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Inversion problem, Legendre transform and inviscid Burgers' equations

Wenhua Zhao*

Department of Mathematics, Illinois State University, Campus Box 4520, Stevenson Hall 303C Normal, IL 61790-4520, USA

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Abstract

Let $F(z) = z - H(z)$ with order $o(H(z)) \geqslant 1$ be a formal map from $\mathbb{C}^n$ to $\mathbb{C}^n$ and $G(z)$ the formal inverse map of $F(z)$. We first study the deformation $F_t(z) = z - tH(z)$ of $F(z)$ and its formal inverse $G_t(z) = z + tN_t(z)$. (Note that $G_{t=1}(z) = G(z)$ when $o(H(z)) \geqslant 2$.) We show that $N_t(z)$ is the unique power series solution of a Cauchy problem of a PDE, from which we derive a recurrent formula for $G_t(z)$. Secondly, motivated by the gradient reduction obtained by de Bondt and van den Essen (A Reduction of the Jacobian Conjecture to the Symmetric Case, Report No. 0308, University of Nijmegen, June 2003, Proc. of the AMS, to appear) and Meng (Legendre Transform, Hessian Conjecture and Tree Formula, math-ph/0308035) for the Jacobian conjecture, we consider the formal maps $F(z) = z - H(z)$ satisfying the gradient condition, i.e. $H(z) = \nabla P(z)$ for some $P(z) \in \mathbb{C}[[z]]$ of order $o(P(z)) \geqslant 2$. We show that, under the gradient condition, $N_t(z) = \nabla Q_t(z)$ for some $Q_t(z) \in \mathbb{C}[[z, t]]$ and the PDE satisfied by $N_t(z)$ becomes the n-dimensional inviscid Burgers' equation, from which a recurrent formula for $Q_t(z)$ also follows. Furthermore, we clarify some close relationships among the inversion problem, the Legendre transform and the inviscid Burgers' equations. In particular the Jacobian conjecture is reduced to a problem on the inviscid Burgers' equations. Finally, under the gradient condition, we derive a binary rooted tree expansion inversion formula for $Q_t(z)$. The recurrent inversion formula and the binary rooted tree expansion inversion formula derived in this paper can also be used as computational algorithms for solutions of certain Cauchy problems of the

* Tel.: +1 309 438 7367; fax: +1 309 438 5866.
E-mail address: wzhao@ilstu.edu (W. Zhao).

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inviscid Burgers' equations and the Legendre transforms of the power series $f(z)$ with $o(f(z)) \geqslant 2$ and $\det \mathrm{Hes}(f)(0) \neq 0$.

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1. Introduction

Let $z = (z_1, z_2, \dots, z_n)$ and $F(z) = z - H(z)$ be a formal map from $\mathbb{C}^n$ to $\mathbb{C}^n$ with $o(H(z)) \geqslant 1$ and $G(z)$ the formal inverse map of $F(z)$. The well-known Jacobian conjecture first proposed by Keller [12] in 1939 claims that, *if $F(z)$ is a polynomial map with the Jacobian $j(F)(z)=1$, the inverse map $G(z)$ must also be a polynomial map*. Despite intense study from mathematicians in more than half a century, the conjecture is still wide open even for the case $n=2$. In 1998, Smale [18] included the Jacobian conjecture in his list of 18 important mathematical problems for the 21st century. For more history and known results on the Jacobian conjecture, see [3,6] and references there. One of natural approaches to the Jacobian conjecture is to derive formulas for the inverse $G(z)$. In literature, formulas which directly or indirectly give the formal inverse $G(z)$ are called inversion formulas. Due to many important applications in other areas, especially in enumerative combinatorics (See, for example, [19,8] and references there.), inversion formulas attracted much attention from mathematicians much earlier than the Jacobian conjecture. The first inversion formula in history was the Lagrange's inversion formula given by Lagrange [14] in 1770, which provides a formula to calculate all coefficients of $G(z)$ for the one-variable case. This formula was generalized to multi-variable cases by Good [9] in 1965. Jacobi [10] in 1830 also gave an inversion formula for the cases $n \leqslant 3$ and later [11] in 1844 for the general case. This formula is now called the Jacobi's inversion formula. Another inversion formula is the Abhyankar–Gurjar inversion formula, which was first proved by Gurjar in 1974 (unpublished), and later Abhyankar [1] gave a simplified proof. By using the Abhyankar–Gurjar inversion formula, Bass et al. [3] in 1982 and Wright [21] in 1989 proved the so-called Bass–Connell–Wright's tree expansion formula. Recently, in [25], this formula has been generalized to a tree expansion formula for formal flows $F(z,t)$ generated by $F(z)$ which provides a uniform formula for all the powers $F^{[m]}(z) = F(z,m)$ ($m \in \mathbb{Z}$) of $F(z)$. Besides the inversion formulas above, there are also many other inversion formulas in literature. See, for example, [8,22] and references there.

Recently, de Bondt and van den Essen [4] and Meng [15] have made a breakthrough on the Jacobian conjecture. They reduced the Jacobian conjecture to polynomial maps $F(z) = z - H(z)$ satisfying the *gradient condition*, i.e. $H(z)$ is the gradient $\nabla P(z)$ of a polynomial $P(z)$. We will refer this reduction as the *gradient reduction* and the condition $H(z) = \nabla P(z)$ the *gradient condition*. One great advantage of the gradient reduction is that, it reduces the Jacobian conjecture that involves n polynomials to a problem that only involves a single polynomial. Note that, by Poincaré lemma, a formal map $F(z)=z-H(z)$ with $(o(H(z)) \geqslant 1)$ satisfies the gradient condition if and only if the Jacobian matrix $JF(z)$ is symmetric. Following the terminology in [4], we also call the formal maps satisfying the gradient condition *symmetric* formal maps. For some further studies on symmetric formal maps, see [4,5,7,15,24,26,27].

In this paper, we first study in Section 2 the deformation $F_t(z) = z - tH(z)$ of $F(z) = z - H(z)$ and its inverse map $G_t(z)$, where t is a formal parameter which commutes with z. It is easy to see that $G_t(z)$ can always be written as $G_t(z) = z + tN_t(z)$ for some $N_t(z) \in \mathbb{C}[[z,t]]^{\times n}$ and $G_{t=1}(z) = G(z)$ when $o(H(z)) \geqslant 2$. We show in Theorem 2.4 that $N_t(z)$ is the unique solution of a Cauchy problem of a PDE, see Eqs. (2.6) and (2.7). The PDE Eq. (2.6) satisfied by $N_t(z)$ has a similar form as the n-dimensional inviscid Burgers' equation (See [16,17] or Eqs. (4.1) and (4.2) in this paper.). By solving the Cauchy problem Eqs. (2.6) and (2.7) recursively, we get a recurrent formula (See Theorem 2.7.) for $N_t(z)$. This recurrent inversion formula not only has more computational efficiency in certain situation than other inversion formulas, but also provides some new understandings on the inversion problem. For some theoretical consequences and applications of this recurrent inversion formula, see [23,24]. Besides the main results described above, some other properties of $N_t(z)$ including the one in Proposition 2.9 that characterizes $N_t(z)$ are also proved in this section.

In Section 3, we consider the case of symmetric formal maps. Let $F(z) = z - H(z)$ with $H(z) = \nabla P(z)$ for some $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geqslant 2$. One can show that, in this case, $N_t(z) = \nabla Q_t(z)$ for some $Q_t(z) \in \mathbb{C}[[z,t]]$. Furthermore, Eq. (2.6) satisfied by $N_t(z)$ in general does become the n-dimensional inviscid Burgers' equation! It can also be simplified to a Cauchy problem Eq. (3.6) in a single formal power series $Q_t(z) \in \mathbb{C}[[z,t]]$ instead of $N_t(z) \in \mathbb{C}[[z,t]]^{\times n}$ in general. By solving the Cauchy problem Eq. (3.6) recurrently, we also get a simplified recurrent formula (See Proposition 3.7.) for $Q_t(z)$. Some other properties of $Q_t(z)$ are also discussed in this section.

In Section 4, we clarify some connections among the inversion problem, the Legendre transform and the inviscid Burgers' equations. In particular, we reduce the Jacobian conjecture to a problem on the inviscid Burgers' equations, see Conjecture 4.1 and Proposition 4.2. More precisely, $\nabla Q_t(z)$ is the unique power series solution of a Cauchy problem of the inviscid Burgers' equations with initial condition $\nabla Q_{t=0}(z) = \nabla P(z) = H(z)$. Note that the inviscid Burgers' equations are master equations for diffusions of air or liquids with viscid constant $c = 0$. It is surprising for us to see that the fate of the Jacobian conjecture is completely determined by behaviors of airs or liquids with viscid constant $c = 0$.

The connection between the inversion problem and the Legendre transform (See [2,15]) is straightforward. For any $f(z) \in \mathbb{C}[[z]]$ of order $o(f(z)) \geqslant 2$, we can always write $f(z) = \frac{1}{2}\sum_{i=1}^{n} z_i^2 - P(z)$ for some $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geqslant 2$. If $\det \operatorname{Hes}(f)(0) \neq 0$, the Legendre transform $\bar{f}(z)$ of $f(z)$ is by definition given by $\bar{f}(z) = \frac{1}{2}\sum_{i=1}^{n} z_i^2 - Q(z)$, where $Q(z)$ is the unique formal power series with $o(Q(z)) \geqslant 2$ such that the formal maps $F(z) = z - \nabla P(z)$ and $G(z) = z - \nabla Q(z)$ are inverse to each other. Hence, the Legendre transform for formal power series $f(z) \in \mathbb{C}[[z]]$ with $o(f(z)) \geqslant 2$, is essentially the inversion problem under the gradient condition. All results and inversion formulas derived in this paper can also be used as computational algorithms for the Legendre transforms of formal power series $f(z) \in \mathbb{C}[[z]]$ with $o(f(z)) \geqslant 2$ and $\det \operatorname{Hes}(f)(0) \neq 0$.

Finally, in Section 5, by using the recurrent formula obtained in Proposition (3.7), we derive a binary rooted tree expansion inversion formula for symmetric maps, see Theorem 5.2. Note that a tree expansion inversion formula for symmetric formal maps has been given by Meng [15] and Wright [24]. The binary rooted tree expansion inversion formula we derive here is different from the one in [15] and Wright [24]. It only involves binary rooted trees.

Two remarks are as follows. First, we will fix $\mathbb{C}$ as our base field. But all results, formulas as well as their proofs given in this paper hold or work equally well for formal power series over any $\mathbb{Q}$-algebra. Secondly, for convenience, we will mainly work on the setting of formal power series over $\mathbb{C}$. But, for polynomial maps or local analytic maps, all formal maps or power series involved in this paper are also locally convergent. This can be easily seen either from the fact that any local analytic map with non-zero Jacobian at the origin has a locally convergent inverse, or from the well-known Cauchy–Kowaleskaya theorem (See [17], for example.) in PDE.

2. A Deformation of formal maps

Once and for all, we fix the following notation and conventions.

(1) We fix $n \geqslant 1$ and set $z = (z_1, z_2, \ldots, z_n)$. For any $\mathbb{Q}$-algebra k, we denote by $k[z]$ (resp. $k[[z]]$) the polynomial algebra (resp. formal power series algebra) over k in z_i $(1 \leqslant i \leqslant n)$.

(2) For any $\mathbb{Q}$-algebra k, by a formal map $F(z)$ from k^n to k^n, we simply mean $F(z) = (F_1(z), F_2(z), \ldots, F_n(z))$ with $F_i(z) \in k[[z]]$ $(1 \leqslant i \leqslant n)$. We denoted by $J(F)$ and $j(F)$ the Jacobian matrix and the Jacobian of $F(z)$, respectively.

(3) We denote by Δ the Laplace operator $\sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2}$. Note that, a polynomial or formal power series $P(z)$ is said to be *harmonic* if $\Delta P = 0$.

(4) For any $k \geqslant 1$ and $U(z) = (U_1(z), U_2(z), \ldots, U_k(z)) \in \mathbb{C}[[z]]^{\times k}$, we set

$$\mathrm{o}(U(z)) = \min_{1 \leqslant i \leqslant k} \mathrm{o}(U_i(z))$$

and, when $U(z) \in \mathbb{C}[z]^{\times k}$,

$$\deg U(z) = \max_{1 \leqslant i \leqslant k} \deg U_i(z).$$

For any $U_t(z) \in \mathbb{C}[t][z]^{\times k}$ or $\mathbb{C}[[z,t]]^{\times k}$ $(k \geqslant 1)$ for some formal parameter t, the notation $\mathrm{o}(U_t(z))$ and $\deg U_t(z)$ stand for the order and the degree of $U_t(z)$ with respect to z, respectively.

(5) For any $P(z) \in \mathbb{C}[[z]]$, we denote by $\nabla P(z)$ the gradient of $P(z)$, i.e. $\nabla P = (\frac{\partial P}{\partial z_1}, \frac{\partial P}{\partial z_2}, \ldots, \frac{\partial P}{\partial z_n})$. We denote by $\mathrm{Hes}\,(P)(z)$ the Hessian matrix of $P(z)$, i.e. $\mathrm{Hes}\,(P)(z) = (\frac{\partial^2 P(z)}{\partial z_i \partial z_j})$.

(6) All n-vectors in this paper are supposed to be column vectors unless stated otherwise. For any vector or matrix U, we denote by U^τ its transpose. The standard $\mathbb{C}$-bilinear form of n-vectors is denoted by $\langle \cdot, \cdot \rangle$.

In this paper, we will fix a formal map $F(z)$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ and always assume that $F(z)$ has the form $F(z) = z - H(z)$ with $o(H(z)) \geqslant 1$. Note that, any formal map $V : \mathbb{C}^n \to \mathbb{C}^n$ with $V(0) = 0$ and $j(V)(0) \neq 0$ can be transformed into the form above by composing with some affine automorphisms of $\mathbb{C}^n$.

Let t be a formal parameter which commutes with z_i $(1 \leqslant i \leqslant n)$. We set $F_t(z) = z - tH(z)$. Since $F_{t=1}(z) = F(z)$, $F_t(z)$ can be viewed as a deformation of the formal map $F(z)$. From now on, we will denote by $G(z)$ and $G_t(z)$ the formal inverses of $F(z)$ and $F_t(z)$, respectively. Note that, $G_t(z)$ can always be written as $G_t(z) = z + tN_t(z)$ for some $N_t(z) \in \mathbb{C}[[z,t]]^{\times n}$ with $o(N_t(z)) \geqslant 1$. Furthermore, when $o(H(z)) \geqslant 2$, $N_t(z)$ actually lies in $\mathbb{C}[t][[z]]^{\times n}$ with $o(N_t(z)) \geqslant 2$, and by the uniqueness of formal inverses, we have $G_{t=1}(z) = G(z)$ in this case. This can be easily proven by using any well-known inversion formulas, for example, the Abhyankar–Gurjar inversion formula [1] or the Bass–Connell–Wright tree expansion formula [3]. We will show in Theorem 2.4 that $N_t(z)$ is the unique solution of a Cauchy problem of PDE, from which we derive a recurrent formula for $N_t(z)$, see Theorem 2.7. We also discuss some other properties of $N_t(z)$ including the one in Proposition 2.9, which characterizes $N_t(z)$, see Proposition 2.10.

Lemma 2.1. *For the formal power series $N_t(z) \in \mathbb{C}[[z,t]]^{\times n}$ defined above, we have the following identities.*

$$N_t(F_t(z)) = H(z), \tag{2.1}$$

$$H(G_t) = N_t(z). \tag{2.2}$$

Proof. Since $z = G_t(F_t)$, we have

$$z = F_t(z) + tN_t(F_t(z)),$$

$$z = z - tH(z) + tN_t(F_t(z)).$$

Therefore,

$$H(z) = N_t(F_t(z)),$$

which is Eq. (2.1). By composing the both sides of Eq. (2.1) with $G_t(z)$ from right, we get Eq. (2.2). □

Lemma 2.2. *The following statements are equivalent.*

(1) $JH(z)$ *is nilpotent.*
(2) $\mathrm{Tr}\, JN_t(z) = 0$.
(3) $JN_t(z)$ *is nilpotent.*

Proof. First, by the fact $JG_t(F_t(z)) = JF_t^{-1}(z)$, we have

$$I + tJN_t(F_t) = (I - tJH)^{-1},$$

$$tJN_t(F_t) = -I + (I - tJH)^{-1} = tJH(I - tJH)^{-1},$$

$$JN_t(F_t) = JH(I - tJH)^{-1} = \sum_{k=1}^{\infty} JH^k(z)t^{k-1}. \tag{2.3}$$

Therefore we have

$$\mathrm{Tr}\, JN_t(F_t) = \sum_{k=1}^{\infty} \mathrm{Tr}\,(JH)^k t^{k-1} \tag{2.4}$$

and, for any $m \geqslant 0$,

$$JN_t^m(F_t) = JH^m(I - tJH)^{-m}, \tag{2.5}$$

since the matrices *JH* and $(I - tJH)^{-1}$ commute with each other.

By using the fact that $F_t(z)$ is an automorphism of the power series algebra $\mathbb{C}[[t]][[z]]$, we see that, (1) $\Leftrightarrow$ (2) follows from Eq. (2.4) and the fact that a matrix *B* is nilpotent if and only if $\mathrm{Tr}\,(B^k) = 0$ $(k \geqslant 1)$; while (1) $\Leftrightarrow$ (3) follows form Eq. (2.5) and the fact $I - tJH(z)$ is invertible in $M_n(\mathbb{C}[[t]][[z]])$. $\square$

By Eq. (2.5) and the fact $N_{t=1}(z) = N(z)$ when $\mathrm{o}(P(z)) \geqslant 2$, it is easy to see that we have the following corollary.

Corollary 2.3. *Let $F(z) = z - H(z)$ with $\mathrm{o}(H(z)) \geqslant 2$ and $G(z) = z + N(z)$ with $\mathrm{o}(N(z)) \geqslant 2$ the formal inverse of $F(z)$. Then, for any $m \geqslant 1$, we have $JH^m(z) = 0$ if and only if $JN^m(z) = 0$. In particular, $JH(z)$ is nilpotent if and only if $JN(z)$ is.*

Theorem 2.4. *For any $H(z) \in \mathbb{C}[[z]]^{\times n}$ and $N_t(z) \in \mathbb{C}[t][[z]]^{\times n}$ with $\mathrm{o}(H(z)) \geqslant 1$ and $\mathrm{o}(N_t(z)) \geqslant 1$, respectively. The following statements are equivalent.*

(1) *The formal map $G_t(z) = z + tN_t(z)$ is the formal inverse of $F_t(z) = z - tH(z)$.*
(2) *$N_t(z)$ is the unique power series solution of the following Cauchy problem of PDE's.*

$$\frac{\partial N_t}{\partial t} = JN_t \cdot N_t, \tag{2.6}$$

$$N_{t=0}(z) = H(z), \tag{2.7}$$

where JN_t is the Jacobian matrix of $N_t(z)$ with respect to z.

Proof. First, we show (1) $\Rightarrow$ (2). By applying $\frac{\partial}{\partial t}$ to the both sides of Eq. (2.1), we get

$$\begin{aligned} 0 &= \frac{\partial N_t(F_t)}{\partial t} \\ &= \frac{\partial N_t}{\partial t}(F_t) + JN_t(F_t)\frac{\partial F_t}{\partial t} \\ &= \frac{\partial N_t}{\partial t}(F_t) - JN_t(F_t)H. \end{aligned}$$

Therefore,

$$\frac{\partial N_t}{\partial t}(F_t) = JN_t(F_t)H.$$

By composing with $G_t(z)$ from right, we get

$$\frac{\partial N_t}{\partial t} = JN_t \cdot H(G_t) = JN_t N_t.$$

Note that $G_{t=0}(z) = z$, for it is the formal inverse of $F_{t=0}(z) = z$. Eq. (2.7) follows immediately from Eq. (2.2) by setting $t = 0$.

To show (2) $\Rightarrow$ (1), we assume that the formal inverse of $F_t(z) = z - tH(z)$ is given by $G_t(z) = z + t\widetilde{N}_t(z)$. By the fact proved above, we know that $\widetilde{N}_t(z)$ also satisfies Eqs. (2.6) and (2.7). We will show in Proposition 2.5 below that the power series solutions of the Cauchy problem Eqs. (2.6) and (2.7) are actually unique. By this fact it is easy to see that (2) $\Rightarrow$ (1) also holds. $\square$

We define the sequence $\{N_{[m]}(z) | m \geqslant 1\}$ by writing

$$N_t(z) = \sum_{m=1}^{\infty} N_{[m]}(z) t^{m-1}. \tag{2.8}$$

Proposition 2.5. *Let* $N_t(z) = \sum_{m=1}^{\infty} N_{[m]}(z) t^{m-1}$ *be a power series solution of Eqs.* (2.6) *and* (2.7). *Then*

$$N_{[1]}(z) = H(z), \tag{2.9}$$

$$N_{[m]}(z) = \frac{1}{m-1} \sum_{\substack{k+l=m \\ k,l \geqslant 1}} JN_{[k]}(z) \cdot N_{[l]}(z) \tag{2.10}$$

for any $m \geqslant 2$.

Proof. First, Eq. (2.9) follows immediately from Eq. (2.7). Secondly, by Eq. (2.6), we have

$$\sum_{m=1}^{\infty} (m-1) N_{[m]}(z) t^{m-2} = \left(\sum_{k=1}^{\infty} JN_{[k]}(z) t^{k-1} \right) \left(\sum_{l=1}^{\infty} N_{[l]}(z) t^{l-1} \right).$$

Comparing the coefficients of t^{m-2} of the both sides of the equation above, we have

$$(m-1) N_{[m]}(z) = \sum_{\substack{k+l=m \\ k,l \geqslant 1}} JN_{[k]}(z) \cdot N_{[l]}(z)$$

for any $m \geqslant 2$. Hence we get Eq. (2.10). $\square$

By using Eqs. (2.9) and (2.10) and the mathematical induction, it is easy to show the following lemma.

Lemma 2.6. (a) $\mathrm{o}(N_{[m]}(z)) \geqslant m+1$ *for any* $m \geqslant 1$.

(b) *Suppose* $H(z) \in \mathbb{C}[z]^{\times n}$, *then, for any* $m \geqslant 1$, $N_{[m]} \in \mathbb{C}[z]^{\times n}$ *with* $\deg N_{[m]}(z) \leqslant (\deg H - 1)m + 1$.

(c) If $H(z)$ *is homogeneous of degree* d, *then*, $N_{[m]}(z)$ *is homogeneous of degree* $(d-1)m+1$ *for any* $m \geqslant 1$.

Note that, by Lemma 2.6(a), the infinite sum $\sum_{m=1}^{\infty} N_{[m]}(z) t_0^{m-1}$ makes sense for any complex number $t = t_0$. In particular, when $t = 1$, $G_{t=1}(z)$ gives us the formal inverse $G(z)$ of $F(z)$.

Theorem 2.7 (*Recurrent inversion formula*). *Let* $\{N_{[m]}(z) | m \geqslant 1\}$ *be the sequence defined by Eqs.* (2.9) *and* (2.10) *recursively. Then the formal inverse of* $F(z) = z - H(z)$ *is given by*

$$G(z) = z + \sum_{m=1}^{\infty} N_{[m]}(z). \tag{2.11}$$

One interesting property of $N_t(z)$ is the following proposition. It basically says that $\{N_t(z) | t \in \mathbb{C}\}$ gives a family of formal maps from $\mathbb{C}^n$ to $\mathbb{C}^n$, which are "closed" under the inverse operation.

Proposition 2.8. *For any* $s \in \mathbb{C}$, *the formal inverse of* $U_{s,t}(z) := z - sN_t(z)$ *is given by* $V_{s,t}(z) := z + sN_{t+s}(z)$. *Actually*, $U_{s,t}(z) = F_{t+s} \circ G_t(z)$ *and* $V_{s,t}(z) = F_t \circ G_{s+t}(z)$.

Proof.

$$\begin{aligned} F_{t+s} \circ G_t(z) &= G_t(z) - (t+s)H(G_t(z)) \\ &= z + tN_t(z) - (t+s)N_t(z) \\ &= z - sN_t(z) \\ &= U_{s,t}(z). \end{aligned}$$

Similarly, we can prove $V_{s,t}(z) = F_t \circ G_{s+t}(z)$. $\square$

Another special property of $N_t(z)$ is given by the following proposition.

Proposition 2.9. *For any* $U(z) \in \mathbb{C}[[z]]$, *the unique power series solution* $U_t(z)$ *in* z *and* t *of the Cauchy problem*

$$\begin{cases} \frac{\partial U_t}{\partial t} = \langle \nabla U_t, N_t \rangle, \\ U_{t=0}(z) = U(z). \end{cases} \tag{2.12}$$

is given by $U_t(z) = U(z + tN_t(z))$.

Proof. By similar arguments as the proof of Proposition 2.5, it is easy to see that the power series solution in z and t of the Cauchy problem Eq. (2.12) is unique. So it will be enough to show that $U_t(z) = U(z + tN_t(z))$ is a solution of Eq. (2.12).

$$\begin{aligned}\frac{\partial U_t}{\partial t} &= \frac{\partial}{\partial t}U(z + tN_t)\\ &= \langle \nabla U(z + tN_t), \frac{\partial}{\partial t}(z + tN_t)\rangle\\ &= \langle \nabla U(z + tN_t), N_t + t\frac{\partial N_t}{\partial t}\rangle\end{aligned}$$

and applying Eq. (2.6):

$$\begin{aligned}&= \langle \nabla U(z + tN_t),\ \ N_t + tJN_tN_t\rangle\\ &= \langle \nabla U(z + tN_t),\ \ (\mathrm{I} + tJN_t)N_t\rangle\\ &= \langle (\mathrm{I} + tJN_t)^\tau \nabla U(z + tN_t),\ \ N_t\rangle\\ &= \langle \nabla (U(z + tN_t)),\ \ N_t\rangle\\ &= \langle \nabla U_t,\ N_t\rangle. \qquad \square\end{aligned}$$

Actually, $N_t(z)$ is characterized by the property in Proposition 2.9.

Proposition 2.10. *For any* $N_t(z) \in \mathbb{C}[[z, t]]^{\times n}$ *with* $\mathrm{o}(N_t(z)) \geqslant 1$, *the following are equivalent.*

(1) $z + tN_t(z)$ *is the formal inverse of* $z - tH(z)$ *for some* $H(z) \in \mathbb{C}[[z]]^{\times n}$.
(2) *Proposition* 2.9 *holds for* $N_t(z)$.

Proof. First, (1) $\Rightarrow$ (2) follows from Proposition 2.9. To show (2) $\Rightarrow$ (1), let $U_{t,i}(z)$ $(1 \leqslant i \leqslant n)$ be the unique power series solution of the Cauchy problem (2.12) with $U(z) = z_i$ and set $\widetilde{U}_t(z) = (U_{t,1}(z), U_{t,2}(z), \ldots, U_{t,n}(z))$. Note that Eq. (2.12) for $U_{t,i}(z)$ $(1 \leqslant i \leqslant n)$ can be written as

$$\frac{\partial \widetilde{U}_t}{\partial t} = J\widetilde{U}_t \cdot N_t. \tag{2.13}$$

By Proposition 2.9, we have

$$\widetilde{U}_t(z) = z + tN_t(z). \tag{2.14}$$

By applying $\frac{\partial}{\partial t}$ to the equation above, we get

$$\frac{\partial \widetilde{U}_t}{\partial t} = N_t + t\frac{\partial N_t}{\partial t}. \tag{2.15}$$

By combining the equation above with Eqs. (2.13) and (2.14), we have

$$N_t + t\frac{\partial N_t}{\partial t} = J\widetilde{U}_t \cdot N_t = (I + tJN_t) \cdot N_t.$$

Therefore, we have

$$\frac{\partial N_t}{\partial t} = JN_t \cdot N_t. \tag{2.16}$$

Set $H(z) = N_{t=0}(z)$. By Theorem 2.4, we see that (1) holds. □

3. The case of symmetric formal maps

Let $F(z) = z - H(z)$ with $\mathrm{o}(H(z)) \geqslant 1$ be a formal map from $\mathbb{C}^n$ to $\mathbb{C}^n$. We say that $F(z)$ is a *symmetric* formal map if its Jacobian matrix $J(F)$ is symmetric. Note that, by Poincaré lemma, it is easy to see that $F(z)$ is symmetric if and only if it satisfies the *gradient condition*, i.e. $H(z) = \nabla P(z)$ for some $P(z) \in \mathbb{C}[[z]]$.

In this section, we study the deformation $F_t(z)$ and its inverse map $G_t(z)$ for symmetric formal maps $F(z)$. Besides some new properties of $N_t(z)$, the main results and formulas for $N_t(z)$ obtained in the previous section will also be simplified.

We first give a different proof for the following lemma which was first proved in [15].

Lemma 3.1. *Let $F(z) = z - H(z)$ with $\mathrm{o}(H(z)) \geqslant 1$ and $j(F)(0) \neq 0$ be a formal map with formal inverse $G(z) = z + N(z)$. Then, $F(z)$ is symmetric if and only if $G(z)$ is.*

Proof. We first assume that $H(z) = \nabla P(z)$ for some $P(z) \in \mathbb{C}[[z]]$. Note that $JH(z) = \mathrm{Hes}\,(P(z))$ is symmetric. By Eq. (2.3), we see that $JN_t(F_t)$ is symmetric. Hence so are $JN_t(z)$ and $JN(z) = JN_{t=1}(z)$. By Poincaré Lemma, we know that $N(z)$ must be the gradient of some $Q(z) \in \mathbb{C}[[z]]$, i.e. $N(z) = \nabla Q(z)$.

By switching $H(z)$ and $N(z)$, we see that the converse also holds. □

Now, for any $P(z) \in \mathbb{C}[[z]]$ with $\mathrm{o}(P(z)) \geqslant 2$, we consider the deformation $F_t(z) = z - t\nabla P(z)$ and its inverse $G_t(z) = z + tN_t(z)$. By applying Lemma 3.1 to $F_t(z)$, we know that $N_t(z) = \nabla Q_t(z)$ for some $Q_t(z) \in \mathbb{C}[[z, t]]$ with $\mathrm{o}(Q_t(z)) \geqslant 2$. We will fix the notation $Q_t(z)$ as above through the rest of this paper unless stated otherwise.

Proposition 3.2. *For any $P(z) \in \mathbb{C}[[z]]$ with $\mathrm{o}(P(z)) \geqslant 2$, the following are equivalent.*

(1) $\mathrm{Hes}\, P(z)$ *is nilpotent*.
(2) $Q_t(z)$ *is harmonic, i.e.* $\Delta Q_t(z) = 0$.
(3) $\mathrm{Hes}\, Q_t(z)$ *is nilpotent*.

Proof. (1) $\Leftrightarrow$ (2) follows from Lemma 2.2. (1) $\Leftrightarrow$ (3) follows from Corollary 2.3. Hence we also have (2) $\Leftrightarrow$ (3). □

Lemma 3.3. *Let $P(z)$, $F_t(z)$, $G_t(z)$ and $Q_t(z)$ as above. Then we have the following identities*.

$$(\nabla Q_t)(F_t) = \nabla P, \tag{3.1}$$

$$(\nabla P)(G_t) = \nabla Q_t. \tag{3.2}$$

Proof. Since $H(z) = \nabla P$ and $N_t(z) = \nabla Q_t$ in our case, the lemma follows immediately from Lemma 2.1. □

Lemma 3.4.

$$Q_t(F_t) = P - \frac{t}{2}\langle \nabla P, \nabla P\rangle, \tag{3.3}$$

$$P(G_t) = Q_t + \frac{t}{2}\langle \nabla Q_t, \nabla Q_t\rangle. \tag{3.4}$$

Proof. For any $1 \leqslant i \leqslant n$, we consider

$$\frac{\partial Q_t(F_t)}{\partial z_i} = \sum_{j=1}^{n} \frac{\partial Q_t}{\partial z_j}(F_t)\frac{\partial F_{t,j}(z)}{\partial z_i}$$

and applying Eq. (3.1) in Lemma 3.3:

$$\begin{aligned}
&= \sum_{j=1}^{n} \frac{\partial P}{\partial z_j}\left(\delta_{i,j} - t\frac{\partial^2 P}{\partial z_i \partial z_j}\right)\\
&= \frac{\partial P}{\partial z_i} - t \sum_{j=1}^{n} \frac{\partial P}{\partial z_j}\frac{\partial^2 P}{\partial z_i \partial z_j}\\
&= \frac{\partial P}{\partial z_i} - \frac{t}{2}\frac{\partial}{\partial z_i} \sum_{j=1}^{n} \frac{\partial P}{\partial z_j}\frac{\partial P}{\partial z_j}\\
&= \frac{\partial}{\partial z_i}\left(P - \frac{t}{2}\langle \nabla P, \nabla P\rangle\right).
\end{aligned}$$

Hence, Eq. (3.3) holds. Eq. (3.4) can be proved similarly by using Eq. (3.2). □

Lemma 3.5.

$$\frac{\partial Q_t}{\partial t}(F_t) = \tfrac{1}{2}\langle \nabla P, \nabla P\rangle. \tag{3.5}$$

Proof. By applying $\frac{\partial}{\partial t}$ to the both sides of Eq. (3.3), we get

$$\begin{aligned}
-\tfrac{1}{2}\langle \nabla P, \nabla P\rangle &= \frac{\partial Q_t}{\partial t}(F_t) + \sum_{j=1}^{n} \frac{\partial Q_t}{\partial z_j}(F_t)\frac{\partial F_{t,j}}{\partial t} \\
&= \frac{\partial Q_t}{\partial t}(F_t) - \sum_{j=1}^{n} \frac{\partial P}{\partial z_j}\frac{\partial P}{\partial z_j} \\
&= \frac{\partial Q_t}{\partial t}(F_t) - \langle \nabla P, \nabla P\rangle.
\end{aligned}$$

Hence, Eq. (3.5) follows. $\square$

Under the gradient condition, Theorem 2.4 becomes the following theorem.

Theorem 3.6. *For any* $Q_t(z) \in \mathbb{C}[[z,t]]$ *with* $o(Q_t(z)) \geqslant 2$ *and* $P(z) \in \mathbb{C}[[z]]$ *with* $o(P(z)) \geqslant 2$, *the following are equivalent.*

(1) $G_t(z) = z + t\nabla Q_t(z)$ *is the formal inverse of* $F_t(z) = z - t\nabla P(z)$.
(2) $Q_t(z)$ *is the unique power series solution of the following Cauchy problem of PDE's.*

$$\begin{cases} \frac{\partial Q_t(z)}{\partial t} = \frac{1}{2}\langle \nabla Q_t, \nabla Q_t\rangle, \\ Q_{t=0}(z) = P(z). \end{cases} \tag{3.6}$$

Note that, (2) $\Rightarrow$ (1) follows from (1) $\Rightarrow$ (2) and the uniqueness of the power series solutions of the Cauchy problem Eq. (3.6). (For a similar argument, see the proof of Theorem 2.4.) While the uniqueness of the power series solutions of Eq. (3.6) can be proved by similar arguments as the proof of Proposition 2.5. (Also see Proposition 3.7 below.) So we only need show (1) $\Rightarrow$ (2), for which we here give two different proofs.

First proof. First note that $J(\nabla Q_t) = \mathrm{Hes}\,(Q_t)$. By replacing $N_t(z)$ by $\nabla Q_t(z)$ in Eqs. (2.6) and (2.7), we get

$$\nabla \frac{\partial Q_t}{\partial t} = \mathrm{Hes}\,(Q_t)\nabla Q_t, \tag{3.7}$$

$$\nabla Q_{t=0}(z) = \nabla P(z). \tag{3.8}$$

Since $o(P(z)) \geqslant 2$ and $o(Q_t(z)) \geqslant 2$, Eq. (3.8) implies $Q_{t=0}(z) = P(z)$. Furthermore, Eq. (3.7) implies that, for any $1 \leqslant i \leqslant n$, we have

$$\frac{\partial}{\partial z_i}\frac{\partial Q_t}{\partial t} = \sum_{j=1}^{n} \frac{\partial^2 Q_t}{\partial z_i \partial z_j}\frac{\partial Q_t}{\partial z_j} = \frac{1}{2}\frac{\partial}{\partial z_i}\langle \nabla Q_t, \nabla Q_t\rangle.$$

Since $o(\frac{\partial Q_t}{\partial t}) \geqslant 2$ and $o(\langle \nabla Q_t, \nabla Q_t\rangle) \geqslant 2$, the PDE in Eq. (3.6) also holds. $\square$

Second proof. By composing with $G_t(z)$ from right to both sides of Eq. (3.5) and applying Eq. (3.2), we have

$$\begin{aligned}\frac{\partial Q_t(z)}{\partial t} &= \tfrac{1}{2}\langle(\nabla P)(G_t), (\nabla P)(G_t)\rangle \\ &= \tfrac{1}{2}\langle \nabla Q_t, \nabla Q_t\rangle.\end{aligned}$$

The initial condition in Eq. (3.6), as proved in the first proof, follows from Eq. (2.7) in Theorem 2.4. □

We define a sequence of formal power series $\{Q_{[m]}(z) \in \mathbb{C}[[z]] | m \geqslant 1\}$ by writing

$$Q_t(z) = \sum_{m=1}^{\infty} Q_{[m]}(z) t^{m-1}. \tag{3.9}$$

From Eq. (3.6), we get $Q_{[1]}(z) = P(z)$. Furthermore, by replacing $Q_t(z)$ by the sum above and comparing the coefficients of t^{m-2} $(m \geqslant 2)$ in Eq. (3.6), we get Eq. (3.11) below. So we obtain the following recurrent formula for the formal power series $\{Q_{[m]}(z) \in \mathbb{C}[[z]] | m \geqslant 1\}$.

Proposition 3.7. *We have the following recurrent formula for* $Q_t(z)$.

$$Q_{[1]}(z) = P(z), \tag{3.10}$$

$$Q_{[m]}(z) = \frac{1}{2(m-1)} \sum_{\substack{k,l \geqslant 1 \\ k+l=m}} \langle \nabla Q_{[k]}(z), \nabla Q_{[l]}(z)\rangle \tag{3.11}$$

for any $m \geqslant 2$. *In particular, when* $P(z)$ *is a polynomial,* $Q_{[m]}(z)$ $(m \geqslant 1)$ *are also polynomials.*

For a uniform non-recurrent formula for $Q_t^k(z)$ $(k \geqslant 1)$ under the condition that $\mathrm{Hes}\,(P)$ is nilpotent, see [26].

4. Relationships with Legendre transform and the inviscid Burgers' equations

In this section, we clarify some close relationships of the inversion problem for symmetric formal maps with the Legendre transform and the inviscid Burgers' equations. In particular, we reduce the Jacobian conjecture to a problem on the Cauchy problem Eq. (4.3), whose PDE is the simplified version of the inviscid Burgers' under the gradient condition.

First let us recall the Legendre transform (See [15,2].). Let $f(z) \in \mathbb{C}[[z]]$ with $o(f(z)) \geqslant 2$ and $\det \mathrm{Hes}\,(f)(0) \neq 0$. Then the formal Legendre transform $\bar{f}(z)$ of $f(z)$ by definition is the unique $\bar{f}(z) \in \mathbb{C}[[z]]$ with $o(\bar{f}(z)) \geqslant 2$ such that the inverse map of the formal map $\nabla f : \mathbb{C}^n \to \mathbb{C}^n$ is given by $\nabla \bar{f}$. Note that, for any $f(z) \in \mathbb{C}[[z]]$ of order $o(f(z)) \geqslant 2$, one can always write $f(z) = \frac{1}{2}\sum_{i=1}^n z_i^2 - P(z)$ for some $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geqslant 2$.

If $\det \operatorname{Hes}(f)(0) \neq 0$, it is easy to check that the Legendre transform $\bar{f}(z)$ of $f(z)$ is given by $\bar{f}(z) = \frac{1}{2}\sum_{i=1} z_i^2 + Q(z)$ for some $\bar{Q}(z) \in \mathbb{C}[[z]]$ with $o(Q(z)) \geqslant 2$. Hence the Legendre transform for $f(z) \in \mathbb{C}[[z]]$ with $o(f(z)) \geqslant 2$ is essentially the inversion problem under the gradient condition. Therefore, the recurrent inversion formula in Proposition 3.7 and the binary rooted tree expansion formula in Theorem 5.2 that will be derived in next section can also be used as computational algorithms for the Legendre transform for formal power series $f(z) \in \mathbb{C}[[z]]$ of $o(f(z)) \geqslant 2$ and $\det \operatorname{Hes}(f)(0) \neq 0$.

Next we consider some relationships of the inversion problem for symmetric formal maps with the inviscid Burgers' equations. The Burgers' equations (See [16,17] and the references there.) are master equations in Diffusion theory. Recall that the *n*-dimensional inviscid Burgers' equation is usually written as

$$\frac{\partial U_t}{\partial t}(z) + (JU_t)^\tau(z) \cdot U_t(z) = 0 \tag{4.1}$$

or

$$\frac{\partial U_t}{\partial t}(z) = (JU_t)^\tau(z) \cdot U_t(z), \tag{4.2}$$

where $U_t(z)$ is a *n*-vector-valued function of (t, z) and $(JU_t)(z)$ denotes the Jacobian matrix of $U_t(z)$ with respect to z.

Note that, for any *n*-vector-valued function $V_t(z)$ of (t, z), $V_t(z)$ satisfies Eq. (4.1) if and only if $-V_t(z)$ satisfies Eq. (4.2). Hence Eqs. (4.1) and (4.2) are equivalent to each other. In this paper, we will refer the PDE (4.2) as the *n*-dimensional inviscid Burgers' equation.

By comparing Eqs. (2.6) and (4.2), we see that, the main PDE Eq. (2.6) for the general inversion problem without the gradient condition is almost the *n*-dimensional inviscid Burgers' equation (4.2) except the transpose part. More interestingly, under the gradient condition, we have $JN_t(z) = \operatorname{Hes}(Q_t)$ which is symmetric and Eq. (2.6) becomes exactly the *n*-dimensional inviscid Burgers' equation Eq. (4.2). The PDE in the Cauchy problem Eq. (3.6) is just a simplified version of the inviscid Burgers' equation (4.2) under the assumption that $U_t(z) = \nabla Q_t(z)$ for some function $Q_t(z)$ of t and z. Motivated by the connections above, we formulate the following conjecture.

Conjecture 4.1. *For any homogeneous polynomial $P(z)$ of degree $d \geqslant 2$ with the Hessian matrix $\operatorname{Hes}(P)$ nilpotent, let $U_t(z)$ be the unique power series solution of the following Cauchy problem of PDE's.*

$$\begin{cases} \frac{\partial U_t}{\partial t}(z) = \frac{1}{2}\langle \nabla U_t(z), \nabla U_t(z)\rangle, \\ U_{t=0}(z) = P(z). \end{cases} \tag{4.3}$$

Then $U_t(z)$ must be a polynomial in both z and t.

Proposition 4.2. *Conjecture* (4.1) *above for $d = 4$ is equivalent to the Jacobian conjecture*.

Proof. First, by using the gradient reduction in [4] and [15] and the homogeneous reduction in [3] on the Jacobian conjecture, we see that the Jacobian conjecture is reduced to polynomial maps $F(z) = z - \nabla P(z)$ with $P(z)$ homogeneous of degree $d = 4$. Secondly, since $P(z)$ is homogeneous, the polynomial map $F(z) = z - \nabla P(z)$ satisfies the Jacobian condition $j(F)(z) = 1$ if and only if the Hessian matrix $\mathrm{Hes}\,(P) = J(\nabla P)$ is nilpotent. Then it is easy to see, using Eqs. (2.8) and (2.11), that the equivalence of Conjecture 4.1 and the Jacobian conjecture follows directly from Theorem 3.6. $\square$

Since the Jacobian conjecture for polynomial maps $F(z)$ of degree $\deg F(z) \leqslant 2$ has been proved by Wang [20], we see that Conjecture 4.1 is true for $d = 2, 3$. It would be very interesting to find some proofs for these results by PDE methods, especially for the case $d = 3$. Understandings of Conjecture 4.1 for $d = 3$ from PDE point view certainly will provide new insights to the Jacobian conjecture.

5. A binary rooted tree expansion inversion formula

In this section, we derive a binary rooted tree inversion expansion formula for symmetric formal maps. (See Theorem 5.2.) First let us fix the following notations and conventions.

By a *rooted tree* we mean a finite 1-connected graph with one vertex designated as its *root*. In a rooted tree there are natural ancestral relations between vertices. We say a vertex w is a child of vertex v if the two are connected by an edge and w lies further from the root than v. We define the *degree* of a vertex v of T to be the number of its children. A vertex is called a *leaf* if it has no children. A rooted tree T is said to be a *binary rooted tree* if every non-leaf vertex of T has exactly two children. When we speak of isomorphisms between rooted trees, we will always mean root-preserving isomorphisms.

Notation: Once and for all, we fix the following notation for the rest of this paper.

(1) We let $\mathbb{T}$ (resp. $\mathbb{B}$) be the set of isomorphism classes of all rooted trees (resp. binary rooted trees). For any $m \geqslant 1$, we let $\mathbb{T}_m$ be the set of isomorphism classes of all rooted trees with m vertices.
(2) We call the rooted tree with one vertex the *singleton*, denoted by $\circ$. For convenience, we also view the empty set as a rooted tree, denoted by $\emptyset$.
(3) For any rooted tree T, we set the following notation:

- rt_T denotes the root vertex of T.
- $|T|$ denotes the number of the vertices of T and $l(T)$ the number of leaves.
- $\alpha(T)$ denotes the number of the elements of the automorphism group $\mathrm{Aut}(T)$.
- $\widehat{T}$ denotes the rooted tree obtained by deleting all the leaves of T.

For any set of rooted trees $T_1, T_2, \ldots, T_d$, we define $B_+(T_1, T_2, \ldots, T_d)$ to be the rooted tree obtained by connecting all roots of T_i $(i = 1, 2, \ldots, d)$ to a single new vertex, which is set to the root of the new rooted tree $B_+(T_1, T_2, \ldots, T_d)$. Note that, for any $T_1, T_2 \in \mathbb{B}$, we have $B_+(T_1, T_2) \in \mathbb{B}$.

Next let us recall *T*-factorial $T!$ of rooted trees *T*, which was first introduced by Kreimer [13]. It is defined inductively as follows:

(1) For the empty rooted tree $\emptyset$ and the singleton $\circ$, we set $\emptyset! = 1$ and $\circ! = 1$.
(2) For any rooted tree $T = B_+(T_1, T_2, \dots, T_d)$, we set

$$T! = |T|\, T_1!\, T_2! \cdots T_d!. \tag{5.1}$$

Note that, for the chains C_m $(m \in \mathbb{N})$, i.e. the rooted trees with *m* vertices and height $m-1$, we have $C_m! = m!$. Therefore the *T*-factorial of rooted trees can be viewed as a generalization of the usual factorial of natural numbers.

Now, for any binary rooted tree *T*, we set

$$\beta(T) = \alpha(T)\, \widehat{T}!. \tag{5.2}$$

Lemma 5.1. (a) *For any non-empty binary rooted tree T, we have*

$$|T| = 2l(T) - 1, \tag{5.3}$$

$$|\widehat{T}| = l(T) - 1. \tag{5.4}$$

(b) *For any* $T \in \mathbb{B}$ *with* $T = B_+(T_1, T_2)$, *we have*

$$\beta(T) = \begin{cases} 2(l(T)-1)\beta(T_1)\beta(T_2) & if\ \ T_1 \simeq T_2, \\ (l(T)-1)\beta(T_1)\beta(T_2) & if\ \ T_1 \not\simeq T_2. \end{cases} \tag{5.5}$$

Proof. (a) First note that Eq. (5.4) follows from Eq. (5.3) and the fact $|\widehat{T}| = |T| - l(T)$. Hence we only need to show Eq. (5.3).

We use the mathematical induction on $|T|$. When $|T|=1$, we have $T=\circ$ and $|T|=l(T)=1$, hence (a) holds.

For any $T \in \mathbb{B}$ with $|T| \geqslant 2$. We write $T = B_+(T_1, T_2)$. Note that $T_1, T_2 \neq \emptyset$ and $|T_i| < |T|$ $(i = 1, 2)$. By our induction assumption, we have

$$\begin{aligned} |T| &= |T_1| + |T_2| + 1 \\ &= (2l(T_1) - 1) + (2l(T_2) - 1) + 1 \\ &= 2(l(T_1) + l(T_2)) - 1 \\ &= 2l(T) - 1. \end{aligned}$$

(b) First note that, we always have

$$\alpha(T) = \begin{cases} 2\alpha(T_1)\alpha(T_2) & \text{if}\ \ T_1 \simeq T_2, \\ \alpha(T_1)\alpha(T_2) & \text{if}\ \ T_1 \not\simeq T_2. \end{cases} \tag{5.6}$$

By Eqs. (5.1) and (5.4), we also have

$$\widehat{T}! = |\widehat{T}|\, \widehat{T}_1!\, \widehat{T}_2! = (l(T) - 1)\, \widehat{T}_1!\, \widehat{T}_2!. \tag{5.7}$$

Then, it is easy to see that Eq. (5.5) follows directly from Eqs. (5.2), (5.6) and (5.7). $\square$

Now we fix $P(z) \in \mathbb{C}[[z]]$ and $Q_t(z) \in \mathbb{C}[[z,t]]$ as in Section 3. We assign a formal power series $Q_T(z) \in \mathbb{C}[[z]]$ for each non-empty binary rooted tree T as follows:

(1) For $T = \circ$, we set $Q_T(z) = P(z)$.
(2) For any binary rooted tree $T = B_+(T_1, T_2)$, we set

$$Q_T(z) = \langle \nabla Q_{T_1}(z), \nabla Q_{T_2}(z) \rangle.$$

Finally we are ready to state and prove the main theorem of this section.

Theorem 5.2. *For any $m \geqslant 1$, we have*

$$Q_{[m]}(z) = \sum_{T \in \mathbb{B}_{2m-1}} \frac{1}{\beta(T)} Q_T(z) = \sum_{\substack{T \in \mathbb{B} \\ l(T)=m}} \frac{1}{\beta(T)} Q_T(z). \tag{5.8}$$

Therefore, by Eq. (3.9) *we have*

$$Q_t(z) = \sum_{T \in \mathbb{B} \setminus \emptyset} \frac{t^{l(T)-1}}{\beta(T)} Q_T(z), \tag{5.9}$$

$$Q(z) = \sum_{T \in \mathbb{B} \setminus \emptyset} \frac{1}{\beta(T)} Q_T(z). \tag{5.10}$$

Proof. Note that, by Eq. (5.3) in Lemma 5.1, we have

$$\mathbb{B}_{2m-1} = \{T \in \mathbb{B} | l(T) = m\}$$

$$\mathbb{B}_{2m} = \emptyset,$$

for any $m \geqslant 1$. Hence the two sums in Eq. (5.8) are equal to each other.

To prove Eq. (5.8), we first set, for any $m \geqslant 1$,

$$V_{[m]}(z) = \sum_{\substack{T \in \mathbb{B} \\ l(T)=m}} \frac{1}{\beta(T)} Q_T(z)$$

and then to show that $V_{[m]}(z) = Q_{[m]}(z)$ $(m \geqslant 1)$. By Proposition 3.7, it will be enough to show that the sequence $\{V_{[m]}(z) \in \mathbb{C}[[z]] | m \geqslant 1\}$ also satisfy Eqs. (3.10) and (3.11).

For the case $m = 1$, since there is only one binary rooted tree T with $l(T) = 1$, namely, $T = \circ$, we have $V_{[1]}(z) = Q_{T=\circ}(z) = P(z) = Q_{[1]}(z)$. Hence we have Eq. (3.10).

For any $m \geqslant 2$, we consider

$$\frac{1}{2(m-1)} \sum_{\substack{k,l\geqslant 1\\ k+l=m}} \langle \nabla V_{[k]}(z), \nabla V_{[l]}(z)\rangle$$
$$= \sum_{\substack{T_1,T_2\in\mathbb{B},\\ l(T_1)=k, l(T_2)=l,\\ k,l\geqslant 1, k+l=m}} \frac{1}{2(m-1)\beta(T_1)\beta(T_2)} \langle \nabla Q_{T_1}(z), \nabla Q_{T_2}(z)\rangle$$
$$= \sum_{\substack{T_1,T_2\in\mathbb{B},\\ l(T_1)=k, l(T_2)=l,\\ k,l\geqslant 1, k+l=m}} \frac{1}{2(m-1)\beta(T_1)\beta(T_2)} Q_{B_+(T_1,T_2)}(z).$$

Note that, the general term in the sum above appears twice when $T_1 \not\simeq T_2$ but only once when $T_1 \simeq T_2$. By applying Eq. (5.5) in Lemma 5.1:

$$= \sum_{\substack{T\in\mathbb{B}\\ l(T)=m}} \frac{1}{\beta(T)} Q_T(z)$$
$$= V_{[m]}(z).$$

Hence we have Eq. (3.11). □

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