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Composition–Diamond lemma for tensor product of free algebras ☆

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ABSTRACT

In this paper, we establish a Composition–Diamond lemma for the tensor product $k\langle X\rangle \otimes k\langle Y\rangle$ of two free algebras over a field. As an application, we construct a Gröbner–Shirshov basis in $k\langle X\rangle \otimes k\langle Y\rangle$ by lifting a given Gröbner–Shirshov basis in the tensor product $k[X] \otimes k\langle Y\rangle$ in which $k[X]$ is the polynomial algebra.

1. Introduction

In [31], A.I. Shirshov established the theory of one-relator Lie algebras $Lie(X|s=0)$. This theory is in full analogy, by statements but not by method, of the celebrated theory of Magnus on one-relater groups [22,23] (see also [24,21]). In particular, A.I. Shirshov provided the algorithmic decidality of the word problem for any one-relator Lie algebra. In order to proceed his ideas, he first created the so-called Gröbner–Shirshov bases theory for Lie algebras $Lie(X|S)$ which are presented by generators and defining relations. The main notion of Shirshov's theory was a notion of composition $(f,g)_w$ of two

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Lie polynomials, $f, g \in Lie(X)$ relative to some associative word w. Based on this notion, he defined an infinite algorithm of adding all non-trivial compositions to some set S until a set S^c is obtained which is closed under compositions. In addition, S and S^c generate the same ideal, i.e., $Id(S) = Id(S^c)$. We now call S^c the Gröbner–Shirshov basis of $Id(S)$.

The following lemma was proved by Shirshov [31].

Let $Lie(X) \subset k\langle X\rangle$ be a free Lie algebra over a field k which is regarded as an algebra of the Lie polynomials in the free algebra $k\langle X\rangle$, and let S be a subset of $Lie(X)$. If $f \in Id(S)$, then $\bar{f} = u\bar{s}v$, where $s \in S^c$, $u, v \in X^*$, $\bar{f}$, $\bar{s}$ are the leading associative words of the corresponding Lie polynomials f, s with respect to the deg-lex order on X^* and X^* is the free monoid generated by X.

The following corollary is an easy consequence of the above lemma.

$Irr(S^c) = \{[u] \mid u \neq a\bar{s}b, s \in S, a, b \in X^*\}$ is a linear basis of the algebra $Lie(X|S) = Lie(X)/Id(S)$, where u is an associative Lyndon–Shirshov word in X^* and $[u]$ is the corresponding non-associative Lyndon–Shirshov word under the Lie brackets $[xy] = xy - yx$.

In order to define the Lie composition $(f, g)_w$ of two monic Lie polynomials, where $\bar{f} = ac$, $\bar{g} = cb$, $c \neq 1$, a, b, c are associative words and $w = acb$, A.I. Shirshov first defined the associative composition $fb - ag$. Then by putting on fb and ag some special brackets $[fb], [ag]$ (see [29]), he obtained the Lie composition $(f, g)_w = [fb] - [ag]$. Following [31], one can easily deduce the same lemma for a free associative algebra, that is, let $S \subset k\langle X\rangle$ and S^c be defined as before. If $f \in Id(S)$, then $\bar{f} = a\bar{s}b$ for some $s \in S^c$, $a, b \in X^*$. This lemma was later formulated by L.A. Bokut [3] as an analogy of Shirshov's Lie composition lemma, and by G. Bergman [1] under the name "Diamond lemma" after the Newman's Diamond lemma for graphs [28].

Nowadays, Shirshov's lemma is named the "Composition–Diamond lemma" for Lie and associative algebras. The formulation of this lemma will be given in the next section of this paper.

This kind of ideas were also independently discovered by H. Hironaka [17] for power series algebras and by B. Buchberger [9,10] for polynomial algebras. The name "Gröbner bases" was suggested by B. Buchberger. The applications of Gröbner bases in mathematics are now well known and are well recognized, particularly in algebraic geometry, computer science and information science.

At present, there are many Composition–Diamond lemmas (CD-lemma for short) for different classes of non-commutative or non-associative algebras. We now list some of them below.

A.I. Shirshov [30] proved himself the first CD-lemma for commutative (anti-commutative) non-associative algebras, and he mentioned that this CD-lemma is also valid for non-associative algebras. It gave a solution of the word problem for these classes of algebras. For non-associative algebras, a version of CD-lemma was proved by A.I. Zhukov in [33].

A.A. Mikhalev [25] proved a CD-lemma for Lie superalgebras.

T. Stokes [32] proved a CD-lemma for left ideals of an algebra $k[X] \otimes E_k(Y)$, the tensor product of a polynomial algebra and an exterior (Grassmann) algebra.

A.A. Mikhalev and E.A. Vasilieva [26] proved a CD-lemma for free supercommutative polynomial algebras.

A.A. Mikhalev and A.A. Zolotykh [27] proved a CD-lemma for $k[X] \otimes k\langle Y\rangle$, the tensor product of a polynomial algebra and a free algebra.

L.A. Bokut, Y. Fong and W.F. Ke [7] proved a CD-lemma for associative conformal algebras.

L. Hellström [16] proved a CD-lemma for a non-commutative power series algebra.

S.-J. Kang and K.-H. Lee [18,19] and E.S. Chibrikov [13] proved a CD-lemma for a module over an algebra (see also [12]).

D.R. Farkas, C.D. Feustel and E.L. Green [15] proved a CD-lemma for path algebras.

L.A. Bokut and K.P. Shum [8] proved a CD-lemma for Γ-algebras.

Y. Kobayashi [20] proved a CD-lemma for algebras based on well-ordered semigroups.

L.A. Bokut, Yuqun Chen and Cihua Liu [5] proved a CD-lemma for dialgebras (see also [4]).

Yuqun Chen, Yongshan Chen and Yu Li [11] proved a CD-lemma for differential algebras.

L.A. Bokut, Yuqun Chen and Jianjun Qiu [6] proved a CD-lemma for associative algebras with multiple linear operators.

Let X and Y be sets and $k\langle X\rangle \otimes k\langle Y\rangle$ be the tensor product of free algebras. In this paper, we will give a Composition–Diamond lemma for the algebra $k\langle X\rangle \otimes k\langle Y\rangle$ (see Theorem 3.4). As a result, we prove a theorem on the pair of algebras $(k[X] \otimes k\langle Y\rangle, k\langle X\rangle \otimes k\langle Y\rangle)$ following the spirit of Eisenbud–Peeva–Sturmfels' theorem [14] on the pair $(k[X], k\langle X\rangle)$. Namely, we construct a Gröbner–Shirshov basis S' in $k\langle X\rangle \otimes k\langle Y\rangle$ by lifting a given Gröbner–Shirshov basis S in the tensor product $k[X] \otimes k\langle Y\rangle$ (in the sense of [27]) such that $(k\langle X\rangle \otimes k\langle Y\rangle)/Id(S') \cong (k[X] \otimes k\langle Y\rangle)/Id(S)$. Also, we give another proof of the Eisenbud–Peeva–Sturmfels' theorem above.

2. Preliminaries

We first cite some known concepts and results in the literature [31,2,3] concerning the Gröbner–Shirshov bases theory of associative algebras.

Let k be a field, $k\langle X\rangle$ the free associative algebra over k generated by X and X^* the free monoid generated by X, where the empty word is the identity which is denoted by 1. For a word $w \in X^*$, we denote the length of w by $|w|$.

A well ordering $>$ on X^* is monomial if it is compatible with the multiplication of the words, that is, for $u, v \in X^*$, we have

$$u > v \quad \Rightarrow \quad w_1 u w_2 > w_1 v w_2, \quad \text{for all } w_1, w_2 \in X^*.$$

A standard example of monomial ordering on X^* is the deg-lex ordering to compare two words first by their degrees and then lexicographically, where X is a well-ordered set.

Let $f \in k\langle X\rangle$ with the leading word $\bar{f}$. Then we call f monic if $\bar{f}$ has coefficient 1.

If f and g are two monic polynomials in $k\langle X\rangle$ and $>$ a well ordering on X^*, then there are two kinds of compositions:

(i) If w is a word such that $w = \bar{f}b = a\bar{g}$ for some $a, b \in X^*$ with $|\bar{f}| + |\bar{g}| > |w|$, then the polynomial $(f, g)_w = fb - ag$ is called the intersection composition of f and g with respect to w.
(ii) If $w = \bar{f} = a\bar{g}b$ for some $a, b \in X^*$, then the polynomial $(f, g)_w = f - agb$ is called the inclusion composition of f and g with respect to w.

Let $S \subset k\langle X\rangle$ with each $s \in S$ monic. Then the composition $(f, g)_w$ is called trivial modulo (S, w) if $(f, g)_w = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$ and $\overline{a_i s_i b_i} < w$. If this is the case, then we write

$$(f, g)_w \equiv 0 \quad mod(S, w).$$

In general, for $p, q \in k\langle X\rangle$, we write $p \equiv q\ mod(S, w)$ which means that $p - q \equiv 0\ mod(S, w)$.

We now call the set S a Gröbner–Shirshov basis in $k\langle X\rangle$ with respect to the monomial ordering $>$ if any composition of polynomials in S is trivial modulo S and corresponding w.

Lemma 2.1 *(Composition–Diamond lemma for associative algebras). Let $S \subset k\langle X\rangle$ be a set of monic polynomials and $>$ a monomial ordering on X^*. Then the following statements are equivalent*:

(i) *S is a Gröbner–Shirshov basis in $k\langle X\rangle$.*
(ii) *$f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in X^*$, where $Id(S)$ is the ideal of $k\langle X\rangle$ generated by S.*
(iii) *$Irr(S) = \{u \in X^* \mid u \neq a\bar{s}b, s \in S,\ a, b \in X^*\}$ is a k-basis of the algebra $A = k\langle X|S\rangle$.*

3. Composition–Diamond lemma for tensor product

Let X and Y be two well-ordered sets, $T = \{yx = xy \mid x \in X, y \in Y\}$. With the deg-lex ordering ($y > x$ for any $x \in X$, $y \in Y$) on $(X \cup Y)^*$, T is clearly a Gröbner–Shirshov basis in $k\langle X \cup Y\rangle$. Then, by Lemma 2.1,

$$N = X^*Y^* = Irr(T) = \{u = u^X u^Y \mid u^X \in X^* \text{ and } u^Y \in Y^*\}$$

is the set of normal words of the tensor product

$$k\langle X\rangle \otimes k\langle Y\rangle = k\langle X \cup Y \mid T\rangle.$$

Let kN be a k-space spanned by N. For any $u = u^X u^Y$, $v = v^X v^Y \in N$, we define the multiplication of the normal words as follows

$$uv = u^X v^X u^Y v^Y \in N.$$

It is clear that kN is exactly the tensor product algebra $k\langle X\rangle \otimes k\langle Y\rangle$, that is, $kN = k\langle X \cup Y|T\rangle = k\langle X\rangle \otimes k\langle Y\rangle$.

Now, we order the set N. For any $u = u^X u^Y$, $v = v^X v^Y \in N$,

$$u > v \quad \Leftrightarrow \quad |u| > |v| \quad \text{or} \quad \big(|u| = |v| \quad \text{and} \quad \big(u^X > v^X \quad \text{or} \quad \big(u^X = v^X \text{ and } u^Y > v^Y\big)\big)\big),$$

where $|u| = |u^X| + |u^Y|$ is the length of u. Obviously, $>$ is a monomial ordering on N. Such an ordering is also called the deg-lex ordering on $N = X^*Y^*$. Throughout this paper, we will adopt this ordering unless otherwise stated.

For any polynomial $f \in k\langle X\rangle \otimes k\langle Y\rangle$, f has a unique presentation of the form

$$f = \alpha_{\bar{f}} \bar{f} + \sum \alpha_i u_i,$$

where $\bar{f}$, $u_i \in N$, $\bar{f} > u_i$, $\alpha_{\bar{f}}$, $\alpha_i \in k$.

The proof of the following lemma is straightforward and we hence omit the details.

Lemma 3.1. *Let $f \in k\langle X\rangle \otimes k\langle Y\rangle$ be a monic polynomial. Then $\overline{ufv} = u\bar{f}v$ for any $u, v \in N$.*

We give here the definition of compositions. Let f and g be two monic polynomials of $k\langle X\rangle \otimes k\langle Y\rangle$ and $w = w^X w^Y \in N$. Then we have the following compositions.

1. Inclusion

1.1. X-inclusion only

Suppose that $w^X = \bar{f}^X = a\bar{g}^X b$ for some $a, b \in X^*$, and $\bar{f}^Y$, $\bar{g}^Y$ are disjoint. Then there are two compositions according to $w^Y = \bar{f}^Y c\bar{g}^Y$ and $w^Y = \bar{g}^Y c\bar{f}^Y$ for $c \in Y^*$, respectively:

$$(f,g)_{w_1} = fc\bar{g}^Y - \bar{f}^Y cagb, \quad w_1 = f^X \bar{f}^Y c\bar{g}^Y$$

and

$$(f,g)_{w_2} = \bar{g}^Y cf - agbc\bar{f}^Y, \quad w_2 = f^X \bar{g}^Y c\bar{f}^Y.$$

1.2. Y-inclusion only

Suppose that $w^Y = \bar{f}^Y = c\bar{g}^Y d$ for $c, d \in Y^*$ and $\bar{f}^X$, $\bar{g}^X$ are disjoint. Then there are two compositions according to $w^X = \bar{f}^X a\bar{g}^X$ and $w^X = \bar{g}^X a\bar{f}^X$ for $a \in X^*$, respectively:

$$(f,g)_{w_1} = fa\bar{g}^X - \bar{f}^X acgd, \quad w_1 = \bar{f}^X a\bar{g}^X f^Y$$

and

$$(f,g)_{w_2} = \bar{g}^X af - cgda\bar{f}^X, \quad w_2 = \bar{g}^X a\bar{f}^X f^Y.$$

1.3. X, Y-inclusion

Suppose that $w^X = \bar{f}^X = a\bar{g}^X b$ for some $a, b \in X^*$ and $w^Y = \bar{f}^Y = c\bar{g}^Y d$ for some $c, d \in Y^*$. Then

$$(f, g)_w = f - acgbd.$$

The transformation $f \mapsto (f, g)_w = f - acgbd$ is said to be the *elimination of leading word* (ELW) of g in f.

1.4. X, Y-skew-inclusion

Suppose that $w^X = \bar{f}^X = a\bar{g}^X b$ for some $a, b \in X^*$ and $w^Y = \bar{g}^Y = c\bar{f}^Y d$ for some $c, d \in Y^*$. Then

$$(f, g)_w = cfd - agb.$$

2. Intersection

2.1. X-intersection only

Suppose that $w^X = \bar{f}^X a = b\bar{g}^X$ for some $a, b \in X^*$ with $|\bar{f}^X| + |\bar{g}^X| > |w^X|$, and $\bar{f}^Y$, $\bar{g}^Y$ are disjoint. Then there are two compositions according to $w^Y = \bar{f}^Y c\bar{g}^Y$ and $w^Y = \bar{g}^Y c\bar{f}^Y$ for $c \in Y^*$, respectively:

$$(f, g)_{w_1} = fac\bar{g}^Y - \bar{f}^Y cbg, \quad w_1 = w^X \bar{f}^Y c\bar{g}^Y$$

and

$$(f, g)_{w_2} = \bar{g}^Y cfa - bgc\bar{f}^Y, \quad w_2 = w^X \bar{g}^Y c\bar{f}^Y.$$

2.2. Y-intersection only

Suppose that $w^Y = \bar{f}^Y c = d\bar{g}^X$ for some $c, d \in Y^*$ with $|\bar{f}^Y| + |\bar{g}^Y| > |w^Y|$, and $\bar{f}^X$, $\bar{g}^X$ are disjoint. Then there are two compositions according to $w^X = \bar{f}^X a\bar{g}^X$ and $w^X = \bar{g}^X a\bar{f}^X$ for $a \in X^*$, respectively:

$$(f, g)_{w_1} = fca\bar{g}^X - \bar{f}^X adg, \quad w_1 = \bar{f}^X a\bar{g}^X w^Y$$

and

$$(f, g)_{w_2} = \bar{g}^X afc - dga\bar{f}^X, \quad w_2 = \bar{g}^X a\bar{f}^X w^Y.$$

2.3. X, Y-intersection

If $w^X = \bar{f}^X a = b\bar{g}^X$ for some $a, b \in X^*$ and $w^Y = \bar{f}^Y c = d\bar{g}^Y$ for some $c, d \in Y^*$ together with $|\bar{f}^X| + |\bar{g}^X| > |w^X|$ and $|\bar{f}^Y| + |\bar{g}^Y| > |w^Y|$, then

$$(f, g)_w = fac - bdg.$$

2.4. X, Y-skew-intersection

If $w^X = \bar{f}^X a = b\bar{g}^X$ for some $a, b \in X^*$ and $w^Y = c\bar{f}^Y = \bar{g}^Y d$ for some $c, d \in Y^*$ together with $|\bar{f}^X| + |\bar{g}^X| > |w^X|$ and $|\bar{f}^Y| + |\bar{g}^Y| > |w^Y|$, then

$$(f, g)_w = cfa - bgd.$$

3. Both inclusion and intersection

3.1. X-inclusion and Y-intersection

There are two subcases to consider.

If $w^X = \bar{f}^X = a\bar{g}^X b$ for some $a, b \in X^*$ and $w^Y = \bar{f}^Y c = d\bar{g}^Y$ for some $c, d \in Y^*$ with $|\bar{f}^Y| + |\bar{g}^Y| > |w^Y|$, then

$$(f, g)_w = fc - adgb.$$

If $w^X=\bar{f}^X=a\bar{g}^Xb$ for some $a,b\in X^*$ and $w^Y=c\bar{f}^Y=\bar{g}^Yd$ for some $c,d\in Y^*$ with $|\bar{f}^Y|+|\bar{g}^Y|>|w^Y|$, then

$$(f,g)_w=cf-agbd.$$

3.2. X-intersection and Y-inclusion

There are two subcases to consider.

If $w^X=\bar{f}^Xa=b\bar{g}^X$ for some $a,b\in X^*$ with $|\bar{f}^X|+|\bar{g}^X|>|w^X|$ and $w^Y=\bar{f}^Y=c\bar{g}^Yd$ for some $c,d\in Y^*$, then

$$(f,g)_w=fa-bcgd.$$

If $w^X=\bar{f}^Xa=b\bar{g}^X$ for some $a,b\in X^*$ with $|\bar{f}^X|+|\bar{g}^X|>|w^X|$ and $w^Y=c\bar{f}^Yd=\bar{g}^Y$ for some $c,d\in Y^*$, then

$$(f,g)_w=cfad-bg.$$

From Lemma 3.1, it follows that for any case of compositions

$$\overline{(f,g)_w}<w.$$

If $Y=\emptyset$, then the compositions of f, g are the same in $k\langle X\rangle$ as mentioned in Section 2.

Let S be a monic subset of $k\langle X\rangle\otimes k\langle Y\rangle$ and $f,g\in S$. A composition $(f,g)_w$ is said to be *trivial modulo* (S,w), denoted by

$$(f,g)_w\equiv 0\quad mod(S,w),$$

if $(f,g)_w=\sum_i\alpha_ia_is_ib_i$, where $a_i,b_i\in N$, $s_i\in S$, $\alpha_i\in k$ and $a_i\bar{s}_ib_i<w$ for any i.

In general, for any $p,q\in k\langle X\rangle\otimes k\langle Y\rangle$, we have $p\equiv q\ mod(S,w)$ if $p-q\equiv 0\ mod(S,w)$.

We call S a *Gröbner–Shirshov basis* in $k\langle X\rangle\otimes k\langle Y\rangle$ if all compositions of elements in S are trivial modulo S and corresponding w.

Lemma 3.2. *Let S be a Gröbner–Shirshov basis in $k\langle X\rangle\otimes k\langle Y\rangle$ and $s_1,s_2\in S$. If $w=a_1\bar{s}_1b_1=a_2\bar{s}_2b_2$ for some $a_i,b_i\in N$, $i=1,2$, then $a_1s_1b_1\equiv a_2s_2b_2\ mod(S,w)$.*

Proof. There are four cases to consider.

Case 1. Inclusion

1.1. X-inclusion only

Suppose that $w_1^X=\bar{s}_1^X=a\bar{s}_2^Xb$, $a,b\in X^*$ and $\bar{s}_1^Y$, $\bar{s}_2^Y$ are disjoint. Then $a_2^X=a_1^Xa$ and $b_2^X=bb_1^X$. There are two subcases to consider: $w_1^Y=\bar{s}_1^Yc\bar{s}_2^Y$ and $w_1^Y=\bar{s}_2^Yc\bar{s}_1^Y$, where $c\in Y^*$.

For $w_1^Y=\bar{s}_1^Yc\bar{s}_2^Y$, we have $w_1=s_1^X\bar{s}_1^Yc\bar{s}_2^Y$, $a_2^Y=a_1^Y\bar{s}_1^Yc$, $b_1^Y=c\bar{s}_2^Yb_2^Y$, $w=a_1w_1b_1^Xb_2^Y=a_1\bar{s}_1ac\bar{s}_2b_2$ and hence

$$\begin{aligned}a_1s_1b_1-a_2s_2b_2&=a_1s_1b_1^Xc\bar{s}_2^Yb_2^Y-a_1^Xaa_1^Y\bar{s}_1^Ycs_2bb_1^Xb_2^Y\\&=a_1\big(s_1c\bar{s}_2^Y-\bar{s}_1^Ycas_2b\big)b_1^Xb_2^Y\\&=a_1(s_1,s_2)_{w_1}b_1^Xb_2^Y\\&\equiv 0\quad mod(S,w).\end{aligned}$$

For $w_1^Y=\bar{s}_2^Yc\bar{s}_1^Y$, we have $w_1=s_1^X\bar{s}_2^Yc\bar{s}_1^Y$, $a_1^Y=a_2^Y\bar{s}_2^Yc$, $b_2^Y=c\bar{s}_1^Yb_1^Y$, $w=a_1^Xa_2^Yw_1b_1$ and hence

$$\begin{aligned}
a_1s_1b_1 - a_2s_2b_2 &= a_1^X a_2^Y \bar{s}_2^Y c s_1 b_1 - a_1^X a a_2^Y s_2 b b_1^X c \bar{s}_1^Y b_1^Y \\
&= a_1^X a_2^Y \big(\bar{s}_2^Y c s_1 - a s_2 b c \bar{s}_1^Y\big) b_1 \\
&= a_1^X a_2^Y (s_1, s_2)_{w_1} b_1 \\
&\equiv 0 \quad mod(S, w).
\end{aligned}$$

1.2. Y-inclusion only

This case is similar to 1.1.

1.3. X, Y-inclusion

We may assume that $\bar{s}_2$ is a subword of $\bar{s}_1$, i.e., $w_1 = \bar{s}_1 = ac\bar{s}_2bd$, $a, b \in X^*$, $c, d \in Y^*$, $a_2^X = a_1^X a$, $b_2^X = bb_1^X$, $a_2^Y = a_1^Y c$ and $b_2^Y = db_1^Y$. Thus, $a_2 = a_1ac$, $b_2 = bdb_1$, $w = a_1w_1b_1$ and hence

$$\begin{aligned}
a_1s_1b_1 - a_2s_2b_2 &= a_1s_1b_1 - a_1acs_2bdb_1 \\
&= a_1(s_1 - acs_2bd)b_1 \\
&= a_1(s_1, s_2)_{w_1} b_1 \\
&\equiv 0 \quad mod(S, w).
\end{aligned}$$

1.4. X, Y-skew-inclusion

Assume that $w_1^X = \bar{s}_1^X = a\bar{s}_2^X b$, $a, b \in X^*$ and $w_1^Y = \bar{s}_2^Y = c\bar{s}_1^Y d$, $c, d \in Y^*$. Then $a_2^X = a_1^X a$, $b_2^X = bb_1^X$, $a_1^Y = a_2^Y c$ and $b_1^Y = db_2^Y$. Thus, $w = a_1^X a_2^Y w_1 b_1^X b_2^Y$ and hence

$$\begin{aligned}
a_1s_1b_1 - a_2s_2b_2 &= a_1^X a_2^Y c s_1 b_1^X d b_2^Y - a_1^X a a_2^Y s_2 b b_1^X b_2^Y \\
&= a_1^X a_2^Y (c s_1 d - a s_2 b) b_1^X b_2^Y \\
&= a_1^X a_2^Y (s_1, s_2)_{w_1} b_1^X b_2^Y \\
&\equiv 0 \quad mod(S, w).
\end{aligned}$$

Case 2. Intersection

2.1. X-intersection only

We may assume that $\bar{s}_1^X$ is at the left of $\bar{s}_2^X$, i.e., $w_1^X = \bar{s}_1^X b = a\bar{s}_2^X$, $a, b \in X^*$ and $|\bar{s}_1^X| + |\bar{s}_2^X| > |w_1^X|$. Then $a_2^X = a_1^X a$ and $b_1^X = bb_2^X$. There are two subcases to consider: $w_1^Y = \bar{s}_1^Y c\bar{s}_2^Y$ and $w_1^Y = \bar{s}_2^Y c\bar{s}_1^Y$, $c \in Y^*$.

For $w_1^Y = \bar{s}_1^Y c\bar{s}_2^Y$, i.e., $w_1 = \bar{s}_1 bc\bar{s}_2^Y$, we have $a_2^Y = a_1^Y \bar{s}_1^Y c$, $b_1^Y = c\bar{s}_2^Y b_2^Y$, $w = a_1\bar{s}_1 ac\bar{s}_2 b_2 = a_1w_1b_2$ and

$$\begin{aligned}
a_1s_1b_1 - a_2s_2b_2 &= a_1s_1bb_2^X c\bar{s}_2^Y b_2^Y - a_1^X a a_1^Y \bar{s}_2^Y c s_2 b_2 \\
&= a_1\big(s_1 bc\bar{s}_2^Y - a\bar{s}_2^Y c s_2\big) b_2 \\
&= a_1(s_1, s_2)_{w_1} b_2 \\
&\equiv 0 \quad mod(S, w).
\end{aligned}$$

For $w_1^Y = \bar{s}_2^Y c\bar{s}_1^Y$, i.e., $w_1 = \bar{s}_2^Y c\bar{s}_1 b$, we have $a_1^Y = a_2^Y \bar{s}_2^Y c$, $b_2^Y = c\bar{s}_1^Y b_1^Y$, $w = a_1^X a_2^Y w_1 b_2^X b_1^Y$ and hence

$$\begin{aligned}
a_1s_1b_1 - a_2s_2b_2 &= a_1^X a_2^Y \bar{s}_2^Y c s_1 b b_2^X b_1^Y - a_1^X a a_2^Y s_2 b_2^X c\bar{s}_1^Y b_1^Y \\
&= a_1^X a_2^Y \big(\bar{s}_2^Y c s_1 b - a s_2 c\bar{s}_1^Y\big) b_2^X b_1^Y
\end{aligned}$$

$$= a_1^X a_2^Y (s_1, s_2)_{w_1} b_2^X b_1^Y$$
$$\equiv 0 \quad mod(S, w).$$

2.2. Y-intersection only

This case is similar to 2.1.

2.3. X, Y-intersection

Assume that $w_1^X = \bar{s}_1^X b = a\bar{s}_2^X$, $w_1^Y = \bar{s}_1^Y d = c\bar{s}_2^Y$, $a, b \in X^*$, $c, d \in Y^*$, $|\bar{s}_1^X| + |\bar{s}_2^X| > |w_1^X|$ and $|\bar{s}_1^Y| + |\bar{s}_2^Y| > |w_1^Y|$. Then $a_2^X = a_1^X a$, $b_1^X = bb_2^X$, $a_2^Y = a_1^Y c$, $b_1^Y = db_2^Y$, $w = a_1 w_1 b_2$ and hence

$$\begin{aligned} a_1 s_1 b_1 - a_2 s_2 b_2 &= a_1 s_1 bb_2^X db_2^Y - a_1^X aa_1^Y cs_2 b_2 \\ &= a_1 (s_1 bd - acs_2) b_2 \\ &= a_1 (s_1, s_2)_{w_1} b_2 \\ &\equiv 0 \quad mod(S, w). \end{aligned}$$

2.4. X, Y-skew-intersection

Assume that $w_1^X = \bar{s}_1^X b = a\bar{s}_2^X$, $w_1^Y = c\bar{s}_1^Y = \bar{s}_2^Y d$, $|\bar{s}_1^X| + |\bar{s}_2^X| > |w_1^X|$, $|\bar{s}_1^Y| + |\bar{s}_2^Y| > |w_1^Y|$, $a, b \in X^*$, $c, d \in Y^*$. Then $a_2^X = a_1^X a$, $b_1^X = bb_2^X$, $a_1^Y = a_2^Y c$, $b_2^Y = db_1^Y$, $w = a_1^X a_2^Y w_1 b_2^X b_1^Y$ and hence

$$\begin{aligned} a_1 s_1 b_1 - a_2 s_2 b_2 &= a_1^X a_2^Y cs_1 bb_2^X b_1^Y - a_1^X aa_2^Y s_2 b_2^X db_1^Y \\ &= a_1^X a_2^Y (cs_1 b - as_2 d) b_2^X b_1^Y \\ &= a_1^X a_2^Y (s_1, s_2)_{w_1} b_2^X b_1^Y \\ &\equiv 0 \quad mod(S, w). \end{aligned}$$

Case 3. Both inclusion and intersection

3.1. X-inclusion and Y-intersection

We may assume that $w_1^X = \bar{s}_1^X = a\bar{s}_2^X b$, $a, b \in X^*$. Then $a_2^X = a_1^X a$ and $b_2^X = bb_1^X$. There two cases to consider: $w_1^Y = \bar{s}_1^Y d = c\bar{s}_2^Y$ and $w_1^Y = c\bar{s}_1^Y = \bar{s}_2^Y d$, where $c, d \in Y^*$, $|\bar{s}_1^Y| + |\bar{s}_2^Y| > |w_1^Y|$.

For $w_1^Y = \bar{s}_1^Y d = c\bar{s}_2^Y$, we have $a_2^Y = a_1^Y c$, $b_1^Y = db_2^Y$, $w = a_1 w_1 b_1^X b_2^Y$ and hence

$$\begin{aligned} a_1 s_1 b_1 - a_2 s_2 b_2 &= a_1 s_1 b_1^X db_2^Y - a_1^X aa_1^Y cs_2 bb_1^X b_2^Y \\ &= a_1 (s_1 d - acs_2 b) b_1^X b_2^Y \\ &= a_1 (s_1, s_2)_{w_1} b_2 \\ &\equiv 0 \quad mod(S, w). \end{aligned}$$

For $w_1^Y = c\bar{s}_1^Y = \bar{s}_2^Y d$, we have $a_1^Y = a_2^Y c$, $b_2^Y = db_1^Y$, $w = a_1^X a_2^Y w_1 b_2^X db_1^Y$ and hence

$$\begin{aligned} a_1 s_1 b_1 - a_2 s_2 b_2 &= a_1^X a_2^Y cs_1 b_1 - a_1^X aa_2^Y s_2 bb_1^X db_1^Y \\ &= a_1^X a_2^Y (cs_1 - as_2 bd) b_1 \\ &= a_1^X a_2^Y (s_1, s_2)_{w_1} b_1 \\ &\equiv 0 \quad mod(S, w). \end{aligned}$$

3.2. X-intersection and Y-inclusion

Assume that $w_1^X = \bar{s}_1^X b = a\bar{s}_2^X$, $a, b \in Y^*$ with $|\bar{s}_1^X| + |\bar{s}_2^X| > |w_1^X|$. Then $a_2^X = a_1^X a$, $b_1^X = bb_2^X$. There are two subcases to consider: $w_1^Y = \bar{s}_1^Y = c\bar{s}_2^Y d$ and $\bar{s}_2^Y = c\bar{s}_1^Y d$, where $c, d \in Y^*$.

For $w_1^Y = \bar{s}_1^Y = c\bar{s}_2^Y d$, we have $a_2^Y = a_1^Y c$, $b_2^Y = db_1^Y$, $w = a_1 w_1 b_2^X b_1^Y$ and hence

$$\begin{aligned} a_1 s_1 b_1 - a_2 s_2 b_2 &= a_1 s_1 b b_2^X b_1^Y - a_1 a c s_2 b_2^X d b_1^Y \\ &= a_1(s_1 b - a c s_2 d) b_2^X b_1^Y \\ &= a_1(s_1, s_2)_{w_1} b_2^X b_1^Y \\ &\equiv 0 \quad mod(S, w). \end{aligned}$$

For $w_1^Y = \bar{s}_2^Y = c\bar{s}_1^Y d$, we have $a_1^Y = a_2^Y c$, $b_1^Y = db_2^Y$, $w = a_1^X a_2^Y w_1 b_2$ and hence

$$\begin{aligned} a_1 s_1 b_1 - a_2 s_2 b_2 &= a_1^X a_2^Y c s_1 b b_2^X d b_2^Y - a_1^X a a_2^Y s_2 b_2 \\ &= a_1^X a_2^Y (c s_1 b d - a s_2) b_2 \\ &= a_1^X a_2^Y (s_1, s_2)_{w_1} b_2 \\ &\equiv 0 \quad mod(S, w). \end{aligned}$$

Case 4. $\bar{s}_1$ and $\bar{s}_2$ are disjoint

For $w = w^X w^Y$, by symmetry, there are two subcases to consider: $w^Y = a_1^Y \bar{s}_1^Y c\bar{s}_2^Y b_2^Y$ and $w^Y = a_2^Y \bar{s}_2^Y c\bar{s}_1^Y b_1^Y$, where $w^X = a_1^X \bar{s}_1^X a\bar{s}_2^X b_2^X$, $a \in X^*$, $a_2^X = a_1^X \bar{s}_1^X a$, $b_1^X = a\bar{s}_2^X b_2^X$ and $c \in Y^*$.

For $w = a_1^X \bar{s}_1^X a\bar{s}_2^X b_2^X a_1^Y \bar{s}_1^Y c\bar{s}_2^Y b_2^Y = a_1\bar{s}_1 ac\bar{s}_2 b_2$, we have $a_2 = a_1\bar{s}_1 ac$, $b_1 = ac\bar{s}_2 b_2$ and hence

$$\begin{aligned} a_1 s_1 b_1 - a_2 s_2 b_2 &= a_1 s_1 a c\bar{s}_2 b_2 - a_1\bar{s}_1 a c s_2 b_2 \\ &= a_1(s_1 - \bar{s}_1) a c s_2 b_2 - a_1 s_1 a c(s_2 - \bar{s}_2) b_2 \\ &\equiv 0 \quad mod(S, w). \end{aligned}$$

For $w = a_1^X \bar{s}_1^X a\bar{s}_2^X b_2^X a_2^Y \bar{s}_2^Y c\bar{s}_1^Y b_1^Y$, we have $a_1^Y = a_2^Y \bar{s}_2^Y c$, $b_2^Y = c\bar{s}_1^Y b_1^Y$ and hence

$$\begin{aligned} a_1 s_1 b_1 - a_2 s_2 b_2 &= a_1^X a_2^Y \bar{s}_2^Y c s_1 a\bar{s}_2^X b_2^X b_1^Y - a_1^X \bar{s}_1^X a a_2^Y s_2 b_2^X c\bar{s}_1^Y b_1^Y \\ &= a_1^X a_2^Y \left(\bar{s}_2^Y c s_1 a\bar{s}_2^X - \bar{s}_1^X a s_2 c\bar{s}_1^Y\right) b_2^X b_1^Y. \end{aligned}$$

Let $s_1 = \sum_{i=1}^n \alpha_i u_{1i}^X u_{1i}^Y$ and $s_2 = \sum_{j=1}^m \beta_j u_{2j}^X u_{2j}^Y$, where $\alpha_1 = \beta_1 = 1$. Then

$$\begin{aligned} \bar{s}_2^Y c s_1 a\bar{s}_2^X - \bar{s}_1^X a s_2 c\bar{s}_1^Y &= \sum_{i=2}^n \alpha_i u_{1i}^X a\bar{s}_2 c u_{1i}^Y - \sum_{j=2}^m \beta_i u_{2j}^Y c\bar{s}_1 a u_{2j}^X \\ &= \sum_{i=2}^n \alpha_i u_{1i}^X a(\bar{s}_2 - s_2) c u_{1i}^Y + \sum_{j=2}^m \beta_j u_{2j}^Y c(s_1 - \bar{s}_1) a u_{2j}^X \\ &\quad + \sum_{i=2}^n \alpha_i u_{1i}^X a s_2 c u_{1i}^Y - \sum_{j=2}^m \beta_j u_{2j}^Y c s_1 a u_{2j}^X \end{aligned}$$

$$\equiv \sum_{i=2}^{n}\sum_{j=2}^{m}\alpha_i\beta_j u_{1i}^X a u_{2j}^X u_{2j}^Y c u_{1i}^Y - \sum_{j=2}^{m}\sum_{i=2}^{n}\alpha_i\beta_j u_{2j}^Y c u_{1i}^Y u_{1i}^X a u_{2j}^X$$
$$\equiv 0 \quad mod(S, w_1)$$

where $w_1 = \bar{s}_2^Y c\bar{s}_1 a\bar{s}_2^X = \bar{s}_1^X a\bar{s}_2 c\bar{s}_1^Y$. Since $w = a_1^X a_2^Y w_1 b_2^X b_1^Y$, we have

$$a_1s_1b_1 - a_2s_2b_2 = a_1^X a_2^Y\big(\bar{s}_2^Y cs_1a\bar{s}_2^X - \bar{s}_1^X as_2c\bar{s}_1^Y\big)b_2^X b_1^Y$$
$$\equiv 0 \quad mod(S, w).$$

This completes the proof. □

Lemma 3.3. *Let $S \subset k\langle X\rangle \otimes k\langle Y\rangle$ with each $s \in S$ monic and $Irr(S) = \{w \in N \mid w \neq a\bar{s}b,\ a, b \in N,\ s \in S\}$. Then for any $f \in k\langle X\rangle \otimes k\langle Y\rangle$,*

$$f = \sum_{a_i\bar{s}_ib_i \leqslant \bar{f}} \alpha_i a_i s_i b_i + \sum_{u_j \leqslant \bar{f}} \beta_j u_j$$

where each $\alpha_i, \beta_j \in k$, $a_i, b_i \in N$, $s_i \in S$ and $u_j \in Irr(S)$.

Proof. Let $f = \sum_i \alpha_i u_i \in k\langle X\rangle \otimes k\langle Y\rangle$, where $0 \neq \alpha_i \in k$ and $u_1 > u_2 > \cdots$. If $u_1 \in Irr(S)$, then let $f_1 = f - \alpha_1 u_1$. If $u_1 \notin Irr(S)$, then there exist some $s \in S$ and $a_1, b_1 \in N$ such that $\bar{f} = a_1\bar{s}_1b_1$. Let $f_1 = f - \alpha_1 a_1 s_1 b_1$. In both cases, we have $\bar{f}_1 < \bar{f}$. Then the result follows from induction on $\bar{f}$. □

By summing up the above lemmas, we arrive at the following theorem.

Theorem 3.4 *(Composition–Diamond lemma for tensor product $k\langle X\rangle \otimes k\langle Y\rangle$). Let $S \subset k\langle X\rangle \otimes k\langle Y\rangle$ with each $s \in S$ monic and $<$ the ordering on $N = X^*Y^*$ as before. Then the following statements are equivalent*:

(i) *S is a Gröbner–Shirshov basis in $k\langle X\rangle \otimes k\langle Y\rangle$.*
(ii) *$f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$ for some $a, b \in N$, $s \in S$.*
(iii) *$Irr(S) = \{w \in N \mid w \neq a\bar{s}b, a, b \in N, s \in S\}$ is a k-linear basis for the factor algebra $k\langle X\rangle \otimes k\langle Y\rangle/Id(S)$.*

Proof. (i) ⇒ (ii). Suppose that $0 \neq f \in Id(S)$. Then $f = \sum \alpha_i a_i s_i b_i$ for some $\alpha_i \in k$, $a_i, b_i \in N$, $s_i \in S$. Let $w_i = a_i\bar{s}_ib_i$ and $w_1 = w_2 = \cdots = w_l > w_{l+1} \geqslant \cdots$. We will prove that $\bar{f} = a\bar{s}b$ for some $a, b \in N$, $s \in S$, by using induction on l and w_1. If $l = 1$, then the result is clear. If $l > 1$, then $w_1 = a_1\bar{s}_1b_1 = a_2\bar{s}_2b_2$. Now, by (i) and Lemma 3.2, we see that $a_1s_1b_1 \equiv a_2s_2b_2\ mod(S, w_1)$. Thus,

$$\alpha_1a_1s_1b_1 + \alpha_2a_2s_2b_2 = (\alpha_1 + \alpha_2)a_1s_1b_1 + \alpha_2(a_2s_2b_2 - a_1s_1b_1)$$
$$\equiv (\alpha_1 + \alpha_2)a_1s_1b_1 \quad mod(S, w_1).$$

By induction on l and w_1, we obtain the desired result.

(ii) ⇒ (iii). For any $0 \neq f \in k\langle X\rangle \otimes k\langle Y\rangle$, by Lemma 3.3, we can express f as

$$f = \sum \alpha_i a_i s_i b_i + \sum \beta_j u_j,$$

where $\alpha_i, \beta_j \in k$, $a_i, b_i \in N$, $s_i \in S$ and $u_j \in Irr(S)$. Then $Irr(S)$ generates the factor algebra. Moreover, if $0 \neq h = \sum \beta_j u_j \in Id(S)$, $u_j \in Irr(S)$, $u_1 > u_2 > \cdots$ and $\beta_1 \neq 0$, then $u_1 = \bar{h} = a\bar{s}b$ for some $a, b \in N$, $s \in S$ by (ii), a contradiction. This shows that $Irr(S)$ is a linear basis of the factor algebra.

(iii) $\Rightarrow$ (i). For any $f, g \in S$, we have $h = (f, g)_w \in Id(S)$. The result is now trivial if $(f, g)_w = 0$. Assume that $(f, g)_w \neq 0$. Then, by Lemma 3.3 and (iii), we have

$$h = \sum_{a_i \bar{s}_i b_i \leqslant \bar{h}} \alpha_i a_i s_i b_i.$$

Now, by noting that $\bar{h} = \overline{(f, g)_w} < w$, we see immediately that (i) holds. $\square$

Remark. Theorem 3.4 is valid for any monomial ordering on X^*Y^*.

Remark. Theorem 3.4 is precisely the Composition–Diamond lemma for associative algebras (Lemma 2.1) when $Y = \emptyset$.

4. Applications

Now, we give some applications of Theorem 3.4.

Example 4.1. Suppose that for the deg-lex ordering, S_1 and S_2 are Gröbner–Shirshov bases in $k\langle X\rangle$ and $k\langle Y\rangle$ respectively. Then for the deg-lex ordering on X^*Y^* as before, $S_1 \cup S_2$ is a Gröbner–Shirshov basis in $k\langle X \cup Y | T\rangle = k\langle X\rangle \otimes k\langle Y\rangle$. It follows that $k\langle X|S_1\rangle \otimes k\langle Y|S_2\rangle = k\langle X \cup Y | T \cup S_1 \cup S_2\rangle$.

Proof. The possible compositions in $S_1 \cup S_2$ are X-including only, X-intersection only, Y-including only and Y-intersection only. Suppose that $f, g \in S_1$ and $(f, g)_{w_1} \equiv 0 \ mod(S_1, w_1)$ in $k\langle X\rangle$. Then in $k\langle X\rangle \otimes k\langle Y\rangle$, $(f, g)_w = (f, g)_{w_1} c$, where $w = w_1 c$ for any $c \in Y^*$. From this it follows that each composition in $S_1 \cup S_2$ is trivial modulo $S_1 \cup S_2$. $\square$

A special case of Example 4.1 is the following:

Example 4.2. Let X, Y be well-ordered sets, $k[X]$ the free commutative associative algebra generated by X. Then $S = \{x_i x_j = x_j x_i \mid x_i > x_j, x_i, x_j \in X\}$ is a Gröbner–Shirshov basis in $k\langle X\rangle \otimes k\langle Y\rangle$ with respect to the deg-lex ordering. Therefore, $k[X] \otimes k\langle Y\rangle = k\langle X \cup Y | T \cup S\rangle$.

In [14], a Gröbner–Shirshov basis in $k\langle X\rangle$ is constructed by lifting a commutative Gröbner basis and adding some commutators. Let $X = \{x_1, x_2, \ldots, x_n\}$, let $[X]$ be the free commutative monoid generated by X and $k[X]$ the polynomial ring. Let

$$S_1 = \{h_{ij} = x_i x_j - x_j x_i \mid i > j\} \subset k\langle X\rangle.$$

Then, consider the natural map $\gamma : k\langle X\rangle \to k[X]$ which maps x_i to x_i and the *lexicographic splitting* of γ, which is defined as the k-linear map

$$\delta : k[X] \to k\langle X\rangle, \qquad x_{i_1} x_{i_2} \cdots x_{i_r} \mapsto x_{i_1} x_{i_2} \cdots x_{i_r} \quad \text{if } i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r.$$

For any $u \in [X]$, we present $u = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$, where $l_i \geqslant 0$.

We use any monomial ordering on $[X]$.

Following [14], we define an ordering on X^* using the ordering $x_1 < x_2 < \cdots < x_n$ as follows: for any $u, v \in X^*$,

$$u > v \quad \Leftrightarrow \quad \gamma(u) > \gamma(v) \quad \text{in } [X] \quad \text{or} \quad \big(\gamma(u) = \gamma(v) \quad \text{and} \quad u >_{lex} v\big).$$

It is easy to check that this ordering is monomial on X^* and $\overline{\delta(s)} = \delta(\bar{s})$ for any $s \in k[X]$. Moreover, for any $v \in \gamma^{-1}(u)$, $v \geqslant \delta(u)$.

For any $m = x_{i_1}x_{i_2}\cdots x_{i_r} \in [X]$, $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r$, denote the set of all the monomials $u \in [x_{i_1+1}, \ldots, x_{i_r-1}]$ by $U(m)$.

The proofs of the following lemmas are straightforward.

Lemma 4.3. *Let $a, b \in X^*$, $a = \delta(\gamma(a))$, $b = \delta(\gamma(b))$ and $s \in k[X]$. If $w = a\delta(\bar{s})b = \delta(\gamma(ab)\bar{s})$, then, in $k\langle X\rangle$,*

$$a\delta(s)b \equiv \delta\big(\gamma(ab)s\big) \quad mod(S_1, w).$$

Proof. Suppose that $s = \bar{s} + s'$ and $h = a\delta(s)b - \delta(\gamma(ab)s)$. Since $a\delta(\bar{s})b = \delta(\gamma(ab)\bar{s})$, we have $h = a\delta(s')b - \delta(\gamma(ab)s')$, and $\bar{h} < w$. By noting that $\gamma(a\delta(s')b) = \gamma(\delta(\gamma(ab)s'))$, $h \equiv 0 \ mod(S_1, w)$. □

Lemma 4.4. *Let $f, g \in k[X]$, $\bar{g} = x_{i_1}x_{i_2}\cdots x_{i_r}$ $(i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r)$ and $w = \delta(\bar{f}\bar{g})$. Then, in $k\langle X\rangle$,*

$$\delta\big((f - \bar{f})g\big) \equiv \sum \alpha_i a_i \delta(u_i g) b_i \quad mod(S_1, w)$$

where $\alpha_i \in k$, $a_i \in [x \in X \mid x \leqslant x_{i_1}]$, $b_i \in [x \in X \mid x \geqslant x_{i_r}]$, $u_i \in U(\bar{g})$ and $\gamma(\sum \alpha_i a_i u_i b_i) = f - \bar{f}$.

Theorem 4.5. *(See [14].) Let the orderings on $[X]$ and X^* be defined as above. If S is a minimal Gröbner basis in $k[X]$, then $S' = \{\delta(us) \mid s \in S, u \in U(\bar{s})\} \cup S_1$ is a Gröbner–Shirshov basis in $k\langle X\rangle$.*

Proof. We will show that all the possible compositions of elements in S' are trivial. Let $f = \delta(us_1)$, $g = \delta(vs_2)$ and $h_{ij} = x_i x_j - x_j x_i \in S'$.

(i) $f \wedge g$

Case 1. f and g have a composition of including, i.e., $w = \delta(u\bar{s}_1) = a\delta(v\bar{s}_2)b$ for some $a, b \in X^*$ and $a = \delta(\gamma(a))$, $b = \delta(\gamma(b))$.

If s_1 and s_2 have no composition in $k[X]$, i.e., $lcm(\bar{s}_1\bar{s}_2) = \bar{s}_1\bar{s}_2$, then $u = u'\bar{s}_2$, $\gamma(ab)v = u'\bar{s}_1$ for some $u' \in [X]$. By Lemmas 4.3 and 4.4, we have

$$\begin{aligned}
(f, g)_w &= \delta(us_1) - a\delta(vs_2)b \\
&\equiv \delta(us_1) - \delta\big(\gamma(ab)vs_2\big) \\
&\equiv \delta\big(u'\bar{s}_2 s_1\big) - \delta\big(u'\bar{s}_1 s_2\big) \\
&\equiv \delta\big(u'(s_1 - \bar{s}_1)s_2\big) - \delta\big(u'(s_2 - \bar{s}_2)s_1\big) \\
&\equiv 0 \quad mod\big(S', w\big).
\end{aligned}$$

Since, in $k[X]$, S is a minimal Gröbner basis, the possible compositions are only intersection. If s_1 and s_2 have composition of intersection in $k[X]$, i.e., $(s_1, s_2)_{w'} = a's_1 - b's_2$, where $a', b' \in [X]$, $w' = a'\bar{s}_1 = b'\bar{s}_2$ and $|w'| < |\bar{s}_1| + |\bar{s}_2|$, then w' is a subword of $\gamma(w)$. Hence, we deduce that $w = \delta(tw') = \delta(ta'\bar{s}_1) = \delta(tb'\bar{s}_2)$ and $u = ta'$, $\gamma(ab)v = tb'$ for some $t \in [X]$. Then

$$\begin{aligned}
(f, g)_w &= \delta(us_1) - a\delta(vs_2)b \\
&\equiv \delta(us_1) - \delta\big(\gamma(ab)vs_2\big) \\
&\equiv \delta\big(ta's_1\big) - \delta\big(tb's_2\big) \\
&\equiv \delta\big(t\big(a's_1 - b's_2\big)\big)
\end{aligned}$$

$$\begin{aligned} &\equiv \delta\big(t(s_1, s_2)_{w'}\big) \\ &\equiv 0 \quad mod\big(S', w\big) \end{aligned}$$

since $t\overline{(s_1, s_2)_{w'}} < tw' = \gamma(w)$.

Case 2. If f and g have a composition of intersection, we may assume that $\bar{f}$ is on the left of $\bar{g}$, i.e., $w = \delta(u\bar{s}_1)a = b\delta(v\bar{s}_2)$ for some $a, b \in X^*$ and $a = \delta\gamma(a)$, $b = \delta\gamma(b)$. Similarly to Case 1, we have to consider whether s_1 and s_2 have compositions in $k[X]$ or not. One can check that both cases are trivial $mod(S', w)$ by Lemmas 4.3 and 4.4.

(ii) $f \wedge h_{ij}$

By noting that $\overline{h_{ij}} = x_i x_j$ cannot be a subword of $\bar{f} = \delta(u\bar{s}_1)$ since $i > j$, only possible compositions are intersection. Suppose that $\bar{s}_1 = x_{i_1} \cdots x_{i_r} x_i$ $(i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r \leqslant i)$. Then $\bar{f} = \delta(u\bar{s}_1) = x_{i_1} v x_i$ for some $v \in k\langle X\rangle$, $v = \delta\gamma(v)$ and $w = \delta(u\bar{s}_1)x_j$.

If $j \leqslant i_1$, then

$$\begin{aligned} (f, h_{ij})_w &= \delta(us_1)x_j - x_{i_1} v(x_i x_j - x_j x_i) \\ &= \delta\big(u(s_1 - \bar{s}_1)\big)x_j + x_{i_1} v x_j x_i \\ &\equiv x_j \delta\big(u(s_1 - \bar{s}_1)\big) + x_j x_{i_1} v x_i \\ &\equiv x_j\big(\delta\big(u(s_1 - \bar{s}_1)\big) + \delta(u\bar{s}_1)\big) \\ &\equiv x_j \delta(us_1) \\ &\equiv 0 \quad mod\big(S', w\big). \end{aligned}$$

If $j > i_1$, then $ux_j \in U(\bar{s}_1)$ and

$$\begin{aligned} (f, h_{ij})_w &= \delta(us_1)x_j - x_{i_1} v(x_i x_j - x_j x_i) \\ &= \delta\big(u(s_1 - \bar{s}_1)\big)x_j + x_{i_1} v x_j x_i \\ &\equiv \delta\big(ux_j(s_1 - \bar{s}_1)\big) + \delta(x_{i_1} v x_i x_j) \\ &\equiv \delta\big(ux_j(s_1 - \bar{s}_1)\big) + \delta(ux_j\bar{s}_1) \\ &\equiv \delta(ux_j s_1) \\ &\equiv 0 \quad mod\big(S', w\big). \end{aligned}$$

Thus, the proof is completed. □

Now we extend γ and δ as follows:

$$\begin{aligned} \gamma \otimes \mathbf{1}:\quad & k\langle X\rangle \otimes k\langle Y\rangle \to k[X] \otimes k\langle Y\rangle, \qquad u^X u^Y \mapsto \gamma\big(u^X\big)u^Y, \\ \delta \otimes \mathbf{1}:\quad & k[X] \otimes k\langle Y\rangle \to k\langle X\rangle \otimes k\langle Y\rangle, \qquad u^X u^Y \mapsto \delta\big(u^X\big)u^Y. \end{aligned}$$

Any polynomial $f \in k[X] \otimes k\langle Y\rangle$ has a presentation $f = \sum \alpha_i u_i^X u_i^Y$, where $\alpha_i \in k$, $u_i^X \in [X]$ and $u_i^Y \in Y^*$.

Let the orderings on $[X]$ and Y^* be any monomial orderings respectively. We order the set $[X]Y^* = \{u = u^X u^Y \mid u^X \in [X], u^Y \in Y^*\}$ as follows. For any $u, v \in [X]Y^*$,

$$u > v \quad \Leftrightarrow \quad u^Y > v^Y \quad \text{or} \quad \big(u^Y = v^Y \quad \text{and} \quad u^X > v^X\big).$$

Now, we order X^*Y^*: for any $u, v \in X^*Y^*$,

$$u > v \quad \Leftrightarrow \quad \gamma\big(u^X\big)u^Y > \gamma\big(v^X\big)v^Y \quad \text{or} \quad \big(\gamma\big(u^X\big)u^Y = \gamma\big(v^X\big)v^Y \quad \text{and} \quad u^X >_{lex} v^X\big).$$

This ordering is clearly a monomial ordering on X^*Y^*.

The following definitions of compositions and the Gröbner–Shirshov basis are taken from [27].

Let f, g be monic polynomials of $k[X] \otimes k\langle Y\rangle$, L the least common multiple of $\bar{f}^X$ and $\bar{g}^X$.

1. Inclusion

Let $\bar{g}^Y$ be a subword of $\bar{f}^Y$, say, $\bar{f}^Y = c\bar{g}^Y d$ for some $c, d \in Y^*$. If $\bar{f}^Y = \bar{g}^Y$ then $\bar{f}^X \geqslant \bar{g}^X$ and if $\bar{g}^Y = 1$ then we set $c = 1$. Let $w = L\bar{f}^Y = Lc\bar{g}^Y d$. We define the composition

$$C_1(f, g, c)_w = \frac{L}{\bar{f}^X} f - \frac{L}{\bar{g}^X} cgd.$$

2. Overlap

Let a non-empty beginning of $\bar{g}^Y$ be a non-empty ending of $\bar{f}^Y$, say, $\bar{f}^Y = cc_0$, $\bar{g}^Y = c_0 d$, $\bar{f}^Y d = c\bar{g}^Y$ for some $c, d, c_0 \in Y^*$ and $c_0 \neq 1$. Let $w = L\bar{f}^Y d = Lc\bar{g}^Y$. We define the composition

$$C_2(f, g, c_0)_w = \frac{L}{\bar{f}^X} fd - \frac{L}{\bar{g}^X} cg.$$

3. External

Let $c_0 \in Y^*$ be any associative word (possibly empty). In the case that the greatest common divisor of $\bar{f}^X$ and $\bar{g}^X$ is non-empty and $\bar{f}^Y$, $\bar{g}^Y$ are non-empty, we define the composition

$$C_3(f, g, c_0)_w = \frac{L}{\bar{f}^X} fc_0\bar{g}^Y - \frac{L}{\bar{g}^X} \bar{f}^Y c_0 g$$

where $w = L\bar{f}^Y c_0 \bar{g}^Y$.

Let S be a monic subset of $k[X] \otimes k\langle Y\rangle$. Then S is called a Gröbner–Shirshov basis (standard basis) in $k[X] \otimes k\langle Y\rangle$ if for any element $f \in Id(S)$, $\bar{f}$ contains $\bar{s}$ as its subword for some $s \in S$.

It is defined as usual that a composition is trivial modulo S and corresponding w. We also have that S is a Gröbner–Shirshov basis in $k[X] \otimes k\langle Y\rangle$ if and only if all the possible compositions of its elements are trivial. A Gröbner–Shirshov basis in $k[X] \otimes k\langle Y\rangle$ is called minimal if for any $s \in S$ and all $s_i \in S \setminus \{s\}$, $\bar{s}_i$ is not a subword of $\bar{s}$.

Similar to the proof of Theorem 4.5, we have the following theorem.

Theorem 4.6. *Let the orderings on $[X]Y^*$ and X^*Y^* be defined as before. If S is a minimal Gröbner–Shirshov basis in $k[X] \otimes k\langle Y\rangle$, then $S' = \{\delta \otimes 1(us) \mid s \in S, u \in U(\bar{s}^X)\} \cup S_1$ is a Gröbner–Shirshov basis in $k\langle X\rangle \otimes k\langle Y\rangle$, where $S_1 = \{h_{ij} = x_i x_j - x_j x_i \mid i > j\}$.*

Proof. We will show that all the possible compositions of elements in S' are trivial.

For $s_1, s_2 \in S$, let $f = \delta \otimes 1(us_1)$, $g = \delta \otimes 1(vs_2)$, $h_{ij} = x_i x_j - x_j x_i \in S'$ and $L = lcm(\bar{s}_1^X, \bar{s}_2^X)$.

1. $f \wedge g$

In this case, all the possible compositions are related to the ambiguities w's (in the following, $a, b \in X^*$, $c, d \in Y^*$).

1.1. X-inclusion only

$$w^X = \delta\big(u\bar{s_1}^X\big) = a\delta\big(v\bar{s}_2^X\big)b, \qquad w^Y = \bar{s}_1^Y c\bar{s}_2^Y \quad \text{or} \quad w^Y = \bar{s}_2^Y c\bar{s}_1^Y.$$

1.2. Y-inclusion only

$$w^X = \delta\big(u\bar{s}_1^X\big)a\delta\big(v\bar{s}_2^X\big) \quad \text{or} \quad w^X = \delta\big(v\bar{s}_2^X\big)a\delta\big(u\bar{s}_1^X\big), \qquad w^Y = \bar{s}_1^Y = c\bar{s}_2^Y d.$$

1.3. X, Y-inclusion

$$w = \delta \otimes 1(u\bar{s}_1) = ac\delta \otimes 1(v\bar{s}_2)bd.$$

1.4. X, Y-skew-inclusion

$$w^X = \delta(u\bar{s}_1^X) = a\delta(v\bar{s}_2^X)b, \qquad w^Y = \bar{s}_2^Y = c\bar{s}_1^Y d.$$

2.1. X-intersection only

$$w^X = \delta\big(u\bar{s}_1^X\big)a = b\delta\big(v\bar{s}_2^X\big), \qquad w^Y = \bar{s}_1^Y c\bar{s}_2^Y \quad \text{or} \quad w^Y = \bar{s}_2^Y c\bar{s}_1^Y.$$

2.2. Y-intersection only

$$w^X = \delta\big(u\bar{s}_1^X\big)a\delta\big(v\bar{s}_2^X\big) \quad \text{or} \quad w^X = \delta(u\bar{s}_2^X)a\delta(v\bar{s}_1^X), \qquad w^Y = \bar{s}_1^Y c = d\bar{s}_2^Y.$$

2.3. X, Y-intersection

$$w^X = \delta(u\bar{s}_1^X)a = b\delta(v\bar{s}_2^X), \qquad w^Y = \bar{s}_1^Y c = d\bar{s}_2^Y.$$

2.4. X, Y-skew-intersection

$$w^X = \delta\big(u\bar{s}_1^X\big)a = b\delta\big(v\bar{s}_2^X\big), \qquad w^Y = c\bar{s}_1^Y = \bar{s}_2^Y d.$$

3.1. X-inclusion and Y-intersection

$$w^X = \delta\big(u\bar{s}_1^X\big) = a\delta\big(v\bar{s}_2^X\big)b, \qquad w^Y = \bar{s}_1^Y c = d\bar{s}_2^Y \quad \text{or} \quad w^Y = c\bar{s}_1^Y = \bar{s}_2^Y d.$$

3.2. X-intersection and Y-inclusion

$$w^X = \delta\big(u\bar{s}_1^X\big)a = b\delta\big(v\bar{s}_2^X\big), \qquad w^Y = \bar{s}_1^Y = c\bar{s}_2^Y d \quad \text{or} \quad w^Y = \bar{s}_2^Y = c\bar{s}_1^Y d.$$

We only check the cases 1.1, 1.2 and 1.3. Other cases are similarly checked.

1.1. X-inclusion only

Suppose that $w^X = \delta(u\bar{s}_1^X) = a\delta(v\bar{s}_2^X)b$, $a, b \in X^*$ and $\bar{s}_1^Y$, $\bar{s}_2^Y$ are disjoint. There are two cases to consider: $w^Y = \bar{s}_1^Y c\bar{s}_2^Y$ and $w^Y = \bar{s}_2^Y c\bar{s}_1^Y$, where $c \in Y^*$. We will only prove the first case and the second one are similar.

If s_1 and s_2 have no composition in $k[X] \otimes k\langle Y\rangle$, i.e., $lcm(\bar{s}_1, \bar{s}_2) = \bar{s}_1\bar{s}_2$, then $u = u'\bar{s}_2^X$, $\gamma(ab)v = u'\bar{s}_1^X$ for some $u' \in [X]$. By the proof of Theorem 4.5, we have

$$\begin{aligned}
(f, g)_w &= \delta \otimes 1(us_1)c\bar{s}_2^Y - \bar{s}_1^Y ca\delta \otimes 1(vs_2)b \\
&\equiv \delta \otimes 1\big(us_1\gamma\big(c\bar{s}_2^Y\big)\big) - \delta \otimes 1\big(\gamma\big(\bar{s}_1^Y c\big)\gamma(ab)vs_2\big) \\
&\equiv \delta \otimes 1\big(u'\bar{s}_2^X s_1 c\bar{s}_2^Y\big) - \delta \otimes 1\big(\bar{s}_1^Y cu'\bar{s}_1^X s_2\big) \\
&\equiv \delta \otimes 1\big(u's_1 c\bar{s}_2\big) - \delta \otimes 1\big(u'\bar{s}_1 cs_2\big) \\
&\equiv \delta \otimes 1\big(u'(s_1 - \bar{s}_1)cs_2\big) - \delta \otimes 1\big(u's_1 c(s_2 - \bar{s}_2)\big) \\
&\equiv 0 \quad mod\big(S', w\big).
\end{aligned}$$

If s_1 and s_2 have composition of external (the elements of S have no composition of inclusion because S is minimal and s_1 and s_2 have no composition of overlap because s_1^Y and s_2^Y are disjoint) in $k[X]\otimes k\langle Y\rangle$, i.e., $C_3(s_1,s_2,c)_{w'}=\frac{L}{\bar{s}_1^X}s_1\gamma(c\bar{s}_2^Y)-\frac{L}{\bar{s}_2^X}\gamma(\bar{s}_1^Y c)s_2=t_2s_1\gamma(c\bar{s}_2^Y)-t_1\gamma(\bar{s}_1^Y c)s_2$ where $gcd(\bar{s}_1^X,\bar{s}_2^X)=t\neq 1$, $\bar{s}_1^X=tt_1$, $\bar{s}_2^X=tt_2$ and $L=tt_1t_2$, $w'=L\gamma(\bar{s}_1^Y c\bar{s}_2^Y)$, then w' is a subword of $\gamma(w)$. Therefore, we have $w=\delta\otimes 1(mw')$ and $u=mt_2$, $\gamma(ab)v=mt_1$ since $ut_1=\gamma(ab)vt_2$ and $gcd(t_1,t_2)=1$. Then

$$\begin{aligned}(f,g)_w&=\delta\otimes 1(us_1)c\bar{s}_2^Y-\bar{s}_1^Y ca\delta\otimes 1(vs_2)b\\&\equiv\delta\otimes 1\bigl(us_1\gamma\bigl(c\bar{s}_2^Y\bigr)\bigr)-\delta\otimes 1\bigl(\gamma\bigl(\bar{s}_1^Y c\bigr)\gamma(ab)vs_2\bigr)\\&\equiv\delta\otimes 1\bigl(mt_2s_1\gamma\bigl(c\bar{s}_2^Y\bigr)\bigr)-\delta\otimes 1\bigl(mt_1\gamma\bigl(\bar{s}_1^Y c\bigr)s_2\bigr)\\&\equiv\delta\otimes 1\bigl(mC_3(s_1,s_2,c)_{w'}\bigr)\\&\equiv 0\quad mod\bigl(S',w\bigr)\end{aligned}$$

since $m\overline{C_3(s_1,s_2,c)_{w'}}<mw'=\gamma(w)$.

1.2. Y-inclusion only

Suppose that $w^Y=\bar{s}_1^Y=c\bar{s}_2^Y d$, $c,d\in Y^*$ and $\delta(u\bar{s}_1^X)$, $\delta(v\bar{s}_2^X)$ are disjoint. Then there are two compositions according to $w^X=\delta(u\bar{s}_1^X)a\delta(v\bar{s}_2^X)$ and $w^X=\delta(v\bar{s}_2^X)a\delta(u\bar{s}_1^X)$ for $a\in X^*$. We only prove the first.

$$\begin{aligned}(f,g)_w&=\delta\otimes 1(us_1)a\delta\bigl(v\bar{s}_2^X\bigr)-\delta\bigl(u\bar{s}_1^X\bigr)ac\delta\otimes 1(vs_2)d\\&\equiv\delta\otimes 1\bigl(us_1\gamma(a)v\bar{s}_2^X-u\bar{s}_1^X\gamma(a)v\gamma(c)s_2\gamma(d)\bigr)\\&\equiv\delta\otimes 1\bigl(u\gamma(a)v\bigl(s_1\bar{s}_2^X-\bar{s}_1^X\gamma(c)s_2\gamma(d)\bigr)\bigr)\\&\equiv\delta\otimes 1\bigl(u\gamma(a)vC_1\bigl(s_1,s_2,\gamma(c)\bigr)_{w'}\bigr)\\&\equiv 0\quad mod\bigl(S',w\bigr)\end{aligned}$$

where $w'=\bar{s}_1^X\bar{s}_2^X\bar{s}_1^Y=\bar{s}_1^X\bar{s}_2^X\gamma(c)\bar{s}_2^Y\gamma(d)$ and $\overline{u\gamma(a)vC_1(s_1,s_2,\gamma(c))_{w'}}<u\gamma(a)vw'=\gamma(w)$.

1.3. X,Y-inclusion

We may assume that $\bar{g}$ is a subword of $\bar{f}$, i.e., $w=\delta\otimes 1(u\bar{s}_1)=ac\delta\otimes 1(v\bar{s}_2)bd$, $a,b\in X^*$, $c,d\in Y^*$. Then $u\bar{s}_1^X=\gamma(ab)v\bar{s}_2^X=mL$ for some $m\in[X]$, $u\bar{s}_1^Y=\gamma(c)\bar{s}_2^Y\gamma(d)$

$$\begin{aligned}(f,g)_w&=\delta\otimes 1(us_1)-ac\delta\otimes 1(vs_2)bd\\&\equiv\delta\otimes 1\bigl(us_1-\gamma(ac)vs_2\gamma(bd)\bigr)\\&\equiv\delta\otimes 1\left(m\frac{L}{\bar{s}_1^X}s_1-m\frac{L}{\bar{s}_2^X}\gamma(c)s_2\gamma(d)\right)\\&\equiv\delta\otimes 1\bigl(mC_1\bigl(s_1,s_2,\gamma(c)\bigr)_{w'}\bigr)\\&\equiv 0\quad mod\bigl(S',w\bigr)\end{aligned}$$

where $w'=L\gamma(c)\bar{s}_2^Y\gamma(d)$ and $\overline{mC_1(s_1,s_2,c)_{w'}}<mw'=\gamma(w)$.

2. $f\wedge h_{ij}$

Similar to the proof of Theorem 4.5, they only have compositions of X-intersection. Suppose that $\bar{s}_1^X=x_{i_1}\cdots x_{i_r}x_i$ $(i_1\leqslant i_2\leqslant\cdots\leqslant i_r\leqslant i)$. Then $\bar{f}=\delta\otimes 1(u\bar{s}_1)=x_{i_1}vx_i\bar{s}_1^Y$ for some $v\in k\langle X\rangle$, and $v=\delta\gamma(v)$ and $w=\delta\otimes 1(u\bar{s}_1)x_j\bar{s}_1^Y$.

If $j \leqslant i_1$, then

$$\begin{aligned}(f, h_{ij})_w &= \delta \otimes 1(us_1)x_j - x_{i_1} v\bar{s}_1^Y (x_i x_j - x_j x_i) \\ &= \delta \otimes 1\big(u(s_1 - \bar{s}_1)\big)x_j + x_{i_1} v x_j x_i \bar{s}_1^Y \\ &\equiv x_j \delta \otimes 1\big(u(s_1 - \bar{s}_1)\big) + x_j x_{i_1} v x_i \bar{s}_1^Y \\ &\equiv x_j\big(\delta \otimes 1\big(u(s_1 - \bar{s}_1)\big) + \delta \otimes 1(u\bar{s}_1)\big) \\ &\equiv x_j \delta \otimes 1(us_1) \\ &\equiv 0 \quad mod\big(S', w\big).\end{aligned}$$

If $j > i_1$, then $ux_j \in U(\bar{s}_1)$ and

$$\begin{aligned}(f, h_{ij})_w &= \delta \otimes 1(us_1)x_j - x_{i_1} v\bar{s}_1^Y (x_i x_j - x_j x_i) \\ &= \delta \otimes 1\big(u(s_1 - \bar{s}_1)\big)x_j + x_{i_1} v x_j x_i \bar{s}_1^Y \\ &\equiv \delta \otimes 1\big(ux_j(s_1 - \bar{s}_1)\big) + \delta \otimes 1\big(x_{i_1} v x_i x_j \bar{s}_1^Y\big) \\ &\equiv \delta \otimes 1\big(ux_j(s_1 - \bar{s}_1)\big) + \delta \otimes 1(ux_j\bar{s}_1) \\ &\equiv \delta \otimes 1(ux_j s_1) \\ &\equiv 0 \quad mod\big(S', w\big).\end{aligned}$$

This completes the proof. □

As an application of the above result, we have now constructed a Gröbner–Shirshov basis for the tensor product $k\langle X\rangle \otimes k\langle Y\rangle$ by lifting a given Gröbner–Shirshov basis in the tensor product $k[X] \otimes k\langle Y\rangle$.

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