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Davenport constant with weights

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ABSTRACT

Let $A \subset \{1, \ldots, n-1\}$ and let $(x_1, \ldots, x_t) \in \mathbb{Z}^t$ be a sequence of integers with a maximal length such that for all $(a_1, \ldots, a_t) \in A^t$, $\sum_{1 \le i \le t} a_i x_i \not\equiv 0 \pmod{n}$. The authors show that for any sequence of integers $(x_1, \ldots, x_{n+t}) \in \mathbb{Z}^{n+t}$, there are $b_1, \ldots, b_n \in A$ and $1 \le k_1 < k_2 < \cdots < k_n \le n+t$ such that

$$\sum_{1 \le i \le n} b_i x_{k_i} \equiv 0 \pmod{n}.$$

1. Introduction

For an abelian group G, the Davenport constant $D(G)$ is defined to be the smallest natural number k such that any sequence of k elements in G has a non-empty subsequence whose sum is zero (the identity element). Another interesting constant $E(G)$ is defined to be the smallest natural number k such that any sequence of k elements in G has a subsequence of length $|G|$ whose sum is zero.

The following result, due to Gao [6], connects these two invariants.

Theorem 1.1. *If G is a finite abelian group of order n, then $E(G) = D(G) + n - 1$.*

For a finite abelian group G and any non-empty $A \subset \mathbb{Z}$, Adhikari and Chen [2] defined the Davenport constant of G with weight A, denoted by $D_A(G)$, to be the least natural number k such that, for any sequence $(x_1, \ldots, x_k)$ with $x_i \in G$, there exists a non-empty subsequence $(x_{j_1}, \ldots, x_{j_l})$ and $a_1, \ldots, a_l \in A$ such that $\sum_{i=1}^{l} a_i x_{j_i} = 0$. Clearly, if G is of order n, it is equivalent to consider A to be a non-empty subset of $\{0, 1, \ldots, n-1\}$ and cases with $0 \in A$ are trivial.

Similarly, for any such set A, for a finite abelian group G of order n, the constant $E_A(G)$ is defined to be the least $t \in \mathbb{N}$ such that for all sequences $(x_1, \ldots, x_t)$ with $x_i \in G$, there exist indices $j_1, \ldots, j_n \in \mathbb{N}$, $1 \le j_1 < \cdots < j_n \le t$, and $\vartheta_1, \ldots, \vartheta_n \in A$ with $\sum_{i=1}^{n} \vartheta_i x_{j_i} = 0$.

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For the group $G = \mathbb{Z}/n\mathbb{Z}$, we write $E_A(n)$ and $D_A(n)$ respectively for $E_A(G)$ and $D_A(G)$. For several sets $A \subset \mathbb{Z}/n\mathbb{Z} \setminus \{0\}$ of weights, exact values of $E_A(n)$ and $D_A(n)$ have been determined: The case $A = \{1\}$ is classical and is covered by the well-known EGZ theorem [5]; the cases $A = \{1, -1\}$ and $A = \{1, 2, \ldots, n-1\}$ were done in [1]; the case $A = (\mathbb{Z}/n\mathbb{Z})^\star = \{a : \gcd(a, n) = 1\}$, was proved in [9,10], settling a conjecture from [1]; the case where $n = p$ is a prime, $A = \{1, \ldots, r\}$, or the set of all quadratic residues (mod p), were solved in [3]. Results in all these known cases, lead Adhikari and Rath [3] to the expectation that for any set $A \subset \mathbb{Z}/n\mathbb{Z}$ of weights, the equality $E_A(n) = D_A(n) + n - 1$ holds (see also Conjecture 4.1 in [9]). In [11], Thangadurai stated the following conjecture

Conjecture 1.1 (*[11] Conjecture 1*). *For any finite abelian group with exponent n and for any non-empty subset A of* $\{1, 2, \ldots, n\}$, *we have*

$$E_A(G) = D_A(G) + |G| - 1.$$

Thangadurai [11] also showed that Conjecture 1.1 holds for some other cases. In [2], Adhikari and Chen proved that Conjecture 1.1 holds for $A = \{a_1, \ldots, a_r\}$ with $\gcd(a_2 - a_1, \ldots, a_r - a_1, n) = 1$.

The main purpose of the present paper is to prove that Conjecture 1.1 holds for cyclic groups. By using the main theorem of Devos, Goddyn and Mohar [4] and a recently proved theorem of the authors [12], we shall prove the following theorem

Theorem 1.2. *For any non-empty set* $A \in \mathbb{Z}$, $E_A(n) = D_A(n) + n - 1$.

Let G be an additive finite abelian group. The *free abelian multiplicative monoid* with base G will be denoted by $\mathcal{F}(G)$. Recall that an element S of $\mathcal{F}(G)$ is a finite product $S = g_1 \ldots g_k = \prod_{i=1}^k g_i = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G)$, where $\mathsf{v}_g(S) \geq 0$ is called the *multiplicity* of g in S and $\sum_{g \in G} \mathsf{v}_g(S) < \infty$. An element S of $\mathcal{F}(G)$ will be called *sequences* over G. We call $|S| = k$ the *length* of S, $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) | g \in G\} \in [0, |S|]$ the maximum of the multiplicities of S, $\mathrm{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\}$ the *support* of S. For every $g \in G$ we set $g + S = (g + g_1) \cdots (g + g_k)$.

We say that S contains some $g \in G$ if $\mathsf{v}_g(S) \geq 1$ and a sequence $T \in \mathcal{F}(G)$ is a *subsequence* of S if $\mathsf{v}_g(T) \leq \mathsf{v}_g(S)$ for every $g \in G$, denoted by $T|S$. Furthermore, by $\sigma(S)$ we denote the sum of S, i.e. $\sigma(S) = \sum_{i=1}^k g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$. For every $k \in \{1, 2, \ldots, |S|\}$, let $\sum_k(S) = \{g_{i_1} + \cdots + g_{i_k} | 1 \leq i_1 < \cdots < i_k \leq |S|\}$, $\sum_{\leq k}(S) = \cup_{i=1}^k \sum_i(S)$, and let $\sum(S) = \sum_{\leq |S|}(S)$.

Let S be a sequence in G. We call S a *zero-sum sequence* if $\sigma(S) = 0$.

Also, we follow the same terminologies and notation as in the survey article [7] or in the book [8].

2. Lemmas

First, we need a result on the sum of l finite subsets of G. If $\mathbf{A} = (A_1, A_2, \ldots, A_m)$ is a sequence of finite subsets of G, and $l \leq m$, we define

$$\sum_l(\mathbf{A}) = \{a_{i_1} + \cdots + a_{i_l} : 1 \leq i_1 < \cdots < i_l \leq m \text{ and } a_{i_j} \in A_{i_j} \text{ for every } 1 \leq j \leq l\}.$$

So $\sum_l(\mathbf{A})$ is the set of all elements which can be represented as a sum of l terms from distinct members of $\mathbf{A}$. The following is the main result of Devos, Goddyn and Mohar [4].

Theorem DGM. *Let* $\mathbf{A} = (A_1, A_2, \ldots, A_m)$ *be a sequence of finite subsets of* G, *let* $l \leq m$, *and let* $H = \mathrm{stab}(\sum_l(\mathbf{A}))$. *If* $\sum_l(\mathbf{A})$ *is nonempty, then*

$$\left|\sum_l(\mathbf{A})\right| \geq |H| \left(1 - l + \sum_{Q \in G/H} \min\{l, |\{i \in \{1, \ldots, m\} : A_i \cap Q \neq \emptyset\}|\}\right).$$

We also need the following new result on Davenport constant [12].

Theorem YZ. *Let* G *be a finite abelian group of order* n, *and* $D(G)$ *the Davenport constant of* G. *Let* $S = 0^{\mathsf{h}(S)} \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G)$ *be a sequence with a maximal multiplicity* $\mathsf{h}(S)$ *attained by* 0 *and*

$|S| \geq n + D(G) - 1$. Then there exists a subsequence S_1 of S with length $|S_1| \geq t + 1 - D(G)$ and $0 \in \sum_k(S_1)$ for every $1 \leq k \leq |S_1|$. In particular, for every sequence S in G with length $|S| \geq n+D(G)-1$, we have

$$0 \in \sum_{km}(S), \quad \textit{for every } 1 \leq k \leq (|S| + 1 - D(G))/m,$$

where m is the exponent of G.

3. Proof of Theorem 1.2

Proof. The proof of $E_A(n) \geq D_A(n) + n - 1$ is easy, so it is sufficient to prove the reverse inequality.

For any non-empty set $A = \{a_1, \dots, a_r\} \subset \mathbb{Z}$ and a cyclic group $G = \mathbb{Z}/n\mathbb{Z}$, let $t = D_A(n) + n - 1$ and $S = x_1 \cdots x_t$ be any sequence over G with length $|S| = t = D_A(n) + n - 1$. Put

$$A_i = Ax_i = \{a_1x_i, \dots, a_rx_i\} \quad \text{for } i = 1, \dots, t$$

and $\mathbf{A} = (A_1, \dots, A_t)$. It suffices to prove that $0 \in \sum_n(\mathbf{A})$.

We shall assume (for a contradiction) that the theorem is false and choose a counterexample (A, G, S) so that $n = |G|$ is minimum, where G is a cyclic group of order n, A is a finite subset of $\mathbb{Z}$ and $S = x_1 \cdots x_t$ is a sequence in G such that

$$0 \notin \sum_n(\mathbf{A}).$$

Next we will show that our assumptions imply $H = \mathrm{stab}(\sum_n(\mathbf{A})) = \{0\}$. Suppose (for a contradiction) that $H = \mathrm{stab}(\sum_n(\mathbf{A})) \neq \{0\}$ and let $\varphi : G \longrightarrow G/H$ denote the canonical homomorphism and $\varphi(x_i)$ the image of x_i for $1 \leq i \leq t$. Let $\mathbf{A}_\varphi = (\varphi(A_1), \dots, \varphi(A_t))$. By our assumption for the minimal of $|G|$, the theorem holds for $(A, \varphi(G), \varphi(S))$. Since $t > |\varphi(G)| + D_A(\varphi(G)) - 1$, $\varphi(G)|n$ and $D_A(G) \geq D_A(\varphi(G))$, repeated applying the theorem to the sequence $\varphi(S) = \varphi(x_1) \cdots \varphi(x_t)$ we have

$$\varphi(0) = \varphi(H) \in \sum_n(\mathbf{A}_\varphi),$$

thus $0 \in H \subset \sum_n(\mathbf{A})$. This contradiction implies that $H = \mathrm{stab}(\sum_n(\mathbf{A})) = \{0\}$.

If there is an element $a \in G$ such that $|\{j \in \{1, \dots, t\} : a \in A_j\}| \geq n$, then $0 \in \sum_n(\mathbf{A})$, a contradiction. Therefore we may assume that for every $a \in G$, $|\{j \in \{1, \dots, t\} : a \in A_j\}| \leq n$. Let r be the number of $i \in \{1, \dots, t\}$ with $|A_i| = 1$, by Theorem DGM and the assumptions, we have

$$\begin{aligned} n - 1 &\geq \sum_n(\mathbf{A}) \geq 1 - n + \sum_{a \in G} \min\{n, |\{j \in \{1, \dots, t\} : a \in A_j\}|\} \\ &= 1 - n + \sum_{i=1}^{t} |A_i| \geq 1 - n + 2(n + D_A(G) - 1 - r) + r. \end{aligned}$$

It follows that

$$r \geq 2D_A(G). \tag{1}$$

Without loss of generality, we may assume that $x_1, \dots, x_r$ are all the elements in $\{x_1, \dots, x_t\}$ such that $|A_i| = 1$, and x_1 is the element in $\{x_1, \dots, x_r\}$ such that a_1x_1 attains the maximal multiplicity in the sequence $S_1 = (a_1x_1) \cdots (a_1x_r)$. Observe that $\sum_n(\mathbf{A}) = \sum_n(A(x_1 - x_u), \dots, A(x_t - x_u))$ for every $1 \leq u \leq r$. Therefore without loss of generality we may assume that $a_1x_1 = 0$ and $\mathsf{v}_0(S_1) = \mathsf{h}(S_1)$ for the sequence

$$S_1 = (a_1x_1) \cdots (a_1x_r) = 0^{\mathsf{h}(S_1)}(a_1x_{\mathsf{h}(S_1)+1}) \cdots (a_1x_r). \tag{2}$$

Let $H_1 = \langle x_1, \dots, x_r \rangle$ be the group generated by $x_1, \dots, x_r$, $H = a_1H_1$. We have the following claim.

Claim. $D_A(G) \geq D_A(H_1) \geq D(H) = |H|$.

The last equality of the Claim follows from the fact that H is a subgroup of the cyclic group G. The first inequality in the Claim is obvious, so we only need to prove that $D_A(H_1) \geq D(H)$. Suppose that $W = y_1 \cdots y_{D(H)-1}$ is a zero-sum free sequence in H. Since $H = a_1H_1$, we have $y_i = a_1w_i$, $w_i \in H_1$, $i = 1, \ldots, D(H)-1$. Further, it is easy to see that $Aw_i = a_1w_i$, $i = 1, \ldots, D(H)-1$ by the definition of H_1, so $w_1 \cdots w_{D(H)-1}$ is a zero-sum free sequence in H_1 with respect to the weight A, thus $D_A(H_1) \geq D(H)$ and the Claim follows.

By the Claim, (1), (2) and Theorem YZ, S_1 has a subsequence S_2 of length $|S_2| = s \geq r + 1 - |H|$ such that $0 \in \sum_l(S_2)$ for every $1 \leq l \leq s$. Without loss of generality, we may assume that $S_2 = (a_1x_1) \cdots (a_1x_s)$.

If $s \geq n$ then $0 \in \sum_n(S_2) \subset \sum_n(\mathbf{A})$, we are done.

If $s < n$, then $|x_{s+1} \cdots x_t| = t - s = n - 1 + D_A(G) - s \geq D_A(G)$. Repeated using the definition of $D_A(G)$, there exists an integer v such that $v \leq n$, $t - s - v \leq D_A(G) - 1$ and

$$0 \in \sum_v ((A_{s+1}, \ldots, A_t)).$$

Since $\sum_l(S_2) = \sum_l((A_1, \ldots, A_s))$ and $0 \in \sum_l((A_1, \ldots, A_s))$ for every $1 \leq l \leq s$, we have

$$0 \in \sum_{v+k}(\mathbf{A}) \quad \text{for every } 0 \leq k \leq s.$$

Therefore $0 \in \sum_n(\mathbf{A})$ since $v + s \geq t + 1 - D_A(G) \geq n$. This completes the proof of the theorem. □

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