is even, whereas if \(r\) is even then \(\overline{\beta}\) is odd and so \(\beta \circ \mu = (\pi \circ \mu) \cup (\overline{\beta} \circ \mu)\) is still even. It should be kept in mind in the preceding discussion that \(\text{dom } \beta\) may contain \(c\); however, whenever this occurs, \((\beta \circ \mu) \setminus (\pi \circ \mu)\), the relative complement inside \(X \times X\), has the appropriate character (even or odd) to guarantee the result.

We now consider the possibility of \([a, b]\) intersecting the range of a chain \(\lambda\) in \(\mathcal{A}\), where \(\lambda = [x_1, \ldots, x_{r+1}]\). If \(a, b\) (in that order) equal any two elements of \(\text{ran } \lambda\), then \(\lambda \circ \mu\) is a chain with rank \(r\), and so \(\beta \circ \mu\) remains even. If \(a = x_t\), where \(t \geq 2\) and \(t\) is even, and if \(\beta\) fixes \(b\), then \(c = x_1\) (since \(\text{rank } x = n - 1\)) and

\[
\lambda \circ \mu = (x_1, x_2, \ldots, x_{t-1}, b) \cup [x_t, x_{t+1}, \ldots, x_{r+1}].
\]

In this case, if \(r\) is even, the result is immediate; otherwise, if \(r\) is odd, \(\mathcal{A} \setminus [\lambda \cup (b)]\) must contain either an odd cycle or an odd chain. Thus, \(\beta \circ \mu\) is even in this case also. If \(a = x_t\), where \(t \geq 3\) and \(t\) is odd, and at the same time \(\beta\) fixes \(b\), then \(c = x_1\) and \(\lambda \circ \mu\) is the disjoint union of an even permutation and an \((r - 2)\)-chain (the latter does not exist when \(r\) is even an \(t = r + 1\)). Once again, if \(r\) is even, the result is immediate, whereas if \(r\) is odd then \(\mathcal{A} \setminus [\lambda \cup (b)]\) must contain an appropriate cycle or chain ensuring that \(\beta \circ \mu\) is even.

To complete the proof we should now investigate the situation where \(\mathcal{A}\) contains a cycle \((y_1, \ldots, y_p)\) and a chain \([z_1, \ldots, z_{q+1}]\), and \(a = y_i\) for some \(i\) while \(b = z_j\) for some \(j\), or vice versa. However, the argument is rather similar to that already presented, so we omit the details.

**Theorem 2.** If \(n\) is odd and \(\alpha \in \mathcal{P}_x\) then \(\alpha \in \mathcal{L}_x\) if and only if \(\alpha\) has non-zero gap and either \(\text{rank } \alpha \leq n - 2\) or \(\alpha\) is an even transformation with rank \(n - 1\).
**Proof.** It only remains to show that if $\alpha \in \mathcal{P}_X$ and rank $\alpha = n - 1$, where $n$ is odd, then $\alpha$ is an even transformation. However, in this case, if $\alpha = \lambda_1 \cdots \lambda_r$ where $\lambda_i$ is nilpotent then each $\lambda_i$ is a chain with even rank $n - 1$. But, as observed in the first paragraph of the proof of Lemma 2, each such chain can be written as a product of an even number of extended 1-chains; hence, by Lemma 10, $\alpha$ is even.

We close this section by illustrating some of the preceding results/algorithms. For example, if $n = 9$ and $\alpha = (1, 2, 3)(4, 5)(6, 7, 8, 9)$ then, by Theorem 2, $\alpha$ can be written as a product of nilpotents: this can be achieved by first writing $(1, 2, 3) \cup \gamma_Y$, where $Y = \{4, 5, 6, 7, 8\}$, as a product of two chains using the algorithm presented in the proof of Lemma 5 and then applying algorithms presented in the proofs of Lemmas 1 and 9 to find chains

$$\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 2 & 9 & 8 & 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} 7 & 1 & 2 & 9 & 8 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 4 & 7 & 8 & 9 \end{pmatrix}$$

whose product equals $\iota_Z \cup (4, 5)[6, 7, 8, 9]$, where $Z = \{1, 2, 3\}$. In fact, we can do better: the algorithms given in the proofs of Lemmas 5 and 9 enable us to weld $(1, 2, 3)$ to $(4, 5)(6, 7, 8, 9)$ and hence $\alpha$ is actually a product of just two chains; we conjecture that this is always possible when $n$ is odd and $\alpha$ is an even transformation with rank $n - 1$. Finally, observe that Theorem 2 implies $(1, 2, 3)(4, 5)(6)[7, 8, 9]$ cannot be expressed as a product of nilpotents.

### 3. The Infinite Case

We now suppose $X$ is infinite and, where convenient, adopt the convention of [1, Vol. 2, p. 241]; namely, if $\alpha \in \mathcal{P}_X$ and we write

$$\alpha = \begin{pmatrix} A_i \\ b_i \end{pmatrix}$$

we take it as understood that the subscript $i$ belongs to some (unmentioned) index set $I$ and that the abbreviation $\{b_i\}$ denotes $\{b_i; i \in I\}$.

Two preliminary results are required before the main result of this section can be stated: we thank Elke Wilkeit and Hans-J. Bandelt for pointing out a mistake in an early draft of this section. In what follows $|X|$ will be infinite and always denoted by $\kappa$: we shall refer to Jech [10] for information on regular and singular cardinals.

**Lemma 11.** If $\lambda$ is a nilpotent in $\mathcal{P}_X$, then $|X \setminus X\lambda| \not\geq \text{cf}(\kappa)$ and either $|X \setminus \text{dom} \lambda| \not\geq \text{cf}(\kappa)$ or there exists $a \in \text{ran} \lambda$ such that $|a\lambda^{-1}| \not\geq \text{cf}(\kappa)$.
Proof. If rank $\lambda < \text{cf}(\mathfrak{A})$ then $|X \setminus \text{ran } \lambda| \geq \text{cf}(\mathfrak{A})$. If in addition $|\text{dom } \lambda| < \text{cf}(\mathfrak{A})$ then $|X \setminus \text{dom } \lambda| \geq \text{cf}(\mathfrak{A})$; however, if $|\text{dom } \lambda| \geq \text{cf}(\mathfrak{A})$ then there must exist some $a \in \text{ran } \lambda$ such that $|a^{-1}| \geq \text{cf}(\mathfrak{A})$ since otherwise dom $\lambda$ would be the disjoint union of sets $Y_i$, $i \in I$ (say), where $|Y_i| < \text{cf}(\mathfrak{A})$ and $|I| < \text{cf}(\mathfrak{A})$, contradicting the fact that $\text{cf}(\mathfrak{A})$ is always regular (see [5, Lemma 3.5]).

Now suppose rank $\lambda \geq \text{cf}(\mathfrak{A})$ and let $m \geq 2$ be the least integer $s$ such that rank $\lambda^s < \text{cf}(\mathfrak{A}) \leq \text{rank } \lambda^{s-1}$ (such an integer exists since $\lambda$ is nilpotent). If $Z = \text{ran } \lambda^{n-1}$ then $|Z| \geq \text{cf}(\mathfrak{A})$ and since $|Z| < \text{cf}(\mathfrak{A})$, either $|Z \cap \text{dom } \lambda| < \text{cf}(\mathfrak{A})$ or, if $\{Y_i; i \in I\} = X/(\lambda \circ \lambda^{-1})$, then $|Z \cap \text{dom } \lambda| \geq \text{cf}(\mathfrak{A})$ but $\{|i \in I; Z \cap Y_i \neq \emptyset\} < \text{cf}(\mathfrak{A})$. In the former case, $|Z \cap \{X \setminus \text{dom } \lambda\}| \geq \text{cf}(\mathfrak{A})$ and so $|X \setminus \text{dom } \lambda| \geq \text{cf}(\mathfrak{A})$, while in the latter case we have

$$\text{cf}(\mathfrak{A}) \leq |Z \cap \text{dom } \lambda| = \left| \bigcup \{Z \cap Y_i; i \in I\} \right|.$$

Since $\text{cf}(\mathfrak{A})$ is regular, $|Z \cap Y_j| \geq \text{cf}(\mathfrak{A})$ for some $j \in I$ and so, if $Y_j \lambda = a$, then $|a^{-1}| \geq \text{cf}(\mathfrak{A})$ as required.

To show $|X \setminus \text{ran } \lambda| \geq \text{cf}(\mathfrak{A})$ when rank $\lambda \geq \text{cf}(\mathfrak{A})$ we first assume $|X \setminus \text{dom } \lambda| \geq \text{cf}(\mathfrak{A})$ and let $\lambda = \{a_p; p \in P\}$. If $|\text{ran } \lambda \cap (X \setminus \text{dom } \lambda)| < \text{cf}(\mathfrak{A})$ then $|X \setminus \text{ran } \lambda \cap (X \setminus \text{dom } \lambda)| \geq \text{cf}(\mathfrak{A})$ and the result follows. So, we also assume that $|\text{ran } \lambda \cap (X \setminus \text{dom } \lambda)| \geq \text{cf}(\mathfrak{A})$ and let $Q = \{p \in P; a_p \in X \setminus \text{dom } \lambda\}$; note that $|Q| \geq \text{cf}(\mathfrak{A})$. We now put $Y_p = a_p \lambda^{-1}$ for each $p \in P$ and observe that $a_p \notin Y_p$ for each $p \in P$ (since $\lambda$ is nilpotent).

Hence, for each $q \in Q$, we can choose $x_q \in Y_q$ so that $\{x_q\} \cap \{a_q\} = \emptyset$ (here we use the convention introduced at the start of this section). If $|\text{ran } \lambda \cap \{x_q\}| < \text{cf}(\mathfrak{A})$ then $|(X \setminus \text{ran } \lambda) \cap \{x_q\}| \geq \text{cf}(\mathfrak{A})$ and the result follows. Thus we may suppose $|\text{ran } \lambda \cap \{x_q\}| = \{a_q\}$ where $|R| \geq \text{cf}(\mathfrak{A})$ and for each $r \in R$ choose $x_r \in Y_r$: if $|\text{ran } \lambda \cap \{x_r\}| < \text{cf}(\mathfrak{A})$, the result follows, and if $|\text{ran } \lambda \cap \{x_r\}| \geq \text{cf}(\mathfrak{A})$, we repeat the argument. Clearly this process must stop (otherwise rank $\lambda^n \geq \text{cf}(\mathfrak{A})$ for all $n \geq 1$) and when it does we obtain $|X \setminus \text{ran } \lambda| \geq \text{cf}(\mathfrak{A})$.

Finally, we assume some $Y_p$, $Y_1$ say, has cardinal $\geq \text{cf}(\mathfrak{A})$. The argument of the last paragraph can be applied verbatim with $Y_1$ taking the place of $X \setminus \text{dom } \lambda$, and after doing this the proof is complete.

For convenience we shall refer to each $a^{-1}$ (or its cardinal), where $a \in \text{ran } \alpha$, as a sink of $\alpha \in \mathcal{P}_X$. With this in mind, we have:

**Lemma 12.** If $\alpha \in \mathcal{L}_X$ then def($\alpha$) $\geq \text{cf}(\mathfrak{A})$ and either the gap or some sink of $\alpha$ has cardinal $\geq \text{cf}(\mathfrak{A})$.

**Proof.** If $\alpha$ is itself nilpotent, we apply Lemma 11. Thus, we may suppose $\alpha = \lambda_1 \cdots \lambda_r$, where each $\lambda_i$ is a non-zero nilpotent and $r \geq 2$. Now, in general, if $\beta, \gamma \in \mathcal{P}_X$ and $|X \setminus X| \geq \text{cf}(\mathfrak{A})$ then $|X \setminus X \gamma| \geq \text{cf}(\mathfrak{A})$ (since
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Xβγ ⊆ Xγ; likewise, if |X \ dom β| ≥ cf(γ) then |X \ dom βγ| ≥ cf(γ) (since dom βγ ⊆ dom β). Hence, in view of Lemma 11, all that remains to be proven is that when there exists z ∈ ran λ with |zλ−1| ≥ cf(λ) then the same is true of x, or alternatively, |X \ dom z| ≥ cf(λ). However, if z ∈ dom λ and we let Y = zλ−1 then Y ⊆ X \ dom x, and the result follows. Hence we can suppose z ∈ dom λ and consider whether zλ2 ∈ dom λ2; a simple induction completes the proof.

Theorem 3. If |X| = k ≥ κ₀ and x ∈ Pₓ has rank ≤ cf(λ) then x is a product of nilpotents if and only if x has non-zero gap, def(λ) ≥ cf(λ), and either the gap or some sink of x has cardinal > cf(λ). In fact, when this occurs, x can be written as a product of three or fewer nilpotents, each with index at most 3.

Proof. We start by supposing both |X \ dom x| ≥ cf(λ) and |X \ ran x| ≥ cf(λ), and consider two subcases: namely, when (X \ ran x) ∩ (X \ dom x) has cardinal ≥, or <, cf(λ). Note that the first of these occurs in particular when rank x < cf(λ); for then |ran x ∩ (X \ dom x)| < cf(λ) and |X \ dom x| ≥ cf(λ) together imply the desired inequality. Now, for the first case, put ran x = {b₁}, A₁ = b₁x⁻¹ and choose c₁ ∈ X \ (ran x ∪ dom x) such that |{c₁}| = rank x. Then

x = \begin{pmatrix} A₁ & c₁ \\ c₁ & b₁ \end{pmatrix}

and the two mappings on the right of this equation are nilpotent with index 2. In the second case, we deduce that rank x = |ran x ∩ (X \ dom x)| = cf(λ) and so there exist c₁ ∈ ran x ∩ (X \ dom x) and d₁ ∈ X \ ran x such that

x = \begin{pmatrix} A₁ & c₁ \\ c₁ & d₁ \end{pmatrix} \begin{pmatrix} c₁ & b₁ \end{pmatrix}

and the three mappings on the right are nilpotent with index 2.

Now assume |X \ ran x| ≥ cf(λ) and there exists z ∈ ran x with |zxa⁻¹| ≥ cf(λ); the argument will proceed essentially as before. If |(X \ ran x) ∩ x⁻¹| ≥ cf(λ), put Y = x⁻¹, (ran x) \ z = {b₁}, A₁ = b₁x⁻¹ and choose x ∉ dom x (note that possibly x = z). Now distinguish an element y and disjoint subsets {c₁} and {d₁} in (X \ ran x) ∩ Y such that y ∉ {c₁} ∪ {d₁} and |{c₁}| = |{d₁}| = (rank x) − 1 (when this number equals 0 the following argument still holds with suitable reinterpretation). Then

x = \begin{pmatrix} Y & A₁ \\ x & c₁ \end{pmatrix} \begin{pmatrix} x & c₁ \\ y & d₁ \end{pmatrix} \begin{pmatrix} y & d₁ \\ z & b₁ \end{pmatrix}

where the mappings on the right are nilpotent with index ≤ 3.
If \(|(X \setminus \text{ran } \alpha) \cap Y| < \text{cf}(\kappa)|\) then \(\text{rank } \alpha = |\text{ran } \alpha \cap Y| = \text{cf}(\kappa)|\). In this event, we choose \(x \notin \text{dom } \alpha\) and \(c_i \in Y \cap [(\text{ran } \alpha) \setminus \{x\}]\), together with a distinguished element \(y\) and a subset \(\{d_i\}\) of \((X \setminus \text{ran } \alpha) \setminus \{x, z\}\) such that \(y \notin \{d_i\}\). With this notation, the previous decomposition of \(\alpha\) remains valid.

Clearly as a consequence we obtain:

Corollary 3. If \(|X| = \kappa\) is infinite and regular then \(\alpha \in \mathcal{L}_X\) if and only if \(\alpha\) has non-zero gap, \(\text{def}(\alpha) = \kappa\), and either the gap or some sink of \(\alpha\) has cardinal equal to \(\kappa\).

Another way of viewing the above result is to say that if \(\kappa\) is regular then \(\mathcal{L}_X\) consists of all \(\alpha \in \mathcal{P}_X\) with rank \(< \kappa\), together with those \(\alpha \in \mathcal{P}_X\) with rank \(\alpha = \text{def}(\alpha) = \kappa\) and either the gap or some sink of \(\alpha\) is \(\kappa\). This follows from the fact that if rank \(\alpha < \kappa\) and

\[
\alpha = \begin{pmatrix} u_i & Y_j \\ v_j & t_j \end{pmatrix},
\]

where \(|Y_j| \geq 2\) for each \(j \in J\), then \(\kappa = |I| + \bigcup Y_j + \text{gap}(\alpha)\).

Before proceeding to consider the case when \(\kappa\) is singular we note that in general if \(\lambda \in \mathcal{P}_X\) is nilpotent and injective then a simple adjustment to the proof of Lemma 11 shows that \(\text{def}(\lambda) = |X|\) and \(\text{gap}(\lambda) = |X|\) (the main variation occurs in the second paragraph of the proof: choose \(m\) to be the least integer \(s\) such that rank \(\lambda^s \leq |X| = \text{rank } \lambda^{s-1}\)). Moreover, it is easy to show that if \(\mu_i\) (\(i = 1, 2\)) are nilpotents satisfying \(\text{def}(\mu_i) = \text{gap}(\mu_i) = |X|\) for \(i = 1, 2\) then the product \(\mu_1 \mu_2\) also possesses this property (the proof is similar to that of Lemma 12). With this in mind, it is then not difficult to modify the proof of Theorem 3 to obtain:

Corollary 4. If \(|X| = \kappa \geq \aleph_0\) and \(\alpha \in \mathcal{A}_X\) then \(\alpha\) is a product of nilpotents in \(\mathcal{A}_X\) if and only if \(\text{def}(\alpha) = \text{gap}(\alpha) = |X|\). Moreover, in this case, \(\alpha\) can be written as a product of three or fewer nilpotents, each with index at most \(2\).

We now turn to the problem of characterising the elements of \(\mathcal{L}_X\) when \(|X|\) is singular. To indicate the extent to which this situation can differ from the case when \(|X|\) is regular, we suppose \(|X| = \kappa\) is singular and write \(X = P \cup Q \cup R\), where \(|P| = |Q| = \kappa\) and \(|R| = \text{cf}(\kappa)|\). Next, write \(Q = \bigcup A_i\), where \(|A_i| < \kappa\) for each \(i \in I\) and \(|I| = \text{cf}(\kappa)|\), and put \(R = \{r_i\}; P = \{p_j\}, Q = \{a_j\}\). Then

\[
\lambda = \begin{pmatrix} p_j & A_i \\ q_j & r_i \end{pmatrix}
\]

is a nilpotent with rank \(\kappa\) whose gap and all sinks have cardinal \(< \kappa\).
An important feature of the above example is that for each cardinal $\alpha < \text{rank } \lambda$, there is some sink of $\lambda$ with cardinal $>\alpha$. Whenever this occurs for $\alpha \in P_\infty$, we shall say that $\alpha$ is spread over its rank. Note that if $\alpha$ is spread over its rank then the "collapse" (see [8]) of $\alpha$ has cardinal $>\text{rank } \alpha$ but the converse is not in general true. Note also that if some sink of $\alpha$ has cardinal $>\text{rank } \alpha$ then $\alpha$ is automatically spread over its rank.

**Lemma 13.** If $\kappa$ is infinite and singular and $\lambda \in P_\kappa$ is nilpotent with rank $\lambda = \varepsilon > \operatorname{cf}(\kappa)$, then $\operatorname{def}(\lambda) = \kappa$ and either gap($\lambda$) $> \varepsilon$ or $\lambda$ is spread over $\varepsilon$.

**Proof.** Suppose rank $\lambda^m < \varepsilon = \text{rank } \lambda^{m-1}$ for $m \geq 2$ (such an integer exists since $\lambda$ is nilpotent). An argument similar to that in the second paragraph of the proof of Lemma 11 then shows that gap($\lambda$) $> \varepsilon$ or $\lambda$ is spread over $\varepsilon$; the only additional thing to observe is that if $J = \{ i \in I : Z \cap Y_i \neq \emptyset \}$, where $|J| < r$ and

$$\varepsilon = |Z \cap \text{dom } \lambda| = \bigcup \{ Z \cap Y_j : j \in J \},$$

then either $|Z \cap Y_j| \geq \varepsilon$ for some $j \in J$ (in which case $\lambda$ is certainly spread over $\varepsilon$) or $|Z \cap Y_j| < \varepsilon$ for each $j \in J$. Since this latter case can only occur when $\varepsilon$ is itself a singular cardinal, it follows that $\lambda$ is spread over $\varepsilon$.

Clearly, if $\varepsilon < \kappa$ then $\operatorname{def}(\lambda) = \kappa$. We now use the information already obtained to prove that $\operatorname{def}(\lambda) = \kappa$ when $\varepsilon = \kappa$.

If the gap or some sink of $\lambda$ has cardinal $\kappa$ then an argument similar to that in the third and fourth paragraphs of the proof of Lemma 11 produces the desired result (just replace $\operatorname{cf}(\kappa)$ throughout the previous argument).

We now define the morass of $\lambda$ to be the set

$$M(\lambda) = \bigcup \{ t^{\lambda^{-1}} : |t^{\lambda^{-1}}| > \operatorname{cf}(\kappa) \}$$

and note that $|M(\lambda)| = \kappa$: this is because $\kappa$, being singular, is the sum of $\operatorname{cf}(\kappa)$ cardinals $\kappa_n > \operatorname{cf}(\kappa)$ and we can now assume $\lambda$ is spread over $\kappa$.

Consider $\lambda \cap M(\lambda)$: if this has cardinal $< \kappa$, we readily deduce that $\operatorname{def}(\lambda) = \kappa$. So we suppose $|\lambda \cap M(\lambda)| = \kappa$ and note that

$$\lambda \cap M(\lambda) = \bigcup \{ \lambda \cap t^{\lambda^{-1}} : i \in I \},$$

where

$$\{ t_i \} = \{ t \in \lambda : |t^{\lambda^{-1}}| > \operatorname{cf}(\kappa) \text{ and } \lambda \cap t^{\lambda^{-1}} \neq \emptyset \}$$

and $|I| \leq \kappa$. In view of the conclusion of the last paragraph we may suppose that $|\lambda \cap t^{\lambda^{-1}}| < \kappa$ for each $i \in I$ (since otherwise some sink of $\lambda$ will
have cardinal $\mathcal{K}$ and the result will follow). If $|\text{ran } \lambda \cap t, \lambda^{-1}| \leq \text{cf}(\mathcal{K})$ for all $i \in I$ then $|I| = \mathcal{K}$ and $|(X \setminus \text{ran } \lambda) \cap t, \lambda^{-1}| > \text{cf}(\mathcal{K})$ for each $i \in I$ (from the way $M(\lambda)$ is defined); since the $t, \lambda^{-1}$ are mutually disjoint, by taking the union over the index set $I$ we can deduce that $\text{def}(\lambda) = \mathcal{K}$. Hence we may suppose that $|\text{ran } \lambda \cap t, \lambda^{-1}| > \text{cf}(\mathcal{K})$ for some $i \in I$ and then put $J = \{i \in I: |A_i| > \text{cf}(\mathcal{K})\}$, where $A_i = \text{ran } \lambda \cap t_i, \lambda^{-1}$ for each $i \in I$. If $|I \setminus J| = \mathcal{K}$, the argument in the last sentence (with $I \setminus J$ in place of $I$) produces the desired result. So we suppose $|I \setminus J| < \mathcal{K}$ and note as a consequence that

$$\mathcal{K} = \bigcup \{A_j: j \in J\}$$

since $i \in I \setminus J$ if and only if $|A_i| \leq \text{cf}(\mathcal{K})$. For each $j \in J$, choose a cross section $B_j$ of $A_j, \lambda^{-1}/(\lambda' \times \lambda')$ and note that $\{A_j\} \cap \{B_j\} = \emptyset$ (even though some $A_j$ and some $B_j$ may have a non-empty intersection); for, if $A_p = B_q$ for $p, q \in J$ then

$$t_p = A_p \lambda = B_q \lambda = A_q,$$

contradicting our supposition that $|A_j| > \text{cf}(\mathcal{K})$ for all $j \in J$. Moreover $|\bigcup B_j| = \mathcal{K}$, where $|B_j| > \text{cf}(\mathcal{K})$ for each $j \in J$.

In the last paragraph of the proof of Lemma 11 we used a “zigzag” argument based on sets of elements. Since we are about to do the same thing using sets of sets, a diagram may be helpful at this point ($\mathcal{B}_n$ denotes $\text{ran } \lambda \cap B_n$):

$$\text{dom } \lambda = \cdots t, \lambda^{-1} \quad B_j \cdots B_n C_n \cdots$$

$$\text{ran } \lambda = \cdots t, \quad A_j \cdots A_n B_n \cdots$$

We now repeat the entire argument of the last paragraph using $\bigcup B_j$ in place of $M(\lambda)$. That is, if $\text{ran } \lambda \cap (\bigcup B_j)$ has cardinal $< \mathcal{K}$ the result will follow, while if its cardinal equals $\mathcal{K}$ then we have

$$\text{ran } \lambda \cap \left(\bigcup B_j\right) = \bigcup \{\text{ran } \lambda \cap B_j: \ell \in L\},$$

where $L = \{j \in J: \text{ran } \lambda \cap B_j \neq \emptyset\}$ and $|L| \leq \mathcal{K}$. If $|\text{ran } \lambda \cap B_j| \leq \text{cf}(\mathcal{K})$ for all $\ell \in L$ then $|L| = \mathcal{K}$ and $|(X \setminus \text{ran } \lambda) \cap B_j| > \text{cf}(\mathcal{K})$ for each $\ell \in L$ (since each $B_j$ has cardinal $> \text{cf}(\mathcal{K})$); since the $B_j$ are mutually disjoint (recall that the $A_j$ were so) we readily obtain $\text{def}(\lambda) = \mathcal{K}$. So we suppose $|\text{ran } \lambda \cap B_j| > \text{cf}(\mathcal{K})$ for some $\ell \in L$ and put

$$N = \{\ell \in L: |\text{ran } \lambda \cap B_\ell| > \text{cf}(\mathcal{K})\}.$$
If $|L \setminus N| = k$ the result follows as before, whereas if $|L \setminus N| < k$ we have

$$k = \left| \bigcup \{ B_n : n \in N \} \right|,$$

where $B_n = \text{ran} \lambda \cap B_n$ and $|B_n| > cf(k)$ for each $n \in N$. Thus, if $C_n$ denotes a cross section of $B_n \lambda^{-1} / (\lambda \cap \lambda^{-1})$ we have $|C_n| > cf(k)$, $| \bigcup C_n| = k$ and moreover $\{ B_n \} \cap \{ C_n \} = \emptyset$; for, if $B_p = C_q$ for $p, q \in N$ then

$$t_p = B_p \lambda^2 = C_q \lambda^2 = B_q \lambda,$$

which is a contradiction since $|B_q \lambda| = |B_q|$. Similar arguments show that in fact $\{ A_j \}$, $\{ B_n \}$, and $\{ C_n \}$ are pairwise disjoint.

The above process can be repeated but must eventually stop (otherwise rank $\lambda^m$ would equal $k$ for all $m \geq 1$) and when it does the proof is complete.

**Lemma 14.** If $k$ is singular and $\alpha \in \mathcal{L}_X$ has rank $r > cf(k)$ then $\text{def}(\alpha) = k$ and either $\text{gap}(\alpha) > r$ or $\alpha$ is spread over $r$.

*Proof.* If $\alpha$ is nilpotent we apply Lemma 13. So we suppose $\alpha = \lambda_1 \cdots \lambda_m$ for some non-zero nilpotents $\lambda$, and some $m \geq 2$.

Now, in general, if $\beta, \gamma \in \mathcal{P}_X$ then $\text{gap}(\beta \gamma) \geq \text{gap}(\beta)$ and in addition if $\text{def}(\gamma) = k$ then $\text{def}(\beta \gamma) = k$. Since $r \leq \min(\text{rank} \lambda_i)$, the result therefore follows if $\text{gap}(\lambda_i) \geq \text{rank}(\lambda_i)$. On the other hand, if $\text{gap}(\lambda_i) = t < l = \text{rank}(\lambda_i)$, we know from Lemma 13 that $\lambda_i$ is spread over $\ell$. So, if $l < t$ then $\lambda_i$ has a sink $Y = z \lambda_i^{-1}$ with cardinal $> t$ and, as in the proof of Lemma 12, we can conclude that either $\text{gap}(\alpha) > t$ or some sink of $\alpha$ has cardinal $> t$ (in which case $\alpha$ is certainly spread over $t$). However, if $t = l$ and $t < r$ then $\lambda_i$ has a sink $Y = z \lambda_i^{-1}$ with cardinal $> \mu$. If every sink of $\lambda_i$ with cardinal $> \mu$ has its image under $\lambda_i$ outside $\text{dom}(\lambda_2 \cdots \lambda_m)$ then $\text{gap}(\alpha) \geq t$ (otherwise we obtain a contradiction by considering a sink of $\lambda_i$ with cardinal $> \mu$, which has its image under $\lambda_i$ inside $\text{dom}(\lambda_2 \cdots \lambda_m)$, in which case $\alpha$ has a sink with cardinal $> \mu$).

To simplify notation in the proof of our next result, we shall regard the index sets $I, J$ as the distinguishing feature of two disjoint sets $\{ x_i \}$ and $\{ x_j \}$: this will only be done in a context where no confusion is likely to occur.

**Theorem 4.** If $|X| = k$ is singular and $\alpha \in \mathcal{P}_X$ has rank $r$ then $\alpha \in \mathcal{L}_X$ if and only if $\alpha$ has non-zero gap, $\text{def}(\alpha) = k$, and either $\text{gap}(\alpha) \geq r$ or $\alpha$ is spread over $r$. Moreover, when this occurs, $\alpha$ can be written as a product of four or fewer nilpotents, each with index at most 3.
Proof. Suppose \( \alpha \in \mathcal{P}_X \). If \( \varepsilon \leq \text{cf}(\kappa) \) then \( \text{def}(\alpha) = \kappa \) and Theorem 3 gives the desired conclusion. If \( \varepsilon > \text{cf}(\kappa) \), we apply Lemma 14 instead.

Conversely, suppose \( \alpha \in \mathcal{P}_X \) has rank \( \varepsilon \) and satisfies the conditions of the Theorem. Write \( \alpha = \left( \begin{array}{cc} u & Y_j \\ v & t_j \end{array} \right) \)

where \( |Y_j| \geq 2 \) for each \( j \in J \) and \( |J \cup J| = \varepsilon \). If \( \varepsilon \) is finite then either the gap or some sink of \( \alpha \) has cardinal \( \geq \text{cf}(\kappa) \), and the result follows by Theorem 3.

Hence we suppose \( \varepsilon \geq \aleph_0 \). If \( \text{gap}(\alpha) \geq \varepsilon \), we consider two subcases: namely, when \( (X \setminus \text{ran} \alpha) \cap (X \setminus \text{dom} \alpha) \) has cardinal \( \geq \varepsilon \) or \( < \varepsilon \). An argument similar to that in the first paragraph of the proof of Theorem 3 then produces the required decomposition of \( \alpha \) (the only additional thing to observe is that when \( |(X \setminus \text{ran} \alpha) \cap (X \setminus \text{dom} \alpha)| < \varepsilon \) then \( |\text{ran} \alpha \cap (X \setminus \text{dom} \alpha)| \geq \varepsilon \)). Likewise, an argument similar to that in the other paragraphs of the proof of Theorem 3 gives the desired result when some sink \( Y \) of \( \alpha \) has cardinal \( \geq \varepsilon \) (once again, the only thing to note is that if \( |(X \setminus \text{ran} \alpha) \cap Y| < \varepsilon \) then \( |\text{ran} \alpha \cap Y| \geq \varepsilon \)).

So, to complete the proof, we suppose \( \alpha \) is spread over \( \varepsilon \), but the gap and each sink of \( \alpha \) has cardinal \( < \varepsilon \). In this event, \( \bigcup Y_j = \varepsilon \) and so \( \varepsilon = \kappa \) (since \( \kappa = \text{gap}(\alpha) + |J| + |\bigcup Y_j| \)).

If \( |I| < \kappa \) then \( |J| = \kappa \) and we can choose \( Y_0 \in \{ Y_j \} \) such that \( \max(|I|, \text{cf}(\kappa)) \leq |Y_0| < \kappa \). Put \( \{ Y_p \} = \{ Y_j \} \setminus Y_0 \) and note that \( \bigcup Y_p = |M| = \kappa \); \( |N| = \text{cf}(\kappa) \), and \( |\bigcup N| = \kappa \): this is possible since \( \kappa \), being singular, is the sum of a strictly increasing sequence of \( \text{cf}(\kappa) \) cardinals \( \kappa \), and we know that for each \( n \in \mathbb{N} \) there exists some \( Y_n \) with cardinal \( > \kappa \). Next write \( \bigcup Y_n = \{ x_m \} \cup \{ x_q \} \), where \( |Q| = \kappa \), and choose disjoint \( \{ x_i \}, \{ x_n \} \subseteq Y_0 \) and some \( z \notin \text{dom} \alpha \). Finally, put

\[
\lambda = \left( \begin{array}{ccc} u & Y_0 & Y_m & Y_n \\ x_i & z & x_m & x_n \\ v & t_0 & t_m & t_n \end{array} \right), \quad \mu = \left( \begin{array}{ccc} x_i & z & x_m & x_n \\ v & t_0 & t_m & t_n \end{array} \right)
\]

and note that \( \lambda \) is nilpotent with index 3, while \( \mu \in \mathcal{X} \) and \( \text{def}(\mu) = \kappa \) (since \( \text{def}(\alpha) = \kappa \) and \( \text{gap}(\mu) = \kappa \) (since \( \{ x_q \} \subseteq X \setminus \text{dom} \mu \)). Also \( \alpha = \lambda \mu \) and hence the result now follows from Corollary 4.

If \( |I| = \kappa \) and \( |J| < \kappa \), we choose \( Y_0 \in \{ Y_j \} \) with \( |J| < |Y_0| < \kappa \) and write \( \{ Y_j \} \setminus Y_0 = \{ Y_p \} \). Next choose \( \{ x_p \} \subseteq Y_0 \) and write \( \bigcup Y_p = \{ x_i \} \cup \{ x_q \} \), where \( |Q| = \kappa \). Now choose \( z \notin \text{dom} \alpha \) and put

\[
\lambda = \left( \begin{array}{ccc} u & Y_p & Y_0 \\ x_i & z & x_p \\ v & t_p & t_0 \end{array} \right), \quad \mu = \left( \begin{array}{ccc} x_i & x_p & z \\ v & t_p & t_0 \end{array} \right).
\]

As before, the result follows after applying Corollary 4.
We now assume \(|I| = \ell\) and \(|J| = \kappa\) and extend the basic idea of the last two paragraphs to this situation. Namely, we choose \(Y_0 \in \{Y_j\}\) with \(\text{cf}(\ell) < |Y_0| < \ell\), put \(\{Y_p\} = \{Y_j\} \setminus Y_0\), and write \(\{t_p\} = \{t_m\} \cup \{t_n\}\) where \(|M| = \ell, \ |N| = \text{cf}(\kappa),\) and \(\bigcup Y_n = \ell\). Next we write \(\bigcup Y_n = \{x_i\} \cup \{x_m\} \cup \{x_o\}\) where \(|Q| = \kappa\) and choose \(\{x_n\} \subseteq Y_0\). Now we put

\[
\lambda = \begin{pmatrix} u_i & Y_m & Y_n & Y_0 \end{pmatrix}, \quad \mu = \begin{pmatrix} x_i & x_m & x_n & z \end{pmatrix}
\]

and observe that the result follows as before.

As before (after Corollary 4) we note that an alternative way of expressing the above result is to say that if \(\ell\) is singular then \(\mathcal{L}_X\) consists of all \(\alpha \in \mathcal{P}_X\) with rank \(< k\), together with the \(\alpha \in \mathcal{P}_X\) with rank \(\alpha = \text{def}(\alpha) = \kappa\) and either \(\text{gap}(\alpha) = \ell\) or \(\alpha\) is spread over \(k\).

Howie ended [8] by using his main result to show that any semigroup can be embedded in an idempotent-generated regular semigroup. We now intend to prove a similar result regarding nilpotents.

When \(|X|\) is infinite and regular, the semigroup \(\mathcal{L}_X\) is regular (in the sense that for each \(\alpha \in \mathcal{L}_X\), there exists \(\beta \in \mathcal{L}_X\) with \(\alpha = \alpha \beta \alpha\)). For, if \(\alpha \in \mathcal{L}_X\) and \(\text{gap}(\alpha) = |X|\), put \(\alpha = \{b_i\}, A_i = b_i \alpha^{-1}\), and then choose \(a_i \in A_i\) for each \(i \in I\). Now define \(\beta \in \mathcal{S}_X\) by letting \(\text{dom} \ \beta = \{b_i\}\) and \(b_i \beta = a_i\) for each \(i \in I\). Since \(\text{def}(\alpha) = |X|\) we have \(\text{gap}(\beta) = |X|\), and since \(X \setminus \text{dom} \ \alpha \subseteq X \setminus \text{dom} \ \beta\) we have \(\text{def}(\beta) = |X|\); thus, by Corollary 4, \(\beta\) is a product of nilpotents in \(\mathcal{S}_X\) and clearly \(\alpha = \alpha \beta \alpha\). When \(\alpha \in \mathcal{S}_X\) and some sink \(B\) of \(\alpha\) has cardinal equal to \(|X|\), put \(B \alpha = a, \ \{y_i\} = (\text{ran} \ \alpha) \setminus a, C_i = y_i \alpha^{-1}\) and then choose \(c_i \in C_i\) for each \(i \in I\). Now define \(\beta \in \mathcal{S}_X\) by letting \(\text{dom} \ \beta = a \cup \{c_i\}\) and \(a \beta = a\), \(y_i \beta = b_i\), for each \(i \in I\). Then as before \(\text{gap}(\beta) = |X|\), while \(B \alpha = \alpha\) implies \(\text{def}(\beta) = |X|\); another application of Corollary 4 leads us to conclude that \(\beta \in \mathcal{L}_X\) and of course \(\alpha = \alpha \beta \alpha\).

An entirely similar argument shows that when \(|X|\) is finite then \(\mathcal{L}_X\) is also regular. For, if \(|X|\) is even and \(\alpha \in \mathcal{L}_X\) then \(\text{gap}(\alpha) \neq 0\) and rank \(\alpha \leq n - 1\). Suppose \(\text{ran} \ \alpha = \{b_i\}, A_i = b_i \alpha^{-1}\) and choose \(a_i \in A_i\) for each \(i = 1, \ldots, r\) where rank \(\alpha = r \leq n - 1\). Then \(\beta \in \mathcal{P}_X\) defined by letting \(\text{dom} \ \beta = \{b_i\}\) and \(b_i \beta = a_i\) satisfies \(\text{gap}(\beta) \neq 0\) and rank \(\beta \leq n - 1\). Hence, by Theorem 1, \(\beta \in \mathcal{L}_X\) and we also have \(\alpha = \alpha \beta \alpha\). On the other hand, if \(|X|\) is odd and \(\alpha \in \mathcal{L}_X\) has rank \(\leq n - 2\) and non-zero gap then the \(\beta\) just defined has rank \(\leq n - 2\) and non-zero gap, and so it belongs to \(\mathcal{L}_X\). If \(\alpha \in \mathcal{L}_X\) is an even transformation with rank \(n - 1\) then \(\alpha^{-1}\) is also and we have \(\alpha = \alpha \alpha^{-1} \alpha\), where \(\alpha^{-1} \in \mathcal{L}_X\).

**Theorem 5.** Any (finite) semigroup can be embedded in a (finite) nilpotent-generated regular semigroup.

**Proof.** If \(S\) is a semigroup and \(|S|\) is infinite we let \(X = S^1 \cup Y\), where
$S^1$ is $S$ with an identity adjoined and $Y$ is a set, disjoint with $S^1$, such that $|X| = |Y|$ and $|X|$ is regular (if $|S|$ is regular, $Y$ could be a copy of $S$ that is disjoint with $S$; and if $|S|$ is singular then $Y$ could be a set disjoint with $S$ and having cardinal equal to $|S|^{-}$, the successor of $|S|$). Define $\rho: S \to \mathcal{P}_X$, $a \to \rho_a$, where $\text{dom} \ \rho_a = S^1$ and $xa = ax$ for all $x \in S^1$. Then $X \setminus \text{dom} \ \rho_a = Y$ and $X \setminus \text{ran} \ \rho_a \supseteq Y \cup 1$. Thus, $\rho$ embeds $S$ into $\mathcal{P}_X$.

When $|S|$ is finite, we put $X = S^1 \cup \{y, z\}$, where $y, z \notin S^1$ and define $\rho$ as before. Since $\text{gap}(\rho_a) \neq 0$ and $\text{rank} \ \rho_a \leq n - 2$ where $|X| = n \geq |S| + 2$, $\rho$ embeds $S$ into $\mathcal{P}_X$ as required.

Finally, we remark that the subsemigroup of $\mathcal{S}_X$ generated by the nilpotents of $\mathcal{S}_X$ is inverse and that the Preston–Vagner method of embedding any inverse semigroup into some $\mathcal{S}_X$ can be easily adapted to show:

**Theorem 6.** Any inverse semigroup can be embedded in a nilpotent-generated inverse semigroup.

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**References**