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On Traces of *d*-stresses in the Skeletons of Lower Dimensions of Piecewise-linear *d*-manifolds

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We show how a d-stress on a piecewise-linear realization of an oriented (non-simplicial, in general) d-manifold in $\mathbb{R}^d$ naturally induces stresses of lower dimensions on this manifold, and discuss implications of this construction to the analysis of self-stresses in spatial frameworks. The mappings we construct are not linear, but polynomial. In the 1860–70s J. C. Maxwell described an interesting relationship between self-stresses in planar frameworks and vertical projections of polyhedral 2-surfaces. We offer a spatial analog of Maxwell's correspondence based on our polynomial mappings. By applying our main result we derive a class of three-dimensional spider webs similar to the two-dimensional spider webs described by Maxwell. We also conjecture an important property of our mappings that is based on the lower bound theorem ($g_2(d+1) = \dim Stress_2 \geq 0$) for d-pseudomanifolds generically realized in $\mathbb{R}^{d+1}$ [10].

1. INTRODUCTION

Let $G(E, V)$ be a framework (possibly infinite), where E is the edge set of the framework, and V the vertex set, in $\mathbb{R}^d$. An (equilibrium) *stress* is an assignment of real numbers $\omega_{ij} = \omega_{ji}$ to the edges, a tension if the sign is positive, or a compression if negative, so that the equilibrium condition

$$\sum_{\{j \mid (ij)\in E\}} \omega_{ij}(\mathbf{v}_j - \mathbf{v}_i) = 0$$

holds at each vertex $\mathbf{v}_i \in V$. The stresses on (E, V) form a linear subspace of $\mathbb{R}^{|E|}$, the left null-space of the *rigidity matrix* $RM(E, V)$. Let M be the $|E| \times |V|$ incidence matrix with entries $M_{ij} = 1$ if and only if $\mathbf{v}_j \in \partial e_i$, but zero otherwise. Then, the ridigity matrix is formed by replacing each entry of M by a d-component row vector, the zero vector when $M_{ij} = 0$, and an edge vector parallel to the ith edge, pointing away from vertex $\mathbf{v}_j$, when $M_{ij} = 1$. The dimension of the space of stresses is equal to $|E| - \operatorname{rank}(RM)$, and, the dimension of the subspace of external loads that can be resolved by the framework is equal to $\operatorname{rank}(RM)$. If all external loads can be resolved, the framework is said to be *statically rigid*. Under these circumstances $\operatorname{rank}(RM) = d|V| - \binom{d+1}{2}$, since the dimension of the space of all possible external loads is $d|V| - \binom{d+1}{2}$. It also follows that the dimension of the space of stresses is $|E| - d|V| + \binom{d+1}{2}$ in the statically rigid case.

The notion of stress on a framework can be naturally generalized to k-stress on a cell complex. This generalization is useful in the combinatorics and geometry of piecewise-linear manifolds, rigidity theory, and the theory of Dirichlet–Voronoi diagrams. Such generalizations have been proposed by Lee [13], Tay *et al.* [24], Crapo and Whiteley [8], and Rybnikov [19, 20].

Consider a piecewise-linear realization K in $\mathbb{R}^N$ of a d-dimensional cell complex $\mathcal{K}$. Denote by $\mathbf{n}(F, C)$ the inner unit normal to a cell C at its facet F.

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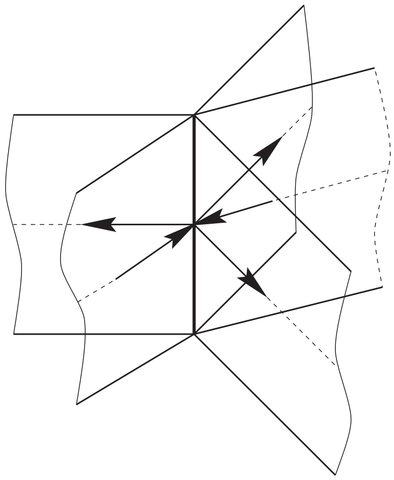

FIGURE 1. Equilibrium of forces at 1-cell.

DEFINITION 1.1. A real-valued function $\omega(\cdot)$ on the $(k-1)$-cells of K is a k-stress if at each (internal) $(k-2)$-cell F of K

$$\sum_{\{C \mid F \subset C\}} \omega(C)\mathrm{vol}_{k-1}(C)\,\mathbf{n}(F, C) = \mathbf{0},$$

where the sum is taken over all $(k-1)$-cells in the star of F. The quantities $\omega(C)$ are the coefficients of the k-stresses, a tension if the sign is positive, a compression if the sign is negative. C need not be convex, but it is important that its boundary is a homology sphere.

It is easy to see that k-stresses form a linear space, and that k-tensions and k-compressions form congruent cones in this linear space. We denote the space of all k-stresses on K by $Stress_k(K)$, the cone of all k-tensions by $Tension_k(K)$. If the coefficients $\omega(C)$ are not all zero, the k-stress ω is called non-trivial. Figure 1 illustrates the geometry of the equilibrium condition for a 3-stress at an edge of a cell complex in $\mathbb{R}^3$.

In the case of stress on a framework, $\omega(e)$ is the force per unit length, and the static force applied at the end points of edge e is $\omega(e)\|e\|$. For a $(k-1)$-cell C a k-stress $\omega(C)$ is the force per unit relative $(k-1)$-volume (area) of C, and the static force applied at a $(k-2)$-face of C is $\omega(C)\ \mathrm{vol}_{k-1}C$. For frameworks the equilibrium condition is written for the star of each vertex of the framework, while for k-stresses on cell complexes the equilibrium condition is formulated for the star of each $(k-2)$-cells, and the summation is over all $(k-1)$-cells. Moreover, the equilibrium of forces in the case of 3-stress has a natural physical interpretation: one can think of plates making contact at a common edge: some plates are under tension, and some under compression, just like the edges in a framework.

The main result of this paper is Theorem 6.2 of Section 6 where we construct a polynomial mapping of degree $d-k+1$ from the space of d-stresses to the space of k-stresses $(0 \le k \le d)$ for a piecewise-linear realization of an oriented d-manifold in $\mathbb{R}^{d+1}$. In general, our mappings are *not bijective*, since for a generic realization of a simplicial sphere in $\mathbb{R}^3$ the dimension of the space of 2-stresses may exceed the dimension of the space of 3-stresses. Below we outline how our research on k-stresses relates to the Maxwell–Cremona theory and its generalizations. This theory served as one of the motivations to study the relationship between stresses of different dimensions.

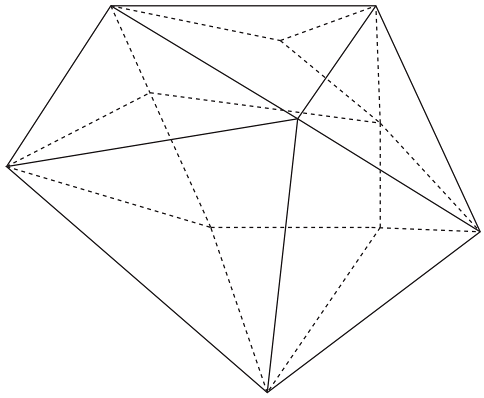

FIGURE 2. 2-sphere realized in $\mathbb{R}^2$.

Let G be a framework in the plane. Suppose there is a polyhedron P in three-space such that the vertical projection on the plane of F takes the vertices of P onto the vertices of G and the edges of P onto the edges of G in a one-to-one manner. Then, as shown by Maxwell (1864, 1869) and Cremona (1872), there is a stress $\omega(P)$ on G completely defined by the values of the dihedral angles of P. Moreover, given a framework G in the plane, if ω is a stress on G and G can be regarded as the graph of a spherical complex, one can find a polyhedron $P(\omega)$ (defined up to the choice of a supporting plane) in three-space such that G is the vertical projection of the 1-skeleton of P. These 19th century results have been extended and put on a rigorous mathematical basis by Crapo and Whiteley [6, 26, 27]. They proved that for piecewise-linear spheres realized in $\mathbb{R}^2$ (like in Figure 2) there is a natural linear isomorphism between the space of stresses on the 1-skeleton and the space of liftings (lifting is the operation that is inverse to projection on the plane) considered up to the addition of an affine function.

Maxwell used a new geometrical tool, the *reciprocal*, in his study of how stresses and projections relate to each other. Roughly speaking, a reciprocal is a planar realization of the dual combinatorial graph of the spherical complex, such that its edges are perpendicular to the corresponding edges of the complex, as pictured in Figure 3. The relationship between a graph and its reciprocal is well illustrated by the relationship between the 1-skeletons of the Delaunay and Dirichlet–Voronoi decompositions for a set of sites in the plane. For spheres and $\mathbb{R}^2$ the linear space of reciprocals is isomorphic to the space of stresses.

As was shown by Crapo, Whiteley and Rybnikov, for certain classes of d-manifolds, including homology spheres, there is a similar connection between the geometry of piecewise-linear d-manifolds realized in $\mathbb{R}^{d+1}$ and stresses supported on the $(d-1)$-cells of their realizations defined by the vertical (or radial) projection on a d-subspace of $\mathbb{R}^{d+1}$. In this case the equilibrium of forces is required not at each vertex, but at each $(d-2)$-cell [6, 19, 26]. Such stresses are called d-stresses because d is the lowest dimension of a manifold for which the space of d-stress is non-trivial in the sense that it essentially depends on the combinatorics and geometry of the manifold. (The space of d-stresses of a closed $(d-1)$-manifold realized in $\mathbb{R}^{d-1}$ is either $\mathbb{R}$ or 0 depending on whether this manifold is orientable or not.)

By an informal conjecture of Baracs and Whiteley there is an analogous correspondence between projections of 4-polyhedra from $\mathbb{R}^4$ onto $\mathbb{R}^3$ and stresses in spatial frameworks [31]. Their idea was, perhaps, motivated by Minkowski's theorem on the vanishing of the sum of normals to a convex polytope at its facets (Theorem 5.1 of Section 5). The projections of d-manifolds with trivial H_1 over $\mathbb{Z}_2$ from $\mathbb{R}^{d+1}$ onto $\mathbb{R}^d$ correspond to d-stresses (see Section 2

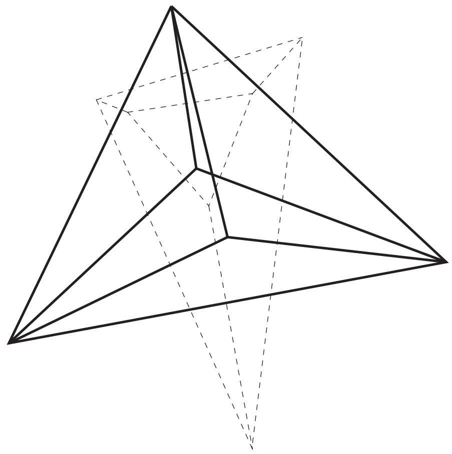

FIGURE 3. Reciprocal for a 2-sphere in $\mathbb{R}^2$.

and [19, 29]); thus, one can reformulate their conjecture as the existence of a natural correspondence between d-stresses on a d-manifold realized in $\mathbb{R}^d$ and stresses on its 1-skeleton (in the general theory of stresses such stresses are called 2-stresses). Theorem 6.2 estublishes a natural one-way connection between d-stresses and k-stresses ($k \leq d$) for oriented piecewise-linear manifolds realized in $\mathbb{R}^d$. Therefore, in some sense this theorem supports the Baracs–Whiteley hypothesis.

Our mappings are well defined not only for simplicial manifolds, but also for general cell-partitions of manifolds. As for the simplicial case, we conjecture that these mappings are *injective* for any generic realization of an orientable closed *simplicial* d-manifold in $\mathbb{R}^d$. However, we are primarily interested in applications of the general theory of stresses and liftings to three-dimensional manifolds and spatial spider webs. From this point of view our construction can be regarded as a natural extension of Maxwell's correspondence between stresses and liftings to spatial frameworks. In the last paragraph of this section we sketch the main ideas and concepts employed in our construction.

Let Δ be a homology d-manifold decomposed into cells, each of which is a simplicial star (see [17, 22] for a discussion of such dissections). The construction of our mapping proceeds systematically in steps. The first step establishes a natural one-to-one correspondence between d-stresses and *reciprocals* (see Section 4). The notion of reciprocal we employ generalizes that of Maxwell (see above). In particular, a segment whose ends are the vertices of the reciprocal corresponding to two adjacent d-cells of Δ is perpendicular to their common facet. This one-to-one correspondence holds only when certain homological restrictions are placed on the manifold, for example, when $H_1(\Delta, \mathbb{Z}_2) = 0$ [19]. Notice that the star of a cell satisfies this condition. Given a d-stress ω, we can construct the corresponding reciprocal $R(\omega, v)$ for the star of each vertex v. If two cells C_1 and C_2 share a face F, the sub-reciprocals of $R(\omega, C_1)$ and $R(\omega, C_2)$ corresponding to F are congruent. Nevertheless, when $H_1(\Delta, \mathbb{Z}_2) \neq 0$, it is generally not possible to construct a global reciprocal (see [19]). Now, one can take for Δ the dual cell-decomposition Δ^*, an idea of which goes back to Poincaré (for details see [17, 22]). We construct a piecewise-linear realization of a barycentric triangulation TD of the dual cell-decomposition Δ^*. Note that the barycentric triangulation of the original cell-decomposition Δ is isomorphic to the barycentric triangulation of Δ^* [22]. We will denote by $C*$ the cell of Δ^* dual to a cell C of Δ. After such a dissection a cell C^* of Δ^* can be regarded as a simplicial star $TD(C^*)$. Naturally, we want to consider only special affine realizations of $TD(C^*)$, namely those where the barycentric triangulation of each k-cell belongs to its affine k-span. Note that $R(\omega, C)$, the subreciprocal of $St(C)$ corresponding to stress ω, defines an affine realization of the barycentric triangulation of C^* up to the location of the barycenters for the k-cells ($k > 0$) of C^*: $TD(C^*)$ lies in the affine span of $R(\omega, C)$, and the vertices of $R(\omega, C)$ belong to the vertex set of $TD(C^*)$. In this case one can introduce a natural summation of volumes of the oriented simplexes of C^*, such that the sum does not depend on the location of the baricenters of the cells (except for the vertices of C^*). When C^* is embedded into $\mathbb{R}^d$ the result is the oriented volume of C^*. That is why we call this function on 'flat' realizations of oriented cells of the dual decomposition the *signed generalized volume*. Evidently, this function can be equally thought of as a function on reciprocals. Using the orientability of Δ and Minkowski's theorem 5.1, we show in Section 6 that the generalized k-volumes of k-cells of Δ^* can be interpreted as the coefficients of $(d-k+1)$-stresses on Δ. As it can be seen from this informal description, the main ingredients of our construction are volumes, reciprocals, duality in homology manifolds, and the notion of orientability. In fact, we suspect that our construction can be generalized for any dimension n, $\lfloor \frac{d+1}{2} \rfloor \leq n \leq d$, thereby providing canonic polynomial mappings from the space of n-stresses to the spaces of k-stresses, $d-n \leq k \leq n$.

1.1. Notation. All complexes that we consider are polyhedra (simplicial complexes) from the topological point of view. However, all theorems in this paper are stated for fixed decompositions of simplicial complexes into polyhedral *cells* (also called blocks or simplicial stars in combinatorial topology [17, 22]) which are not necessarily simplexes. We assume that all complexes have at most a countable number of cells. Cells of co-dimension 1 are referred to as *facets*. We denote the star of a cell C by $St(C)$, and the k-dimensional skeleton of a complex $\mathcal{K}$ by $Sk^k(\mathcal{K})$.

We shall consider more general constructions than embeddings or immersions of cell complexes into Euclidean space, such as *piecewise-linear* (PL—*throughout the text*) *realizations* . Such general construction can be helpful, for example, for studying frameworks with bar intersections, polyhedral scenes, splines over triangulations (in the planar case this point of view was adopted in [6, 24, 26]; in the three-dimensional case such PL-realizations were considered by Crapo and Whiteley in [6, 28]), and in the case of general dimension by Tay, White, and Whiteley [24]. For example, a Schlegel d-diagram is a PL-realization of a $(d+1)$-polytope P in $\mathbb{R}^d$ obtained by the radial projection of P onto one of its facets. In all geometric discussions cell complexes will be considered as PL-realizations, rather than abstract combinatorial objects.

Recall that one can identify an abstract combinatorial cell complex $\mathcal{K}^d$ with its embedding into $\mathbb{R}^{2d+1}$ (since $\mathcal{K}^d$ is a polyhedron). More formally, a PL-realization of a combinatorial simplicial complex $\mathcal{K}^d \subset \mathbb{R}^{2d+1}$ with a fixed decomposition into polyhedral cells is a continuous PL-mapping r of $\mathcal{K}^d$ in $\mathbb{R}^N$ $(N \geq d)$ such that *the closure of each k-cell, $k = 0, \ldots, d$ is embedded by r into $\mathbb{R}^N$ as a 'flat' (lying in an affine k-subspace) k-polyhedron*.

If Δ is a PL-realization of a polyhedron with a specific cell-decomposition, we shall frequently abuse notation and make no distinction between the polyhedron, its cell-decomposition, and the PL-realization. If we refer to the metric, projective, or affine properties of a cell complex, these should be understood as the properties of its fixed PL-realization. However, when we consider the combinatorial or homological properties of a cell complex, we are referring to its abstract combinatorial structure. We will often use the notion of link below. Notice that in the case of a non-simplicial cell-decomposition, the link of a cell is defined through the barycentric triangulation.

A homology d-sphere is a polyhedron P with the homology groups of a standard d-sphere such that for any n-cell C of P the link of C has the homology groups of a standard $(d-n)$-sphere. A homology d-disk is a polyhedron P with the homology groups of a standard d-disk such that for any n-cell C of P the link of C has the homology groups of a standard $(d-n)$-sphere (if C is interior) or disk (if C is the boundary). A *homology d-manifold* (with boundary) is a cell complex such that the link of each k-cell, is either a homology $(d-k-1)$-sphere (or a homology $(d-k-1)$-disk). A manifold is closed if each facet is adjacent to exactly two d-cells. All statements in the paper are formulated for both closed manifolds and for manifolds with boundary, unless stated otherwise. Since we consider manifolds from the combinatorial point of view, a manifold is always understood to be a *homology* manifold. Throughout the paper we include 'good' decompositions of $\mathbb{R}^n$ (like, for example, weighted Dirichlet–Voronoi diagrams) into the class of homology manifolds.

2. Stresses

Stresses on frameworks have a natural generalization to k-stresses on cell complexes of higher dimensions (see Definition 1.1). As in the case of frameworks, the linear space of k-stresses can be characterized as the left null space of a geometric matrix RM_k which is constructed as follows. Let M_k be the incidence matrix for the k- and $(k-1)$-cells of K, where the rows are indexed by the k-cells and the columns by $(k-1)$-cells. Thus $M_k(i, j) = 1$ if and only if $C_j^{k-1} \subset \partial C_i^k$, but is equal to 0 otherwise. The matrix RM_k is obtained by replacing unit entries of M_k by the corresponding positively oriented unit normal vectors, and zero entries by the zero vector; these replacement vectors are taken to be row vectors. The left null-space of RM_k is the space of $(k+1)$-stresses (the vectors of this space have the number of components equal to the number of k-cells).

The notion of k-stress on simplicial complexes was introduced by Lee [13]. For a simplicial complex a k-stress can be interpreted as an element of a certain quotient of the face-ring of the complex K. Let K be a simplicial complex in $\mathbb{R}^d$, with vertex set $\mathbf{v}_1, \dots, \mathbf{v}_n$. Then, in Lee's terminology the space of *affine* k-stresses on K is the linear subspace of polynomials of degree k of R/V, where R is the Stanley–Reisner ring of K, and V is the ideal generated by linear forms $\sum_{i=1}^n v_{ki} x_i$ $(k = 1, \dots, d)$, and $\sum_{i=1}^n x_i$ (see [13, 24]). For a simplicial complex K in $\mathbb{R}^d$ our k-stress on K is the same as Lee's affine k-stress on K. In fact, Lee considered two types of stress: linear and affine. Lee formulated most of his theorems in terms of so-called *linear* stresses. For generic realizations of K in $\mathbb{R}^d$ the space of our k-stresses is isomorphic to the space of Lee's *linear* k-stresses for K realized generically in $\mathbb{R}^{d+1}$. The equilibrium condition defining a linear stress says that the sum of normals $\mathbf{n}(F, C)$ weighted by $\omega(C)$ lies in the linear span of F. Higher-dimensional stresses were also considered by Tay, White, and Whiteley [24] and Rybnikov [19, 20]. Our terminology is in good agreement with the terminology in these papers.

If f_k denotes the number of simplexes of dimension k in K, then g_k and h_k are defined as follows:

$$g_k(K, d) = \sum_{j=-1}^{k-1} (-1)^{k+j-1} \binom{d-j}{d-k+1} f_j$$

$$h_k(K, d) = \sum_{j=0}^{k} (-1)^{j+k} \binom{d-j}{d-k} f_{j-1}.$$

Let Δ be a simplicial homology sphere. For a generic realization of Δ in $\mathbb{R}^{d+1}$ the dimension of the space of k-stresses is $g_k(\Delta, d+1)$, if $k \le \lfloor \frac{d+1}{2} \rfloor$, and 0, if $k > \lfloor \frac{d+1}{2} \rfloor$ (see [13]); for a generic realization of Δ in $\mathbb{R}^d$ the dimension of the space of k-stresses is $h_k(\Delta, d+1)$ [13, 24].

There is no similar algebraic theory of stresses for non-simplicial manifolds. The main barrier is the absence of an analog of the notion of a face ring for non-simplicial complexes.

The notion of stress can be defiend for PL-realizations of cell complexes where each cell is realized in Euclidean space as a simplicial star with possible self-intersections. All results of our paper hold for such realizations of cell complexes. Let $\mathcal{K}^d$ be a cell complex where a barycentric triangulation is fixed for each cell. For $\mathcal{K}^d$, consider a PL-realization K^d in $\mathbb{R}^N$ such that the triangulation of each cell C of $\mathcal{K}^d$ is realized in an affine subspace of dimension $\dim(C)$. Pick a (combinatorial) orientation for each $(k-1)$-cell of $\mathcal{K}^d$. Denote by $\mathbf{n}(S^{k-2}, C^{k-1})$ the unit normal to the oriented cell C^{k-1} at its simplicial facet S^{k-2} whose orientation is induced by the orientation of C^{k-1}. To define the notion of k-stress on such realization of K^d we have to formulate the equilibrium conditions for each simplex of the barycentric triangulation of each $(k-2)$-cell. However, it is easy to see that if the equilibrium condition holds for one simplex of C^{k-2}, it holds for all other simplexes of C^{k-2} (when we pick another $(k-2)$-simplex from the triangulation of C^{k-2} all normals either change their direction to the opposite, or stay the same).

DEFINITION 2.1. A real-valued function $\omega(\cdot)$ on (non-embedded, in general) oriented $(k-1)$-cells of K^d is a k-stress, if for each $(k-2)$-simplex S^{k-2} of each (internal) $(k-2)$-cell C^{k-2} of K^d

$$\sum_{C^{k-1}} \omega(C^{k-1})\, \mathbf{n}(S^{k-2}, C^{k-1}) \mathrm{vol}_{k-1}(S^{k-1}) = 0,$$

where C^{k-1} ranges over all oriented $(k-1)$-cells such that $S^{k-2} \subset \partial\, C^{k-1}$, and S^{k-1} stands for the simplex of C^{k-1} such that $S^{k-2} \subset \partial S^{k-1}$.

To be precise we would like to make the following technical remarks.

REMARK 2.2. When we discuss stresses on polyhedral stars we neglect the absence of the equilibrium at the cells that belong to the boundary of the star.

REMARK 2.3. The notion of stress is well defined for fans and cell-decompositions of $\mathbb{R}^d$ with non-compact cells. In this case the volumes of $(k-1)$-cells should be left out of the formula, and a stress coefficient no longer has the meaning of force per unit relative volume (area).

3. ORIENTABILITY AND GENERALIZED VOLUME

In this section $\mathbb{R}^d$ denotes Euclidean affine space with a fixed coordinate system. Consider an oriented, simplicial $(d-1)$-manifold Δ realized in $\mathbb{R}^d$. We introduce a generalized volume function, Vol_d, which assumes positive, negative or zero values on such manifolds. In the case where the manifold Δ bounds a d-dimensional body, and the orientation of Δ is chosen appropriately, $\mathrm{Vol}_d(\Delta)$ is the standard Euclidean volume of the body. Let $F = (v_1, \ldots, v_d)$ be an oriented $(d-1)$-simplex in $\mathbb{R}^d$. We denote by $[\mathbf{v}_1(F) - \mathbf{p}, \ldots, \mathbf{v}_d(F) - \mathbf{p}]$ the matrix whose columns are d-vectors pointing from point $\mathbf{p} \in \mathbb{R}^d$ to the vertices of F.

DEFINITION 3.1. Let Δ be a closed oriented simplicial manifold of co-dimension 1 in $\mathbb{R}^d$. Then

$$\mathrm{Vol}_d(\Delta) = \frac{1}{d!} \sum_{F \subset \Delta} \det [\mathbf{v}_1(F) - \mathbf{p}, \ldots, \mathbf{v}_d(F) - \mathbf{p}]$$

where the summation ranges over all oriented $(d-1)$-faces of Δ.

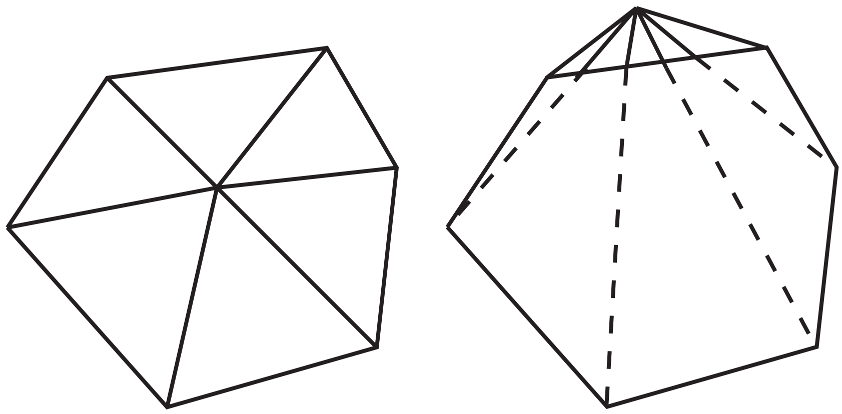

FIGURE 4. Two realizations of a star.

The value of $\mathrm{Vol}_d(\Delta)$ is independent of the the choice of point $\mathbf{p}$. That is why it is normally written for $\mathbf{p} = \mathbf{0}$. This formula can be rewritten as

$$\mathrm{Vol}_d(\Delta) = \frac{1}{d} \sum_{F \in \Delta} d(\mathbf{p}, \mathrm{aff}(F)) \mathrm{Vol}_{d-1}(F, \mathbf{p}) \tag{1}$$

where $d(\mathbf{p}, \mathrm{aff}(F))$ stands for the distance between $\mathbf{p}$ and the affine hull of F. The generalized $(d-1)$-volume $\mathrm{Vol}_{d-1}(F, \mathbf{p})$ is computed with respect to the orientation of $\mathrm{aff}(F)$ induced by the vector $\mathbf{v}_i(F) - \mathbf{p}$ (i is arbitrary), i.e., with respect to an orthonormal coordinate frame $[\mathbf{e}_1 \dots \mathbf{e}_{d-1}]$ in $\mathrm{aff}(F)$ such that $[\mathbf{v}_i(F) - \mathbf{p}, \mathbf{e}_1 \dots \mathbf{e}_{d-1}]$ is positively oriented in $\mathbb{R}^d$.

Let $\mathbb{S}^{d-1}$ be an oriented simplicial sphere, and let D be a cell-decomposition of $\mathbb{S}^{d-1}$ which is the result of an amalgamation of some of the simplexes of $\mathbb{S}^{d-1}$ into blocks (see Section 1 and [17, 22]). Consider a realization of the simplicial complex $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ such that each block lies in the affine span of its vertex set. For example, a block can be realized as a convex polytope partitioned into simplexes or as a simplicial star with self-intersections (see Figure 4). Then $\mathrm{Vol}_d(\mathbb{S}^{d-1})$ does not depend on the positions of the baricenters of the blocks of all dimensions greater than 0; this can be shown by induction in d. The case of $d = 1$ is obvious. The induction step follows from an application of Formula 1.

In Section 6 we will use the following observation.

REMARK 3.2. Let B be a d-dimensional cell complex such that the closures of all its faces, including B, are cones over homology spheres. In other words B is a homology ball. An example of such a complex would be a convex polytope. Suppose a barycentric triangulation of B is realized in $\mathbb{R}^d$ so that the affine dimension of the vertex set of each cell of B equals the dimension of this cell (the cell structure of a convex polytope would serve as a simple example). Then the generalized d-volume of the boundary of B does not depend on the positions of the baricenters of its faces, provided the baricenter of each face of B lies in the affine span of the vertex set of this face. We call the realizations of these baricenters in $\mathbb{R}^d$ virtual baricenters.

For discussion of the algebraic properties of the generalized volume $\mathrm{Vol}_d(\mathbb{S}^{d-1})$ as function of the edge lengths see [5].

4. Combinatorial Dual Graph and Reciprocals

Let $F(V, E)$ be a framework realized in $\mathbb{R}^2$, and assume that graph (V, E) can be regarded as the 1-skeleton of a spherical complex Δ. Suppose that this framework is in a state of static equilibrium. Consider a vertex of $F(V, E)$. The sum of vectors of stresses applied to this vertex is equal to zero; therefore, when rotated 90° clockwise they form a polygon (self-intersecting in general). It was first noticed by Maxwell (and proved by Whiteley [26]) that the positions of rotated edges of $F(V, E)$ can be adjusted so that they form a reciprocal graph (often called simply *reciprocal*). Each edge of this reciprocal corresponds to an edge of $F(V, E)$ and each vertex to a cell of Δ. One can introduce a similar notion for PL-realizations of d-manifolds in $\mathbb{R}^d$ (for more information see [2, 8, 19, 28]). In this section we will explore connections between the d-stresses and the generalization of Maxwell's reciprocal for d-manifolds.

The *combinatorial dual graph* $\mathcal{G}(\mathcal{M}^d)$ of a manifold $\mathcal{M}^d$ is defined as follows. The vertices of $\mathcal{G}$ are d-cells of $\mathcal{M}^d$, and the edges of $\mathcal{G}$ are internal $(d-1)$-cells of $\mathcal{M}^d$. Two vertices share an edge if and only if the corresponding d-cells are adjacent.

A *reciprocal* of a PL-realization M of a manifold Δ in $\mathbb{R}^d$ is a rectilinear realization R in $\mathbb{R}^d$ of the combinatorial dual graph $\mathcal{G}(\Delta)$ such that the edges of R are perpendicular to the corresponding facets. If none of the edges of a reciprocal collapses into a point, the reciprocal is called non-degenerate. Reciprocals were originally considered by Maxwell [15] in connection with stresses in plane frameworks. He, and almost at the same time, L. Cremona [9] noticed that reciprocals corresponded to equilibrium stresses on 1-skeletons of polyhedral spheres drawn in the plane. Reciprocals were later studied in [2, 6, 7, 21, 26]. Crapo and Whiteley gave an explicit treatment of the theory of reciprocals, stresses and liftings for 2-manifolds in [6–8].

To illustrate the concept of reciprocal let us consider the case where the realization M is an embedding. Let $v(C_1)$ and $v(C_2)$ be vertices of a reciprocal R corresponding to adjacent d-cells C_1 and C_2. Call the edge $[v(C_2)v(C_1)]$ *properly oriented* if $\mathbf{v}(C_2) - \mathbf{v}(C_1)$ is cooriented with an outer normal to C_1 at the facet shared with C_2. Otherwise call $[v(C_2)v(C_1)]$ improperly oriented. A hexagonal reciprocal for the embedded star of a vertex in a 2-manifold is shown in Figure 5. One can see that edges ef, cd are improperly oriented, and edges ab, cb, de, and fa are properly oriented). If all edges of R are properly oriented R is called a convex reciprocal (since the cycles of R corresponding to the stars of the $(d-2)$-cells are convex in this case).We refer to reciprocals of stars of the manifold as local reciprocals.

Evidently, reciprocals with one fixed vertex form a linear space. Denote this complex by $\mathrm{Rec}(M)$. If M is an embedding, then convex reciprocals form a cone $C\mathrm{Rec}(M)$ in the linear space $\mathrm{Rec}(M)$. The following theorem by Rybnikov [19] explains connections between reciprocals and stresses in the case of general dimension. We will utilize this theorem in the proof of our main theorem from Section 6.

Theorem 4.1. *Let M be a PL-realization of a homology d-manifold Δ in $\mathbb{R}^d$ with trivial first homology group over $\mathbb{Z}/2\mathbb{Z}$. Then there is an isomorphism between* $\mathrm{Stress}_d(M)$ *and* $\mathrm{Rec}(M)$. *Non-zero coefficients of stresses correspond to non-vanishing edges of a reciprocal. If M is an embedding of Δ into $\mathbb{R}^d$, then one can interpret properly oriented edges as corresponding to tensed facets, and improperly oriented edges as corresponding to compressed facets.*

Let B be a d-dimensional cell complex which is the cone over a homology sphere (not necessarily simplicial). Obviously, $B \setminus \partial B$ can be regarded as a star St. Let R be a reciprocal for St and denote by $R(C)$ a sub-reciprocal of R corresponding to a face $C \in St$. The vertex

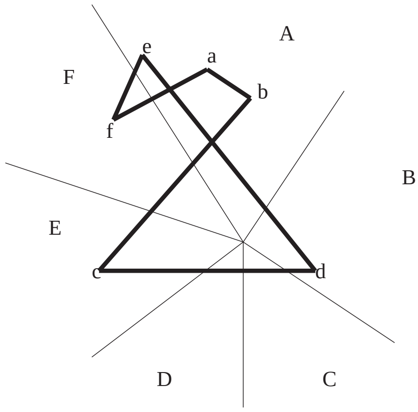

FIGURE 5. Non-convex reciprocal.

set of R is a realization of the vertex set of a complex dual to St. Denote it by St^*. For each cell C ($k = 1 \le \dim(C) \le d$) of St^* choose an arbitrary point $\mathbf{v}bc(C, R)$ on the plane $\mathrm{aff}(R(C))$, and call it the **v***irtual* **b***ari***c***enter* of $R(C)$. The vertices of R and the points $\mathbf{vbc}(C, R)$, $k = 1 \le \dim(C) \le d$ define a PL-realization of St^*. We know from Remark 3.2 that if a barycentric triangulation of St^* is realized in $\mathbb{R}^d$ so that the affine dimension of the vertex set of each cell of St^* equals the dimension of this cell (the cell structure of a convex polytope would serve as a simple example), then the generalized volume of oriented simplicial sphere $\partial\, St^*$ does not depend on the positions of the virtual baricenters of its faces, provided the virtual baricenter of each face of St^* lies in the affine span of the vertex set of this face. We can sum up this observation in the following proposition which will be of great use in the following section.

PROPOSITION 4.2. *Let R be a reciprocal for an oriented d-dimensional star St realized in $\mathbb{R}^d$. Then the generalized volume* $\mathrm{Vol}_d(R)$ *is well defined.*

5. MINKOWSKI'S THEOREM AND STRESSES

In this section we give an application of the well-known Minkowski theorem (see, for example, [32]) to stresses on polyhedral partitions of $\mathbb{R}^d$.

THEOREM 5.1 (Minkowski). *Let P be a convex polytope in $\mathbb{R}^d$, and denote by $\{\mathbf{n}(F)\}$ the inner unit normals to facets of P. Then*

$$\sum_{F \subset \partial P} \mathrm{vol}_{d-1}(F)\, \mathbf{n}(F) = \mathbf{0}.$$

Notice, that Minkowski's theorem has a well-known physical interpretation: a convex polytope immersed floating in a fluid is in a static equilibrium if and only if the sum of inward forces applied at its facets is zero. This fact was already known to Rankine in 1864.

If we choose a (combinatorial) orientation for P and denote by $\mathrm{Vol}_{d-1}(F, \mathbf{n}(F))$ the generalized volume of an oriented facet F with respect to the orientation of $\mathrm{aff}(F)$ induced by $\mathbf{n}(F)$, then the above formula can be rewritten as

$$\sum_{F \subset \partial P} \mathrm{Vol}_{d-1}\,(F, \mathbf{n}(F))\;\; \mathbf{n}(F) = \mathbf{0}.$$

Notice that in the last formula the directions of normals $\mathbf{n}(F)$ need not agree, since $\mathrm{Vol}_{d-1}(F)$ is computed with respect to the orientation induced by $\mathbf{n}(F)$. Flipping the normal changes the sign of $\mathrm{Vol}_{d-1}(F)$.

Let $St(v)$ be the star of a vertex v in a polyhedral partition of $\mathbb{R}^d$.

DEFINITION 5.1. A *dual convex polytope of $St(v)$* is a d-dimensional polytope D in $\mathbb{R}^d$ satisfying the following conditions:

(1) There is a one-to-one correspondence $\mathcal{I}$ between the m-dimensional faces of D and the $(d-m)$-dimensional faces of $St(v)$ $(0 \leq m \leq d)$.
(2) If $D^s \subseteq D^t$ are faces of D corresponding to faces F^{d-s} and F^{d-t} of $St(v)$, then $F^{d-t} \subseteq F^{d-s}$. In other words, the mapping $\mathcal{I}$ induces an isomorphism between the face lattices of D and $St(v)$.
(3) For $0 \leq m \leq d$ each m-dimensional face of D is perpendicular to the corresponding $(d-m)$-dimensional face of $St(v)$.
(4) $Sk^1(D)$ is a convex reciprocal graph for the star $St(v)$ (see Section 4).

The convexity of the dual polytope immediately follows from Conditions 1–4. It is worth mentioning that the dual polytope does not always exist (see [19] for various necessary and sufficient conditions). Suppose that there is a d-tension ω on St(v) (all coefficients of d-stresses are strictly positive). In this case by results of [19] and [21] there is a convex polytope $D(\omega)$ dual to $St(v)$. The d-tension ω defines such polytope uniquely up to translation: the lengths of the edges of $D(\omega)$ are equal to the corresponding coefficients of ω. By the Minkowski theorem cited above the sum of facet normals of a convex polytope scaled by the facet volumes is zero. Therefore, one can interpret the volumes of m-faces, $1 \leq m \leq d-1$, of $D(\omega)$ as coefficients of $(d-m+1)$-stresses on $(d-m)$-dimensional cells of $St(v)$. Thus, a d-tension on the star $St(v)$ induces an $(d-m)$-tension on $St(v)$, $1 \leq m \leq d-1$. It is easy to see that the constructed mappings are polynomial. Now, let ω be a d-tension on a cell-decomposition Δ of $\mathbb{R}^d$. We just described how ω induces tensions of lower dimensions on the stars of all vertices of Δ. In fact, for any $k < d$ the stress ω induces a k-tension on Δ. If F is a $(k-1)$-face of Δ, then the restriction of ω to the star $St(v)$ of a vertex v of F defines the coefficient of k-tension for F. If u is another vertex of F, then the restriction of ω to $St(u)$ gives exactly the same values of the induced k-tension on F, since the faces of the dual polytopes of $St(v)$ and $St(u)$ corresponding to F are equal (up to translation).

PROPOSITION 5.2. *Let Δ be a cell-decomposition of (a polyhedral region in) $\mathbb{R}^d$. For $k = 1, \ldots, d-1$ there is a polynomial mapping of degree $d-k+1$ from the cone of d-tensions on Δ to the cone of k-tensions on Δ. An all non-zero d-tension is always mapped to an all non-zero k-tension.*

By construction, a d-tension is mapped to a 2-tension on the 1-skeleton of Δ:

COROLLARY 5.3. *Let G be the* 1*-skeleton of a cell-decomposition Δ of $\mathbb{R}^d$ by convex polyhedra. If there is a convex surface which projects onto Δ, then G supports a positive equilibrium stress at all edges and, therefore, is an infinite spider web.*

It turns out that the mappings from Proposition 5.2 can be extended from the cone of tensions to all of the space of d-stresses, and the above construction can be carried out for arbitrary PL-realizations of orientable d-manifolds (not necessary embeddings). In order to formally establish this, we will need the concept of generalized volume introduced in Section 3.

The Minkowski theorem can be formulated for simplicial spheres arbitrarily realized in $\mathbb{R}^d$and, as we will see in Section 6, even for a larger class of polyhedral (not necessarily simplicial) spheres realized in $\mathbb{R}^d$ with self-intersections.

We need the following lemma.

LEMMA 5.4. *Let Δ be an oriented simplicial manifold realized in $\mathbb{R}^d$. For each oriented $(d-1)$-simplex F pick a unit normal vector $\mathbf{n}(\mathrm{aff}(F))$, and let $\mathrm{Vol}_{d-1}(F, \mathbf{n}(\mathrm{aff}(F)))$ be the generalized volume of F computed in $\mathrm{aff}(F)$ equipped with an orientation induced by $\mathbf{n}(F)$. Then*

$$\sum_{F\subset\Delta} \mathrm{Vol}_{d-1}(F, \mathbf{n}(\mathrm{aff}(F))\, \mathbf{n}(\mathrm{aff}(F) = \mathbf{0}.$$

PROOF. The orientation of Δ induces an orientation on a cone with Δ as base. Thus if F_1 and F_2 are two adjacent $(d-1)$-faces of Δ, the orientations of the cone over their common facet are opposite. Therefore the above formula can be rewritten as

$$\sum_{F\subset\Delta} \sum_{s^{d-1}\subset\partial\, 0\cdot F} \mathrm{Vol}_{d-1}(s^{d-1}, \mathbf{n}(\mathrm{aff}(s^{d-1}))\, \mathbf{n}(\mathrm{aff}(s^{d-1})) = \mathbf{0}.$$

where s^{d-1} stands for a facet of the cone $\mathbf{0}\cdot F$, and $\mathbf{n}(\mathrm{aff}(s^{d-1}))$ is an arbitrary unit normal to hyperplane $\mathrm{aff}(s^{d-1})$. Applying Minkowski's theorem to each d-simplex $\mathbf{0}\cdot F$ we get the required formula. $\square$

The interplay between stresses and volumes for simple and simplicial convex polytopes was also discussed by McMullen [18] and Lee [13].

6. TRACES OF d-STRESSES IN LOWER DIMENSIONS

REMARK 6.1. Let Δ be an orientable homology $(d-1)$-manifold in Euclidean space of dimension d. An orientation of Δ induces the orientation of normals to Δ at the cells of maximal dimension by the following rule. Let $(v_1(S), \ldots, v_d(S))$ be an oriented simplex of Δ. If frame $[\mathbf{v}_1(S), \ldots, \mathbf{v}_d(S)]$ is positively oriented, then the corresponding normal to Δ at S has positive scalar product with all these vectors. Conversly, a consistent choice of the field of normals to Δ at their simplexes of maximal dimension determines an orientation of Δ (e.g., outer normals for a convex polytope; see Figure 6).

In the case of an orientable d-manifold it is possible to fix the orientation of cells so that they form a d-cycle. By the above remark such orientation of cells induces the orientation of frames of normals corresponding to flags of cells. Thus, if Δ is an orientable d-manifold in $\mathbb{R}^d$ and the orientations of d-cells are picked up in such a way that it turns Δ into a d-cycle, any two flags of equal length having d-cells as maximal elements and distinct at only one position have corresponding frames of opposite orientation. A face-to-face partition of $\mathbb{R}^d$ provides a transparent example. Each of the two possible orientations of the partition correspond to flags of either inner or outer normals.

THEOREM 6.2. *Let Δ be an orientable homology d-manifold realized in $\mathbb{R}^d$. Then for $k = 1, \ldots, d-1$ there is a polynomial mapping $\mathfrak{p}_k$ of degree $d-k+1$ from $\mathrm{Stress}_d(\Delta)$ to $\mathrm{Stress}_k(\Delta)$.*

PROOF. For a cell-decomposition of a homology manifold there is so-called dual cell-decomposition (also called dual block decomposition). Consider the barycentric triangulation $T(\Delta)$ of the original cell-decomposition. Each cell of the original decomposition is a

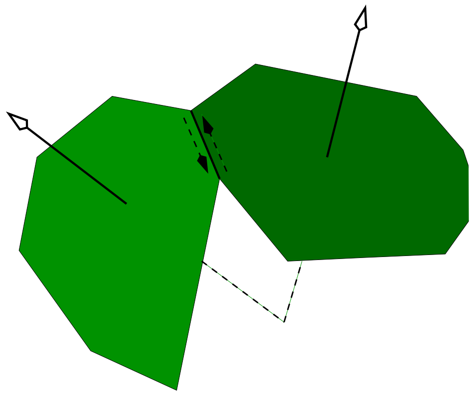

FIGURE 6. Orientation.

simplicial star in the barycentric triangulation. The $(d-k)$-simplexes of $T(\Delta)$ sharing the baricenter of a k-cell C form the dual cell (also called a block) for C. This dual cell is a homology $(d-k)$-disk. The boundary of the dual cell is a homology sphere (for more details on the geometrical duality in homology manifolds see [17, 22]).

Let v be a vertex of Δ, and let D_v be the d-dimensional cell (block) corresponding to v in the dual decomposition of Δ. Obviously, the boundary of D_v is the link $Lk(v)$ of v in Δ. Each cell of Δ or Δ^* is itself an orientable homology manifold, namely a homology disk. Thus, an orientation of triangulation $T(\Delta)$ induces in a natural way orientation on cell complexes Δ and Δ^*.

Let R be a (Euclidean) reciprocal for $St(v) \subset \Delta$ (see Section 4). By Theorem 4.1, the linear space of d-stresses on $St(v)$ is naturally isomorphic to the space of reciprocals with one fixed vertex. It turns out that one can introduce the notion of generalized 'k-volume' ($k = 0, \ldots, d$) for the sub-reciprocals of R, corresponding to the stars of cells of $St(v)$ (we refer to them as 'faces' of R). It is natural to call this function the k-volume, because when a reciprocal can be regarded as the vertex set of a convex k-polytope, the absolute value of this function is equal to the k-volume of the polytope. We keep the same notation for the k-volume of a reciprocal that we used for generalized volumes, i.e., Vol_k.

Let C^{d-k} be a $(d-k)$-cell from the (open) star of v. Obviously, $St(C^{d-k}) \subset St(v)$. The subreciprocal $R(C^{d-k})$ of R corresponding to this star spans an affine k-plane perpendicular to C^{d-k}. $R(C^{d-k})$ can be regarded as a realization of the vertex set of a cell of Δ^* dual to C^{d-k}. Thus, it makes sense to talk about the (combinatorial) orientation of $R(C^{d-k})$. Recall that a k-cell of the dual decomposition corresponds to a $(d-k)$-cell of Δ. Choose a flag of full length in C^{d-k}. This flag corresponds to some simplex S of $T(\Delta)$ whose vertices are the 'baricenters' of the flag cells. Denote by C^k the k-cell of Δ^* dual to C^{d-k}. The iterative coning of C^k with vertices of S is a cell from an amalgamation of the triangulation $T(\Delta^*)$. This amalgamation consists of (non-simplicial, in general) blocks of the form $v_0(\cdots(v_{d-k}\cdot C^k))$ constructed by the $(d-k)$-fold iterated coning of k-cells of Δ^*: here C^k is a k-cell of Δ^*, $\{v_0, \ldots, v_{d-k}\}$ is the set of barycenters corresponding to a flag of full length of the cell of Δ dual to C^k, and $v_i \cdot (\ldots)$ stays for the cone with apex v_i over base $(\ldots)$. Note that the orientation of $T(\Delta) = T(\Delta^*)$ induces an orientation on $v_0 \cdots (v_{d-k} \cdot C^k)$. Therefore, the choice of a flag in C^{d-k} determines an orientation for C^k.

A flag of faces of C^{d-k} corresponds to an ordered $(d-k)$-tuple of normals to the faces of C^{d-k}. Denote it by $[N]$. This $(d-k)$-tuple induces an orientation of affine subspace spanned by $R(C^{d-k})$ by the following rule. A frame N' in $\mathrm{aff}(R(C^{d-k}))$ is said to be cooriented with the frame $[N]$ if $[N, N']$ is cooriented with the coordinate frame of $\mathbb{R}^d$. Therefore $\mathrm{Vol}_{d-k}(R(C^{d-k})$ is well defined provided a flag of cells in C^{d-k} (see Section 3) has been fixed. We have to show that $\mathrm{Vol}_{d-k}(R(C^{d-k}))$ does not depend on the choice of flag in C^{d-k}. It is enough to show that for two flags in C^{d-k} that differ in one position the $\mathrm{Vol}_{d-k}(R(C^{d-k}))$ is the same, since any two flags in C^{d-k} can be connected by a sequence of alterations. Obviously, two flags that differ in one position induce opposite combinatorial orientations on $R(C^{d-k})$. But on the other hand it means that the $(d-k)$-tuples of vectors corresponding to these flags have opposite orientations. Thus the generalized k-volume of $R(C^{d-k})$ is well defined and does not depend on the choice of a flag of faces in C^{d-k}.

Let ω be a d-stress on Δ. Since the star of a $(d-k)$-cell of Δ is a homology d-disk, a d-stress restricted to the star of a vertex generates a k-dimensional reciprocal for this star (see Section 4). The distance between two vertices of the reciprocal corresponding to two adjacent d-cells equals the absolute value of stress on their common facet. Let $R(C)$ be the reciprocal of the star of a $(d-k)$-cell C corresponding to the stress ω. Let us interpret $\mathrm{Vol}_k(R(C))$ as the value of $(d-k+1)$-stress on C (recall that $(d-k)$-cells bear $(d-k+1)$-stresses). We have to check the equilibrium condition at every $(d-k-1)$-cell of Δ. Let F be a $(d-k-1)$-cell of Δ. Construct the reciprocal $R(F)$ for $St(F)$ corresponding to the d-stress ω. Notice that if $F \subset C$, then the sub-reciprocal or $R(F)$ corresponding to the star of C is equal to $R(C)$ (up to translation). Let $\mathbf{n}(F, C)$ denote the fixed unit normal to C at F whose orientation is induced by the orientation of Δ as it was explained in the beginning of this section. In the case where Δ is embedded into $\mathbb{R}^d$ we can think of $\mathbf{n}(F, C)$ as an inward unit normal:

$$\sum_{\{C \mid F \subset C\}} \mathrm{Vol}_k\,(R(C), \mathbf{n}(F, C))\,\mathbf{n}(F, C) = \sum_{\{R(C) \mid F \subset C\}} \sum_{S \subset C} \mathrm{Vol}_k(S, \mathbf{n}(F, C))\mathbf{n}(F, C)$$

where S is an oriented $(d-k)$-simplex from a barycentric triangulation of $R(C)$ arbitrarily realized in $\mathrm{aff}(R(C))$. By Minkowsi's theorem the last quantity is always zero. □

REMARK 6.3. Recall our assumption that each cell has an underlying structure of a simplicial star. The above theorem still holds if the cells are not embedded, but realized as simplicial stars with self-intersections in such a way that the triangulation of each cell lies in the affine plane spanned by this cell.

One way to show this is as follows.

PROOF. One can extend a d-stress ω on Δ to a stress on the PL-realization of its barycentric triangulation $D(\Delta)$: set $\omega(S^{d-1}) = 0$ for any $(d-1)$-simplex S^{d-1} which does not belong to the triangulation of a $(d-1)$-cell of Δ, and set $\omega(S^{d-1}) = \omega(C^{d-1})$ if S is a $(d-1)$-simplex of a $(d-1)$-cell C^{d-1}. All simplicial cells of the barycentric triangulation are, indeed, embedded. Reorient (if necessary) all $(d-1)$-simplexes in the barycentric triangulation so that the positive direction of normal is always inwards. The space of d-stresses of the reoriented complex is isomorphic to the original space of d-stresses. This reorientation is required, because we want to use the definition of stress for complexes with embedded cells. By Theorem 4.1 there is a corresponding reciprocal $R(s)$ for $D(\Delta)$. Now, we can define the polynomial mappings $\mathfrak{p}_k$ for $D(\Delta)$. By construction of the reciprocal, (geometric) cycles of $R(\omega)$ corresponding to simplexes that belong to the same cell are congruent. Consider a $(k-1)$-cell C of Δ. The constructed k-stress $\mathfrak{p}_k(\omega)$ takes on the same values on any two $(k-1)$-simplexes of C

that can be connected by a cell-facet path of $(k-1)$-simplexes of C, such that any two adjacent simplexes do not overlap; $\mathfrak{p}_k(\omega)$ takes on opposite values otherwise. Therefore $\mathfrak{p}_k(\omega)$ can be regarded as a k-stress on the original cell-partition of Δ. $\square$

Another way to prove the theorem for the case of self-intersecting cells is to directly adopt the proof of Theorem 6.2. The only use of the notion of the inner/outer normal in the proof of Theorem 6.2 was when we geometrically defined the orientation of the cells of the dual partition Δ^*. In the general case we just have to pick some combinatorial orientation for Δ^*. The rest of the proof is virtually unchanged.

As explained in the proof of Theorem 6.2, a Euclidean reciprocal R can be naturally regarded as the 1-skeleton of a PL-realization of the dual partition of Δ. To define the realization completely we just have to specify the positions of the baricenters of the cells of the dual partition. Now, we can ask the same questions about liftings, reciprocals, and stresses about the PL-realization of the dual partition. Notice, that in studies of liftings, stresses, and reciprocals the positions of the baricenters are not important. The above generalization is natural, since the class of PL-realization where each cell is realized as a simplicial star is closed under duality, whereas a dual complex for a PL-realization with embedded cells can have cells with self-intersections.

One should notice that the orientability of Δ is essential for our construction. Only in the case of orientable manifold the constructed mappings can be correctly defined.

Since the generalized $(d-k+1)$-volume of R can be expressed (non-uniquely) as a homogeneous polynomial of degree $d-k+1$ in the (oriented) lengths of the edges of R, and the absolute values of the edges of R equal to the absolute values of corresponding d-stresses (see Section 3), the constructed mappings $\mathfrak{p}_k$ from $Stress_d(\Delta)$ to $Stress_k(\Delta)$, $k = 1, \ldots, d-1$ are polynomial of degree $d-k+1$. The coefficients of these polynomials depend on the geometry of Δ. According to Connelly, Sabitov, and Waltz [5] the 3-volume of an orientable simplicial 2-surface in $\mathbb{R}^3$ is an algebraic integer over the ring generated by the squared lengths of the surface edges. It means that if ω is a d-stress on a $(d-2)$-primitive oriented Δ, the value of $(d-2)$-stress $\mathfrak{p}_{d-2}(s)$ on each $(d-3)$-cell of Δ is an algebraic integer over the ring generated by the squared values of ω on the $(d-1)$-cells of the star of this cell. It would be interesting to know if there are any implications of this fact for the algebraic geometry of our mappings $\mathfrak{p}_k$.

There are other canonical mappings between the spaces of stresses of different dimensions. For instance, according to Stanley [23] and Lee [13], for homology d-spheres in $\mathbb{R}^d$ the space of k-stresses has the same dimension as the space of $(d-k+1)$-stresses, if $k \leq \lfloor \frac{d+1}{2} \rfloor$. This is due to the existence of natural isomorphisms between the spaces of k-stresses and $(d-k+1)$-stresses. These isomorphisms play an important role in Stanley's and McMullen's proofs of the g-theorem [18, 23]. They are linear (see [13, 23]), whereas our mappings are not; ours are polynomial. In general, our mappings are *not bijective*, since for a generic realization of a simplicial sphere in $\mathbb{R}^3$ the dimension of the space of 2-stresses may exceed the dimension of the space of 3-stresses. For example, using the results of Lee [13] one can show that for a generic realization in $\mathbb{R}^3$ of the boundary of the four-dimensional cross-polytope $\dim(Stress_2) = 6$, but $\dim(Stress_3) = 4$. In fact, in some sense, our mappings are almost never bijective.

7. Stresses on Frameworks

Maxwell [15, 16] discovered the 'convex stress' induced by projection of a convex polytope on the plane (see Figure 7). Crapo and Whiteley gave a rigorous treatment of Maxwell's theorem [6, 26, 27].

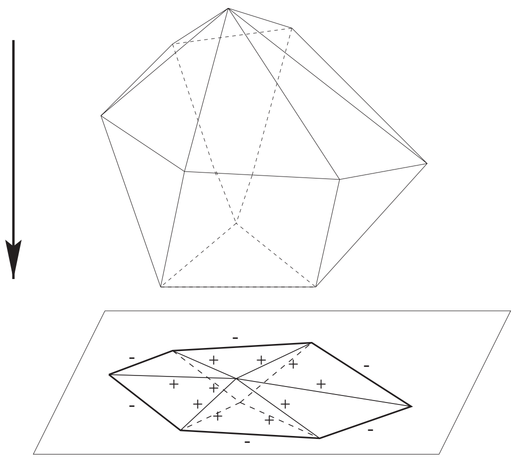

FIGURE 7. Maxwell's convex stress.

THEOREM 7.1 (Maxwell). *The vertical projection of a strictly convex polyhedron, with no faces vertical, produces a plane framework with a stress that is negative on the boundary edges and positive on all edges interior to this boundary polygon.*

Now we can formulate a partial analog of Maxwell's theorem on convex stresses and projections of spatial polyhedra. It immediately follows from our main theorem.

THEOREM 7.2. *Let P^4 be a strictly convex polytope in $\mathbb{R}^4$ without vertical faces, and let G be the projection of $Sk^1(P^4)$ onto $\mathbb{R}^3 \subset \mathbb{R}^4$. Then G supports a stress ω which is positive on all edges of G that are in the interior of the projection. If all the edges of P^4 that project on the boundary of the projection are incident to exactly three 3-cells of P^4, then in addition ω is negative on all edges of G that are on the boundary of the projection.*

PROOF. The vertical projection of our polytope P^4 induces the realization of its complex in $\mathbb{R}^3$; denote this realization by P^3. Using our main theorem, we construct the mapping $\mathfrak{p}_2 : Stress_3(P^3) \to Stress_2(P^3)$ for the realization P^3. P^4 can be thought of as the union of two polyhedral 'lids': the upper one and the lower one. These lids share common boundary. The upper lid is convex down, the lower lid is convex up. Obviously, since the upper and the lower lids are convex, the reciprocals for the 'interior' (with respect to the body of the realization P^3) edges of P^3 are convex (1-skeletons of convex polytopes); it is easy to see that they must have the generalized 2-volumes of the same sign. The reciprocals of the 'boundary' edges need not be convex; however, if a 'boundary' edge has a simplicial reciprocal, its volume ought to have the sign opposite to signs of the volumes of the reciprocals of the 'interior' edges. Therefore, if all the edges of P^4 that project on the boundary of the projection are incident to exactly three 3-cells of P^4, then the reciprocals of the 'boundary' edges of P^3 are all triangles and their generalized 2-volumes have the same sign. Thus, in this special case the 'interior' edges bear negative stress. □

Theorem 7.2 can be formulated in the case of general dimension. The proof repeats one-to-one the arguments of Theorem 7.2; notice that $(d-1)$-simplexes in the reciprocal of the projection play the role of triangles.

THEOREM 7.3. *Let P^d be a strictly convex polytope in $\mathbb{R}^d$ without facets parallel to x_d-axis. Let G be the projection of $Sk^1(P^d)$ onto $\{x_d = 0\} \subset \mathbb{R}^d$. Then G supports a self-stress ω which is positive on all edges of G that are in the interior of the projection. If all the edges of P^d that project on the boundary of the projection are incident to exactly d facets of P^d, then, in addition, ω is negative on all edges of G that are on the boundary of the projection.*

After the paper has been accepted for publication Robert Connelly told the authors that the above Theorem 7.2 has important applications to Lovasz's theorem on connections between the Colin de Verdiere graph matrices and Steinitz's theorem on graphs of convex 3-polytopes. (see [11, 14] for interesting new prospectives).

Recall, that Maxwell's correspondence states also that any equilibrium stress can be interpreted as one induced by the projection of a spatial polytope. At the CMS winter meeting of 1998 Connelly asked if the following conjecture is true for our correspondence.

CONJECTURE 7.4. *Let M^3 be a homology sphere realized in $\mathbb{R}^3$ and let ω_2 be a stress* (2-*stress*) *on the* 1-*skeleton of M^3. There is a* 3-*stress ω_3 on M^3 such that $\mathfrak{p}_2(\omega_3) = \omega_2$.*

As was mentioned in the Introduction, the generic realization of the boundary of the four-dimensional cross-polytope O_4 provides a counterexample. According to Lee [13] dim $(Stress_3(O_4)) = 4$, but $\dim(Stress_2(O_4)) = 6$ (stresses on a framework are 2-stresses). Since the mappings are algebraic the image of the space of 3-stresses cannot cover the space of 2-stresses. It would be interesting to give a geometric or algebraic (in the simplicial case) interpretation for those 2-stresses that can be interpreted as the images of 3-stresses under the above mapping.

A cell-decomposition of a closed d-manifold is called k-primitive if the star of each k-cell has $d - k + 1$ d-cells (some authors call 0-primitive decompositions *simple*; our terminology goes back to Voronoi and Delaunay). The meaning of this definition is that in a decomposition of $\mathbb{R}^d$ by convex polyhedra, $d - k + 1$ is the minimal possible number of d-cells making contact in a k-cell. When a k-primitive cell-decomposition of $\mathcal{M}^d$ is assumed to be fixed, we will refer to this k-primitive decomposition of $\mathcal{M}^d$ as *k-primitive manifold $\mathcal{M}^d$*. If a PL-realization of a sphere $\mathbb{S}^d$ in $\mathbb{R}^d$ can be lifted to a convex polytope in $\mathbb{R}^{d+1}$, then 0-primitive vertices of $\mathbb{S}^d$ correspond to simple vertices of this convex polytope. The notion of k-primitive decomposition naturally arises in studies of space-fillers, lattice polytopes and stereohedra. For example, the affine equivalence between space-fillers and Dirichlet domains of lattices was proved by Voronoi only for 0-primitive (simple) tilings. The existence of a lattice Dirichlet domain which is affinely isomorphic to a space-filler Π is equivalent to the existence of a d-stress with some special symmetries on the lattice tiling $T(\Pi)$ by Π (Voronoi) [19, 21, 25].

A *spider web* is a framework (with vertices at infinity usually allowed) supporting a stress strictly positive on all edges. Spider webs in $\mathbb{R}^2$ naturally appear from projections of convex surfaces. Planar and spatial spider webs serve as a tool for investigating various problems about dense packing of equal balls in $\mathbb{R}^2$ and $\mathbb{R}^3$ [1, 6, 7]. There are interesting applications of the theory of stresses in frameworks to physics, chemistry and engineering (see [1, 3, 6, 7, 30]). Since any $(d-3)$-primitive decomposition of $\mathbb{R}^d$ is the projection of a convex surface [19, 21], we have the following corollary.

COROLLARY 7.5. *The* 1-*skeleton of a* $(d - 3)$-*primitive decomposition of $\mathbb{R}^d$ by convex polyhedra is always a spider web.*

For closed manifolds a similar statement is as follows.

COROLLARY 7.6. *Let M be a realization in $\mathbb{R}^d$ of a $(d-3)$-primitive manifold Δ with trivial $H_1(\Delta, \mathbb{Z}_2)$. Suppose the body $|M|$ of this realization is convex and M is a double cover of $int\ |M|$. Then the* 1*-skeleton of M admits a convex stress.*

PROOF. By Rybnikov's theorem [19, 20] M can be lifted to a convex polytope in $\mathbb{R}^{d+1}$. By Theorem 7.3 the vertical projection of this polytope on $\mathbb{R}^d$ induces a stress on the 1-skeleton of M, positive on the 'interior' edges, and negative on the 'boundary' edges of M. □

CONJECTURE 7.7. *Let Δ be an oriented simplicial homology d-manifold. Then for any realization of Δ in a general position in $\mathbb{R}^d$ mappings $\mathfrak{p}_k$, $k = 1, \ldots, d$ have Jacobians of rank* $\min(\dim Stress_k(\Delta), \dim Stress_d(\Delta))$ *at almost all points $\omega \in Stress_d(\Delta)$.*

One can ask about generic properties of the mappings $\mathfrak{p}_k$ only if the combinatorial structure of Δ is preserved under any small perturbations of the coordinates of its vertices. In addition to the simplicial case, it is also true when $d = 2$ and when Δ is 0-primitive (simple). It is plausible that in the last two cases, for general realizations of Δ the mappings $\mathfrak{p}_k$ have Jacobians of maximal possible ranks.

A stronger form of the above conjecture is as follows.

CONJECTURE 7.8. *Let Δ be a closed oriented simplicial homology d-manifold. Then for any realization of Δ in a general position in $\mathbb{R}^d$ mappings $\mathfrak{p}_k$, $k = 1, \ldots, d$ are injective.*

A necessary condition for our Conjecture 7.7 about the Jacobians of $\mathfrak{p}_k$'s is $\dim(Stress_d) \leq \dim(Stress_k)$ for $k \leq d$. Below we give a count that demonstrates that this condition holds for $k = 2$ (i.e. when a d-stress is mapped to a stress on the 1-skeleton). The dimension of the space of d-stresses on a simplicial d-pseudomanifold in $\mathbb{R}^d$ is at least $f_0 - d - 1$ (follows from [4]) and is equal to $f_0 - d - 1$ if Δ is a manifold with $H_1(\Delta, \mathbb{Z}_2) = 0$ [19]. By the result of Fogelsanger [10] the 1-skeleton of a generic realization of a d-pseudomanifold in $\mathbb{R}^{d+1}$ is statically rigid. It means that $Sk^1(\Delta)$ can resolve any external load in $\mathbb{R}^{d+1}$ (see the Introduction). Thus $\dim Stress_2(\Delta, d+1) = f_1 - (d+1) f_0 + \binom{d+2}{2} = g_2(\Delta, d+1) \geq 0$ (the lower bound theorem for general simplicial pseudomanifolds).

For Conjecture 7.7 to be true, it is necessary that

$$\dim Stress_2(\Delta, d) \geq \dim Stress_d(\Delta, d) = f_0 - d - 1.$$

Let us verify this: $\dim Stress_2(\Delta, d) - (f_0 - d - 1) = f_1 - (d+1) f_0 + \binom{d+2}{2} = \dim Stress_2(\Delta, d+1) = g_2(\Delta, d+1) \geq 0$, as shown above. For connections between the rigidity theory and the lower bound theorem see the paper of Kalai [12].

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