Generalized quasi-variational inequalities: 
Duality under perturbations

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Abstract

The aim of this paper is to investigate, in the case of finite dimensional spaces, the stability of a duality scheme as well as of generalized Kuhn–Tucker conditions previously introduced by the authors for generalized quasi-variational inequalities with multifunction of the constraints described by a finite number of inequalities.

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1. Introduction

In a previous paper [1], a duality scheme and generalized Kuhn–Tucker conditions for generalized quasi-variational inequalities have been presented when the problem is defined in a real Banach space and the constraints are described by a finite number of inequalities (a typical situation for real world applications like environmental problems, see, for example, [2,3]). Note that

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these results extend the Lagrangian duality theory for constrained optimization problems (see, for example, [4,5]) and for variational inequalities in finite dimensional spaces [6,7]. More precisely, if \( X \) is a real Banach space and \( X^* \) is the dual of \( X \), let \( (VP) \) be the following variational problem:

\[
(VP) \quad \begin{cases}
\text{Find } x \in K(x) \text{ such that there exists } x^* \in T(x) \\
\text{satisfying } \langle x^*, y - x \rangle \geq 0, \ \forall y \in K(x),
\end{cases}
\]

where \( T \) is a set-valued operator from \( X \) to \( X^* \), \( K \) is defined on \( X \) by

\[
K(x) = \{ z \in X \mid f_j(x, z) \leq 0, \ \text{for all } j = 1, 2, \ldots, m \} \tag{1}
\]

and the following assumption is satisfied:

\( (H) \) the extended real valued function \( f_j(x, \cdot) \), defined on \( X \), is proper, closed and convex [8], for all \( j = 1, \ldots, m \).

In particular, the point \( x \) is said to be a solution to \((VP)\) and the pair \((x, x^*)\) is said to solve \((VP)\).

As defined in [1], the Dual Problem of the problem \((VP)\) (in short \((DVP)\)), is the following generalized variational inequality:

\[
(DVP) \quad \begin{cases}
\text{Find } u \in R^m_+ \text{ such that there exists } d \in G(u) \\
\text{satisfying } \langle d, v - u \rangle \geq 0, \ \text{for all } v \in R^m_+, 
\end{cases}
\]

with

\[
F(x, y) = (f_1(x, y), \ldots, f_m(x, y))
\]

and

\[
G(u) = \left\{ -F(x, x) \mid 0 \in T(x) + \sum_{j=1}^{m} u_j \partial_2 f_j(x, x) \right\},
\]

where \( \partial_2 f_j(x, t) \) is the subdifferential [5,8] of the function \( f_j(x, \cdot) \) at the point \( t \), that is:

\[
\partial_2 f_j(x, t) = \{ z \in X^* \mid f_j(x, y) \geq f_j(x, t) + \langle z, y - t \rangle, \ \forall y \in X \}.
\]

Moreover, a point \( (x, u) \in X \times R^m_+ \) is said to satisfy the generalized Kuhn–Tucker conditions associated to the problem \((VP)\) if:

\[
(KT)_1 \ x \in K(x); \\
(KT)_2 \ 0 \in T(x) + \sum_{j=1}^{m} u_j \partial_2 f_j(x, x); \\
(KT)_3 \ F(x, x) \in N_{R^m_+}(u).
\]

Note that the stability of the primal problem \((VP)\) with respect to perturbations has been investigated in [9] for monotone or pseudomonotone operators and generic set-valued constraints.

The aim of this paper is to investigate the stability of the dual problem \((DVP)\) as well as of the generalized Kuhn–Tucker conditions under perturbations on the operators and on the set-valued constraints. We now assume that \( X \) is a finite dimensional space and consider two sequences \((T_n)_n\) and \((K_n)_n\) of set-valued functions such that:

\[
K_n(x) = \{ y \in X \mid f_j^n(x, y) \leq 0, \ \text{for all } j = 1, 2, \ldots, m \}, \tag{2}
\]

where, for all \( n \in N \), the following assumption is satisfied:
(H_n) \ f^n_j(x, \cdot) : X \to R \text{ is a convex function for all } j = 1, \ldots, m \text{ and for all } n \in N.

Let us denote with (VP)_n the perturbed generalized quasi-variational inequality:

\begin{align*}
(\text{VP})_n \quad \begin{cases}
\text{Find } x_n \in K_n(x_n) \text{ such that there exists } x_n^* \in T_n(x_n)
\text{satisfying } \langle x_n^*, y - x_n \rangle \geq 0, \forall y \in K_n(x_n),
\end{cases}
\end{align*}

and with (DVP)_n the dual of (VP)_n, that is:

\begin{align*}
(\text{DVP})_n \quad \begin{cases}
\text{Find } u_n \in R^m_+ \text{ such that there exists } d_n \in G_n(u_n)
\text{satisfying } \langle d_n, v - u_n \rangle \geq 0, \text{ for all } v \in R^m_+,
\end{cases}
\end{align*}

where

\begin{align}
G_n(u) = \left\{-F_n(x_n, x_n) \mid 0 \in T_n(x_n) + \sum_{j=1}^m u_j \partial^2 f^n_j(x_n, x_n)\right\},
\end{align}

and

\begin{align*}
F_n(x, y) = (f^n_1(x, y), \ldots, f^n_m(x, y)).
\end{align*}

Finally, the point \( (x_n, u_n) \in X \times R^m_+ \) satisfies the generalized Kuhn–Tucker conditions associated to (VP)_n if:

\begin{align*}
(KT)_{1,n} x_n \in K(x_n); \\
(KT)_{2,n} 0 \in T_n(x_n) + \sum_{j=1}^m u_j \partial^2 f^n_j(x_n, x_n); \\
(KT)_{3,n} F_n(x_n, x_n) \in N_{R^m_+}(u_n).
\end{align*}

In Section 2 the stability of the dual problem is investigated while in Section 3 the attention is focused on the generalized Kuhn–Tucker conditions. More precisely sufficient conditions are given in order to guarantee that, if a sequence of solutions to the perturbed dual problem (DVP)_n converges to a point \( u \), then \( u \) is a solution to the dual problem (DVP) and if a sequence of points ((x_n, u_n))_n which satisfy the conditions (KT)_{1,n}, (KT)_{2,n}, (KT)_{3,n} converges to \( (x, u) \), then the point \( (x, u) \) satisfies (KT)_{1}, (KT)_{2}, (KT)_{3}. Significative examples are given to illustrate the results. Finally, in Section 4, an application to pseudo-game, where coupled constraints are involved, is presented.

Recall that convergence results for dual problems have been given for optimization problems ([10–12], . . .) and for variational inequalities involving maximal monotone operators and a particular kind of perturbations [13].

2. Stability of the dual problem

First, we recall that the problem (DVP) is called the Dual of (VP) since the following holds:

**Theorem 1.** [1] Assume that (H) is satisfied and \( x \) is a point of \( X \) such that \( E(x) = \bigcap_{j=1}^m \text{dom}(f_j(x, \cdot)) \) is an open subset of \( X \).

1. If \( (x, u) \), with \( u \in R^m_+ \), satisfies the generalized Kuhn–Tucker conditions:
   \begin{align*}
   (KT)_1 & \ x \in K(x); \\
   (KT)_2 & \ 0 \in T(x) + \sum_{j=1}^m u_j \partial^2 f_j(x, x);
   \end{align*}
\[(KT)_3 \ F(x,x) \in N_{R^m_+}(u); \]
then \(x\) is a solution to (VP) and \(u\) is a solution to (DVP).

(2) If \(x\) is a solution to (VP) and:
\[\exists \bar{y} \in X \text{ such that } f_j(x, \bar{y}) < 0 \text{ for all } j = 1, \ldots, m,\]
then there exists a point \(u \in R^m_+\) such that \((x,u)\) satisfies the generalized Kuhn–Tucker conditions \((KT)_1\) to \((KT)_3\) (and therefore \(u\) solves (DVP)).

Note that, since the Dual Problem is a generalized variational inequality defined in a finite dimensional space and with constraints described by a cone, it could be simpler to solve than the primal problem. In light of the previous theorem, to solve (VP) one could solve the dual problem (DVP) and find the solutions of the primal problem using the generalized Kuhn–Tucker condition \((KT)_2\).

Now, let us recall some definitions and preliminary results.
Let \((f_n)_n\) be a sequence of extended real valued functions defined on \(X\).

The sequence \((f_n)_n\) \(-\longrightarrow\) converges \([14]\) (or \(epiconverges [15]\)) to a function \(f\) on \(X\) to \(f\) whenever:

\(-\) for any point \(x\) of \(X\) and any sequence \((x_n)_n\) converging to \(x\) in \(X\), it results:
\[f(x) \leq \liminf_{n \to \infty} f_n(x_n);\]
\(-\) for any point \(x\) of \(X\) there exists a sequence \((\bar{x}_n)_n\) converging to \(x\) in \(X\), such that
\[\limsup_{n \to \infty} f_n(\bar{x}_n) \leq f(x).\]

The sequence \((f_n)_n\) \textit{continuously converges} to a function \(f\) on \(X\) (see, for example, \([16]\)) if:

\(-\) for any point \(x\) of \(X\) and any sequence \((x_n)_n\) converging to \(x\) in \(X\), it results:
\[f(x) = \lim_{n \to \infty} f_n(x_n).\]

A sequence of set-valued operators \((T_n)_n\) is \textit{uniformly bounded} (see, for example, \([17]\)) if for any sequence \((x_n)_n\) contained in a bounded set, there exists a positive number \(M\) such that, for any sequence \((x^*_n)_n\) with \(x^*_n \in T_n(x_n)\) for all \(n \in N\), we have
\[\|x^*_n\| \leq M \quad \text{for all } n \in N.\]
A multifunction \(M\) is \textit{sequentially subcontinuous} (see, for example, \([18]\)) at a point \(x_0\) if, for any sequence \((x_n)_n\) converging to \(x_0\) and any sequence \((y_n)_n\) such that \(y_n \in M(x_n)\) for any \(n\), there exists a subsequence of \((y_n)_n\) which converges to a point \(y_0\).
A sequence of sets \((S_n)_n\) is said to be \textit{sequentially subcontinuous} if the multifunction defined by
\[n \in N \to S_n\]
is sequentially subcontinuous; that is, if, for any sequence \((x_n)_n\) with \(x_n \in S_n\) for all \(n \in N\), there exists a subsequence of \((x_n)_n\) which converges to a point \(x\).
A sufficient condition for the sequential subcontinuity of a sequence of sets \((S_n)_n\) is that there exists a bounded subset \(S\) of \(X\), such that
\[S_n \subseteq S \quad \text{for } n \text{ sufficiently large.} \quad (4)\]
This condition is not necessary as shown by the following example.
Example 2. Let $X = ]-\infty, 0]$ and, for all $n \in N$, define

$$S_n = \{ x \in X \mid f_n(x) = \left((-1)^n(x + 1)\right)^{(-1)^n} \leq 0 \}.$$  

So, if $n$ is even, we have $S_n = ]-\infty, -1]$ while if $n$ is odd it results $S_n = ]-1, 0]$. Obviously it does not exist a bounded subset of $X$ which satisfies condition (4) but the sequence of sets $(S_n)_n$ is sequentially subcontinuous.

Finally, given a sequence of sets $(S_n)_n$ we denote with $\text{Limsup}_{n \to \infty} S_n$ the set of the limits of sequences $(s_h)_h$ with $s_h \in S_{n_h}$ for a subsequence $(n_h)_h$ and with $\text{Liminf}_{n \to \infty} S_n$ the set of the limits of sequences $(s_n)_n$ with $s_n \in S_n$ for all $n \in N$ (see, for example, [19]).

We will use also the following lemma:

Lemma 3. Let $(D_n)_n$ and $(C_n)_n$ be two sequences of sets. If for all $n \in N$, $D_n \subseteq D$, with $D$ bounded set, then

$$\text{Limsup}_{n \to \infty} (D_n + C_n) \subseteq \text{Limsup}_{n \to \infty} D_n + \text{Limsup}_{n \to \infty} C_n.$$  

Proof. Let $x \in \text{Limsup}_{n \to \infty} (D_n + C_n)$, then there exists a sequence $(x_h)_h$ with $x_h = d_h + c_h$ converging to $x$ with $d_h \in D_{n_h}$, $c_h \in C_{n_h}$ for a subsequence $(n_h)_h$ and for all $h \in N$. Since the sequence $(d_h)_h$ is bounded, there exists a subsequence $(d_{hk})_k$ of $(d_h)_h$ converging to a point $d$ when $k \to \infty$. So also $c_{hk} = x_{hk} - d_{hk}$ converges to the point $c = x - d$ and $x = d + c$ belongs to $\text{Limsup}_{n \to \infty} D_n + \text{Limsup}_{n \to \infty} C_n$. □

In the following, for all $n \in N$, we denote by $B_n$ the set defined as

$$B_n = \{ x \in X \mid f^n_j (x, x) \leq 0, \text{ for all } j = 1, 2, \ldots, m \}$$  

so $B_n$ is the set of fixed points of the multifunction $K_n$ for all $n \in N$.

Then we have

Theorem 4. Assume that $(H)$ and $(H_n)$ are satisfied and denote with $\hat{f}^n_j$ and $\hat{f}_j$ the functions defined on $X$ respectively as $\hat{f}^n_j (x) = f^n_j (x, x)$ and $\hat{f}_j (x) = f_j (x, x)$. If:

(i) the sequence $(T_n)_n$ is uniformly bounded or, for all $j = 1, \ldots, m$, the sequence $(\partial_2 f^n_j)_n$ is uniformly bounded;
(ii) $\text{Limsup}_{n \to \infty} \text{graph} T_n \subseteq \text{graph} T$;
(iii) the sequence $(B_n)_n$ (with $B_n$ defined as in (5)) is sequentially subcontinuous;
(iv) $\hat{f}^n_j$ and $\hat{f}_j$ are convex real valued functions for all $n \in N$ and for all $j = 1, \ldots, m$;
(v) the sequence $(\hat{f}^n_j)_n$ $\Gamma^-$-converges (or epiconverges) to $\hat{f}_j$, for all $j = 1, \ldots, m$;

then, whenever $u_n$ is a solution to the dual problem $(DVP)_n$ and the sequence $(u_n)_n$ converges to $u$, $u$ is a solution to the dual problem $(DVP)$.

Proof. Let $u$ be the limit of a sequence $(u_n)_n$ where $u_n$ is a solution to $(DVP)_n$, then, for all $n \in N$, there exists a point $d_n \in G_n(u_n)$ such that

$$\langle d_n, v - u_n \rangle \geq 0, \text{ for all } v \in R^m_+.$$  

(6)
By the definition of $G_n$ given in (3), there exists a point $x_n$, such that
\[ d_n = -\left( f_1^n(x_n, x_n), \ldots, f_m^n(x_n, x_n) \right) \]  
and
\[ 0 \in T_n(x_n) + \sum_{j=1}^m u_j^n \partial_2 f_j^n(x_n, x_n) \]  
and so, in light of Theorem 1, $x_n$ is a solution to $(VP)_n$ for all $n \in N$. Since $x_n \in B_n$ for all $n \in N$, by assumption (iii) there exists a subsequence $(x_{n_h})_h$ of $(x_n)_n$ converging to a point $x$.

Since $\Gamma^-$-convergence and continuous convergence agree on the class of real valued convex functions on $\mathbb{R}^n$ [16, Corollary 2.2], the sequence $(\hat{f}_j^n)_n$ converges continuously to $\hat{f}_j$ for each $j = 1, \ldots, m$ and the sequence $(\hat{f}_1^n + \cdots + \hat{f}_m^n)_n$ converges continuously to the function $\hat{f}_1 + \cdots + \hat{f}_m$. So the sequence $(d_n)_n$ converges to the point $d = -(f_1(x,x), \ldots, f_m(x,x))$.

Moreover, since $\sum_{j=1}^m u_j^n \partial_2 f_j^n(x_n, x_n) = \partial_2 \left( \sum_{j=1}^m u_j^n f_j^n(x_n, x_n) \right)$ and the sequence
\[ \left( \sum_{j=1}^m u_j^n f_j^n(x_n) \right)_n = \left( \sum_{j=1}^m u_j^n f_j^n(x_n, x_n) \right)_n \]
continuously converges to the function $\sum_{j=1}^m u_j \hat{f}_j(x) = \sum_{j=1}^m u_j f_j(x, x)$, from [15, Theorem 3.66], it follows that
\[ \operatorname{Limsup}_{n \to \infty} \partial_2 \left( \sum_{j=1}^m u_j^n f_j^n(x_n, x_n) \right) \subseteq \sum_{j=1}^m u_j \partial_2 f_j(x, x). \]

Then, since the sequence $(T_n)_n$ or, for all $j = 1, \ldots, m$, the sequences $(\partial_2 f_j^n)_n$ are uniformly bounded, we have that one of the sequences of sets $(T_n(x_n))_n$ or $(\partial_2 (\sum_{j=1}^m u_j^n f_j^n(x_n, x_n)))_n$ is contained in a bounded set, so by Lemma 3 it follows that
\[ 0 \in \operatorname{Limsup}_{n \to \infty} \left( T_n(x_n) + \sum_{j=1}^m u_j^n \partial_2 f_j^n(x_n, x_n) \right) \]
\[ = \operatorname{Limsup}_{n \to \infty} \left( T_n(x_n) + \partial_2 \sum_{j=1}^m u_j^n f_j^n(x_n, x_n) \right) \]
\[ \subseteq \operatorname{Limsup}_{n \to \infty} T_n(x_n) + \operatorname{Limsup}_{n \to \infty} \partial_2 \sum_{j=1}^m u_j^n f_j^n(x_n, x_n) \]
\[ \subseteq T(x) + \sum_{j=1}^m u_j \partial_2 f_j(x, x). \]

So $d$ belongs to $G(u)$. Passing to the limit in (6), one has
\[ \langle d, v - u \rangle \geq 0, \quad \text{for all } v \in R^m_+, \]
and the theorem is proved. \( \Box \)

For generalized variational inequalities, the stability has been also obtained [17, Theorem 3.3], instead of condition (ii) of Theorem 4, with the alternative conditions:
(i) the operator $T_n$ is pseudomonotone for all $n \in N$;
(ii) for any $x \in X$ and any $x^* \in T(x)$ there exists a sequence $(x_n^*)_n$ converging to $x^*$ such that $x_n^* \in T_n(x)$ for all $n \in N$.

This is not possible for the stability of the dual problem, as shown by the following example.

**Example 5.** Let us consider as primal problem the generalized quasi-variational inequality $(VP)$ associated to the following operators:

$$T(x) = x$$

and

$$K(x) = \left\{ y \in X \mid f_1(x, y) = y - 2x \leq 0 \text{ and } f_2(x, y) = x - y \leq 0 \right\}.$$

The dual problem $(DVP)$ associated to $(VP)$ is:

$$\begin{cases}
\text{Find } u \in R_+^2 \text{ such that there exists } d \in G(u) \\
\text{satisfying } \langle d, v - u \rangle \geq 0, \text{ for all } v \in R_+^2,
\end{cases}$$

where $G$ is the operator defined on $R_+^2$ as

$$G(u_1, u_2) = (u_2 - u_1, 0).$$

The solutions of $(DVP)$ are all the points $(0, u)$ and $(u, u)$ with $u \geq 0$.

Now let us consider, for all $n \in N$, the perturbed primal problem $(VP)_n$ connected with the following pseudomonotone operators:

$$T_n(x) = \begin{cases}
    x & \text{if } x \neq 0, \\
    \{0, 1\} & \text{if } x = 0,
\end{cases}$$

and

$$K_n(x) = K(x).$$

For all $n \in N$, the dual of $(VP)_n$ is the following variational inequality:

$$\begin{cases}
\text{Find } u_n \in R_+^2 \text{ such that there exists } d_n \in G_n(u_n) \\
\text{satisfying } \langle d_n, v - u_n \rangle \geq 0, \text{ for all } v \in R_+^2,
\end{cases}$$

where $G_n$ is the multifunction defined on $R_+^2$ as

$$G_n(u_1, u_2) = \begin{cases}
    (u_2 - u_1, 0) & \text{if } u_2 - u_1 \notin \{0, 1\}, \\
    (0, 0) & \text{if } u_2 - u_1 = 0, \\
    \{(0, 0), (1, 0)\} & \text{if } u_2 - u_1 = 1.
\end{cases}$$

Note that the alternative conditions (i) and (ii) are satisfied and the point $(1, 2)$ is a solution to each perturbed dual problem $(DVP)_n$ but it is not a solution to $(DVP)$. So the stability of the dual problem with respect to the considered perturbations is not obtained.

**Remark 6.** In the case of generalized variational inequalities, the stability of the dual problem has been investigated in [13] for a particular kind of perturbations, that is when $T_n(x) = T(x) + \epsilon_n \overline{T}(x)$ where $\overline{T}$ is a multifunction from $X$ to $X$ and $\epsilon_n \to 0$. 
3. Stability of the generalized Kuhn–Tucker conditions

Concerning the stability of the generalized Kuhn–Tucker conditions we have:

**Proposition 7.** Suppose that assumptions (H), (Hn) and (i)–(ii)–(iv)–(v) of Theorem 4 are satisfied. If, for all \( n \in \mathbb{N} \), the point \( (x_n, u_n) \) satisfies the generalized Kuhn–Tucker conditions \((KT)_{1,n}\), \((KT)_{2,n}\), \((KT)_{3,n}\) and the sequence \( (x_n, u_n) \) converges to \( (x, u) \), then \( (x, u) \) satisfies the generalized Kuhn–Tucker conditions \((KT)_{1}\), \((KT)_{2}\), \((KT)_{3}\).

**Proof.** Since \( x_n \in K_n(x_n) \) and the sequence of functions \( (f_j^n) \) \( \Gamma^- \)-converges to \( f_j \) for all \( j = 1, \ldots, m \), it follows that \( x \in K(x) \). Moreover, as in Theorem 4, we have that

\[
0 \in \operatorname{Limsup}_{n \to \infty} \left( T_n(x_n) + \sum_{j=1}^{m} u_j^n \partial_2 f_j^n(x_n, x_n) \right) 
\leq T(x) + \sum_{j=1}^{m} u_j \partial_2 f_j(x, x)
\]

so condition \((KT)_2\) is satisfied.

Moreover, since \( F_n(x_n, x_n) = (f_1^n(x_n, x_n), \ldots, f_m^n(x_n, x_n)) \) converges to \( F(x, x) \), from

\[
\langle F_n(x_n, x_n), v - u_n \rangle \leq 0 \quad \text{for all } v \in \mathbb{R}_+^m
\]

it follows that

\[
\langle F(x, x), v - u \rangle \leq 0 \quad \text{for all } v \in \mathbb{R}_+^m.
\]

Therefore \( F(x, x) \in N_{K_n^m}(u) \) and the proposition is proved. \( \square \)

Note that the stability of the generalized Kuhn–Tucker conditions is obtained under weaker assumptions with respect to the stability of the primal problem. Concerning the primal problem, we have the following result which is a natural extension of Proposition 3.1 in [17], obtained for generalized variational inequalities.

**Proposition 8.** Assume that:

(i) the sequence \( (T_n) \) is uniformly bounded;
(ii) \( \operatorname{Limsup}_{n \to \infty} \text{graph } T_n \subseteq \text{graph } T \);
(iii) the sequence of multifunctions \( K_n \) converges to \( K \) in the sense of Painlevé–Kuratowski [19], that is,

\[
\operatorname{Limsup}_{n \to \infty} K_n(x_n) \subseteq K(x) \subseteq \operatorname{Liminf}_{n \to \infty} K_n(x_n) \quad \forall x_n \to x.
\]

Whenever \( x_n \) is a solution to the problem \((VP)_n\) and the sequence \( (x_n) \) converges to \( x \), then \( x \) is a solution to the dual problem \((VP)\).

**Proof.** Let \( x \) be the limit of a sequence \( (x_n) \) where \( x_n \) is a solution to \((VP)_n\) for all \( n \in \mathbb{N} \). Then there exists a sequence \( (x_n^*) \), with \( x_n^* \in T_n(x_n) \) such that

\[
\langle x_n^*, y_n - x_n \rangle \geq 0 \quad \text{for all } y_n \in K_n(x_n). \quad (9)
\]
Since \( x_n \in K_n(x_n) \) for all \( n \in \mathbb{N} \), it follows that \( x \in K(x) \). Since the sequence \( (x_n)_n \) is bounded and the sequence \( (T_n)_n \) is uniformly bounded, there exists a subsequence \( (x_{n_k})_k \) of \( (x_n)_n \) which converges to a point \( x^* \) and, in light of the assumption (ii), \( x^* \in T(x) \). Now, let us consider a point \( y \in K(x) \). From (iii), there exists a sequence \( (\bar{y}_n)_n \) converging to \( y \) such that, for any \( n \) sufficiently large, \( \bar{y}_n \in K_n(x_n) \). So \( \langle x^*_{n_k}, \bar{y}_{n_k} - x_{n_k} \rangle \) converges to \( \langle x^*, y - x \rangle \) and, in light of (9) and for the arbitrariness of \( y \), we have

\[
\langle x^*, y - x \rangle \geq 0 \quad \text{for all } y \in K(x),
\]

that is \( x \) is a solution to (VP). \( \square \)

If the multifunctions which define the constraints are as in \((H)\) and \((H_n)\), sufficient conditions for assumption (iii) of Proposition 8 are given by the following result:

**Theorem 9.** [18, Proposition 3.3.3] Assume the following conditions:

(i) for any \( j = 1, \ldots, m \), any \( (x, y) \in X \times X \) and any \( (x_n, y_n) \) converging to \( (x, y) \), one has

\[
\liminf_{n \to \infty} f^n_j(x_n, y_n) \geq f_j(x, y);
\]

(ii) for any \( j = 1, \ldots, m \), any \( (x, y) \in X \times X \) such that \( f_j(x, y) < 0 \) and any sequence \( (x_n) \) converging to \( x \), there exists a sequence \( \bar{y}_n \) converging to \( y \) such that

\[
\limsup_{n \to \infty} f^n_j(x_n, \bar{y}_n) \leq f_j(x, y);
\]

(iii) for any \( x \in X \) there exists \( y \in Y \) such that \( f_j(x, y) < 0 \) for all \( j = 1, \ldots, m \);

(iv) for any \( j = 1, \ldots, m \), any \( n \in \mathbb{N} \) and any \( x \in X \) the function \( y \to f_j(x, y) \) is strictly quasi-convex [20];

then the sequence of set-valued operators \((K_n)\) (defined by (2)) converges to \( K \) in the sense of Painlevé–Kuratowski.

In the following example, it is shown how one can use the stability of the generalized Kuhn–Tucker conditions to obtain information on the solutions of the primal problem in a case in which the stability of the generalized quasi-variational inequalities is not guaranteed. More precisely, let \( x_n \) be a solution to the perturbed primal problem \((VP)_n\) and assume that \((x_n, u_n)\) satisfies conditions \((KT)_{1,n}, (KT)_{2,n}, (KT)_{3,n} \). Whenever the sequence \((x_n, u_n)_n\) converges to \((x, u)\) and \((x, u)\) satisfies the conditions \((KT)_1, (KT)_2, (KT)_3 \), we have that \( x \) is a solution to \((VP)\).

**Example 10.** Let us consider the generalized quasi-variational inequality \((VP)\) associated to the following operators:

\[
T(x) = \begin{cases} 
-x & \text{if } x \geq 0, \\
x & \text{if } x < 0,
\end{cases}
\]

and

\[
K(x) = \{ y \in X \mid f_1(x, y) = y - 2x \leq 0 \text{ and } f_2(x, y) = x - y \leq 0 \}.
\]

The dual problem \((DVP)\) associated to \((VP)\) is

\[
\begin{cases} 
\text{Find } u \in R^2_+ \text{ such that there exists } d \in G(u) \\
\text{satisfying } \langle d, v - u \rangle \geq 0, \text{ for all } v \in R^2_+,
\end{cases}
\]
where $G$ is the operator defined on $\mathbb{R}^2_+$ as
\[
G(u_1, u_2) = \begin{cases} 
(u_2 - u_1, 0), (u_1 - u_2, 0) & \text{if } u_2 - u_1 \leq 0, \\
\emptyset & \text{otherwise.}
\end{cases}
\]
The solutions of $(DVP)$ are all the points $(u, u)$ with $u \geq 0$, the unique solution of $(VP)$ is 0 and $(0, u, u)$ with $u \geq 0$ are the points which satisfy the Kuhn–Tucker conditions $(KT)_1, (KT)_2, (KT)_3$.

Now let us consider, for all $n \in \mathbb{N}$, the perturbed primal problem $(VP)_n$ connected with the following operators:
\[
T_n(x) = \begin{cases} 
-x & \text{if } x \geq 0 \text{ and } x \neq 1, \\
\{-x, n\} & \text{if } x = 1, \\
x & \text{if } x < 0,
\end{cases}
\]
and
\[
K_n(x) = K(x).
\]
The dual $(VP)_n$ is the variational inequality on $\mathbb{R}^2_+$, characterized by the following set-valued operator:
\[
G_n(u_1, u_2) = \begin{cases} 
(u_2 - u_1, 0), (u_1 - u_2, 0) & \text{if } u_2 - u_1 \leq 0, \\
(1, 0) & \text{if } u_2 - u_1 = n, \\
\emptyset & \text{otherwise.}
\end{cases}
\]
The solutions of $(DVP)_n$ are the points $(u, u)$ with $u \geq 0$ and $(0, n)$, so the solution of $(VP)_n$ are the points 0 and 1, while $(0, u, u)$ with $u \geq 0$ and $(1, 0, n)$ are the points which satisfy the Kuhn–Tucker conditions $(KT)_{1,n}, (KT)_{2,n}, (KT)_{3,n}$.

Since the sequence of operators $(T_n)_n$ is not uniformly bounded, one cannot say a priori anything about the stability of the primal problem with respect to the considered perturbations. In fact, the primal problem is not stable since 1 is a solution to $(VP)_n$ for all $n \in \mathbb{N}$ but 1 is not a solution to $(VP)$. However assumptions of Theorem 4 hold and therefore the Kuhn–Tucker conditions are stable. So, since 0 is a solution to $(VP)_n$ and $(0, u, u)$ converges for all $u \geq 0$, we can conclude that 0 is a solution to $(VP)$.

4. Application to convergence of Social Nash Equilibria

As in [1] we consider a $N$-person noncooperative pseudo-game [21], that is a game with coupled constraints. Let $X_i \subset \mathbb{R}^{m_i}$ be the strategy set of player $i$, $J_i$ from $X = X_1 \times \cdots \times X_N$ to $R$ be his payoff function and $K_i$ be the set-valued function which define his constraints depending on the strategies of the others players. Assume that $i$ wants to minimize his payoff function. We recall that a Social Nash Equilibrium of the pseudo-game $\Gamma = \{X_i, J_i, K_i\}$ is a point $x \in X$ such that
\[
J_i(x) \leq J_i(y_i, x_{-i}) \quad \text{for all } y_i \in K_i(x_{-i}) \text{ and for all } i = 1, \ldots, N.
\]
It is well known that, under suitable assumptions, the Social Nash Equilibrium problem can be put into the form of a generalized quasi-variational inequality (see, for example, [9,22]). More precisely, if we assume that the following condition is satisfied:
\[
(C) \quad \text{for every } x_{-i} \in X_{-i} \text{ the function } J_i(\cdot, x_{-i}) \text{ is convex and bounded from below on } X_i, \text{ for } i = 1, \ldots, N,
\]
then, a point $x$ is a Social Nash Equilibrium of $\Gamma$ if and only if $x$ is a solution to the following problem:

\[
\begin{align*}
&\text{(SNE)} \\
&\begin{cases}
\text{Find } x \in K(x) \text{ such that there exists } x^* \in T(x) \\
\text{satisfying } \langle x^*, y - x \rangle \geq 0, \text{ for all } y \in K(x),
\end{cases}
\end{align*}
\]

where

\[
T(x) = \partial_x J_1(x) \times \cdots \times \partial_x J_N(x)
\]

and

\[
K(x) = K_1(x_{-1}) \times \cdots \times K_N(x_{-N})
\]

so we can apply to (SNE) the duality result given in Theorem 1. See [1, Example 3.1].

Now, for all $i = 1, \ldots, N$, let $(J_i,n)$ be a sequence of real valued functions on $X$ which satisfy, for all $n \in \mathbb{N}$, the assumption (C) and $(K_i,n)$ be a sequence of multifunctions from $X$ to $X_{-i}$. We can consider the perturbed pseudo-game $\Gamma_n = (J_i,n, X_i, K_i,n)$ for $i = 1, \ldots, N$.

As a consequence of Theorem 4 and Proposition 8, we obtain a result about the convergence of Social Nash Equilibria and its dual. Note that a condition on the payoff functions that imply the convergence required for the operators $T$ and $T_n$ in Theorem 4 and Proposition 8 is given by the following proposition.

**Proposition 11.** Let us assume that:

(i) for all $x \in X$ and for all sequence $(x_n)_n$ converging to $x$ we have

\[
\liminf_{n \to \infty} J_i,n(x_n) \geq J_i(x) \quad i = 1, \ldots, N;
\]

(ii) for all $i = 1, \ldots, N$, for all $x \in X$ and for all sequences $(x_{-i,n})_n$ converging to $x_{-i}$, there exists a sequence $(\tilde{x}_i,n)_n$ converging to $x_i$ such that

\[
\limsup_{n \to \infty} J_i,n(\tilde{x}_i,n, x_{-i,n}) \leq J_i(x).
\]

Then

\[
\limsup_{n \to \infty} \text{graph } T_n \subseteq \text{graph } T.
\]

**Proof.** Let $(x, z)$ be the limit of a sequence $((x_n, z_n))_n$, with $(x_n, z_n) \in \text{graph } T_n$. In light of (i) and (ii) for all $y_i \in X_i$ there exists $(\tilde{y}_i,n)_n$ converging to $y_i$ such that for all $i = 1, \ldots, N$:

\[
J_i(y_i, x_{-i}) \geq \limsup_{n \to \infty} J_i,n(\tilde{y}_i,n, x_{-i,n})
\]

\[
\geq \liminf_{n \to \infty} [J_i,n(x_i,n, x_{-i,n}) + \langle z_i,n, \tilde{y}_i,n - x_i,n \rangle]
\]

\[
\geq J_i(x) + \langle z_i, y_i - x_i \rangle.
\]

So $z$ belongs to $T(x)$ and $\limsup_{n \to \infty} \text{graph } T_n \subseteq \text{graph } T$. \(\square\)
References