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Nonvanishing cohomology and classes of Gorenstein rings[☆]

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Abstract

We give counterexamples to the following conjecture of Auslander: given a finitely generated module M over an Artin algebra Λ, there exists a positive integer n_M such that for all finitely generated Λ-modules N, if $\mathrm{Ext}^i_\Lambda(M,N)=0$ for all $i \gg 0$, then $\mathrm{Ext}^i_\Lambda(M,N)=0$ for all $i \geqslant n_M$. Some of our examples moreover yield homologically defined classes of commutative local rings strictly between the class of local complete intersections and the class of local Gorenstein rings.

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0. Introduction

In this paper we give examples on the vanishing of Ext and Tor which simultaneously disprove a conjecture of Auslander and identify new classes of

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commutative local (meaning also Noetherian) rings lying between the class of local complete intersections and the class of local Gorenstein rings.

The following conjecture of Auslander appears in [1, p. 795] and [11]: *let Λ be an Artin algebra. For every finitely generated Λ-module M there exists an integer n_M such that for all finitely generated Λ-modules N, if $\mathrm{Ext}^i_\Lambda(M,N)=0$ for all $i \gg 0$, then $\mathrm{Ext}^i_\Lambda(M,N)=0$ for all $i \geqslant n_M$.*

Auslander's conjecture is known to hold, for example, when Λ is a group ring of a finite group over a field, by [8, 2.4], or when Λ is a local complete intersection (see the discussion later in the introduction). Part (1) of our main theorem below gives a counterexample to Auslander's Conjecture over a commutative self-injective Koszul k-algebra. Part (2) is relevant in the context of recent research on refinements of the Gorenstein condition, as we shall explain shortly.

Theorem. *Let k be a field which is not algebraic over a finite field. Then there exist commutative finite dimensional self-injective Koszul k-algebras A with Hilbert series $\sum(\mathrm{rank}_k\, A_i)t^i = 1+5t+5t^2+t^3$ and finitely generated graded A-modules M with linear resolution and $\sum(\mathrm{rank}_k\, M_i)t^i = 2t+8t^2+2t^3$ such that the following hold:*

(1) *For each positive integer q there exists a finitely generated graded A-module N_q with linear resolution and $\sum(\mathrm{rank}_k (N_q)_i)t^i = 1+t$ satisfying*

$$\mathrm{Ext}^i_A(M,N_q)\neq 0 \quad \textit{if and only if} \quad i=0,\ q-1,\ q.$$

(2) *There exists a finitely generated graded A-module V with linear resolution and $\sum(\mathrm{rank}_k\, V_i)t^i = 2+2t$ satisfying*

$$\mathrm{Tor}^A_i(M,V)=0 \textit{ for all } i>0 \quad \textit{and} \quad \mathrm{Ext}^i_A(M,V)\neq 0 \textit{ for all } i>0.$$

It is easy to see that Auslander's Conjecture holds when Λ is a commutative local ring with maximal ideal $\mathfrak{m}$ satisfying $\mathfrak{m}^2=0$. We show in a corollary of our main theorem (Corollary 3.3(2)) that the conjecture already fails for a commutative local ring $(B,\mathfrak{m})$ with $\mathfrak{m}^3=0$.

The rings A in our main theorem and the rings B in the corollary are constructed by Gasharov and Peeva in [10] to give a counterexample to an unrelated conjecture of Eisenbud. We turned to these rings since they admit modules of infinite complete intersection dimension (see [6] for the definition), which is a necessary property of any module yielding a counterexample to Auslander's Conjecture (cf. [7]).

Recall that a *local complete intersection* is a local ring R whose completion with respect to the maximal ideal $\mathfrak{m}$ is a quotient of a regular local ring by a regular sequence. Let $\mathscr{CI}$ denote the class of all such rings. Let $\mathscr{GOR}$ represent the class of local Gorenstein rings, and $\mathscr{TE}$ the class of commutative local rings R which have the following property: $\mathrm{Tor}^R_i(M,N)=0$ for all $i\gg 0$ implies $\mathrm{Ext}^i_R(M,N)=0$ for all $i\gg 0$, for all finitely generated R-modules M and N. One of the main theorems of Avramov and Buchweitz [3] gives the first inclusion in the chain

$$\mathcal{CI} \subseteq \mathcal{TE} \subseteq \mathcal{GOR}.$$

(The second inclusion is clear: just take $N = R$ in the definition of the class $\mathcal{TE}$.) In [3], the authors remark that 40 years of research in commutative algebra have not produced a class of local rings intermediate between $\mathcal{CI}$ and $\mathcal{GOR}$, and ask whether either inclusion above is strict. In a recent paper, Huneke and Jorgensen [14] prove that the first inclusion is strict. Part (2) of our main theorem shows that so is the second.

In [14] an *AB ring* is defined to be a local Gorenstein ring R with the property that $\mathrm{Ext}^i_R(M,N) = 0$ for all $i \gg 0$ implies $\mathrm{Ext}^i_R(M,N) = 0$ for all $i > \dim R$, for all finitely generated R-modules M and N. Let $\mathcal{AB}$ denote the class of all AB rings. It is shown in [3] (cf. also [14]) that $\mathcal{CI} \subseteq \mathcal{AB}$, and subsequently in [14] (cf. also [21]) it is shown that this inclusion is strict. Note that the condition defining AB rings is a strengthening of Auslander's Conjecture. Therefore, part (1) of our main theorem also shows that $\mathcal{AB}$ lies properly between $\mathcal{CI}$ and $\mathcal{GOR}$.

The paper is organized as follows. In Section 1, we give some positive results on Auslander's Conjecture for commutative non-Gorenstein rings. In particular, we prove that Golod rings, and commutative local rings which are "small" in various senses satisfy (a strong form of) Auslander's Conjecture.

The rings A and B and the corresponding modules are defined in Section 2. Here we also explain our method for computing homology and cohomology.

The main theorem above is an immediate consequence of Corollary 3.3(1) and Proposition 3.9 proved in Section 3. We also give there the corresponding statements for the ring B, and compare these with the results of Section 1, noting that our examples involving the ring B are "smallest" in various senses where one can expect Auslander's Conjecture to fail.

In Section 4, we discuss classes of homologically defined local Gorenstein rings, including the ones described above. We give local Gorenstein rings which are known to satisfy Auslander's Conjecture, and we compare these rings to our examples from Section 3.

1. Some commutative rings for which Auslander's Conjecture holds

In this section, R denotes a commutative local ring, with maximal ideal $\mathfrak{m}$ and residue field k.

As is evidenced by results of [3,14], Auslander's Conjecture is relevant and interesting in the context of commutative local rings (of possibly nonzero Krull dimension). It turns out that all the commutative local rings for which Auslander's Conjecture is known to hold actually satisfy a stronger condition, which we call the *uniform Auslander condition* (uac):

(uac) *There exists an integer n such that for all finitely generated R-modules M and N, if* $\mathrm{Ext}^i_R(M,N) = 0$ *for all* $i \gg 0$ *then* $\mathrm{Ext}^i_R(M,N) = 0$ *for all* $i \geqslant n$.

In this section, we prove that (**uac**) holds for certain rings which are small in various senses. Let edim R denote the minimal number of generators of $\mathfrak{m}$ and $\lambda(R)$ denote the length of R.

1.1. Proposition. *The local ring* $(R, \mathfrak{m})$ *satisfies* (**uac**) *if any one of the following conditions holds.*

(1) $\mathfrak{m}^2 = 0$.
(2) $\operatorname{edim} R - \dim R \leqslant 2$.
(3) $\mathfrak{m}^3 = 0$ *and* $\operatorname{edim} R = 3$.
(4) $\mathfrak{m}^3 = 0$ *and* $\lambda(R) \leqslant 7$.

Recall that the *Poincaré series* of M over R is the formal power series

$$P_M^R(t) = \sum_{i=0}^{\infty} b_i(M) t^i \in \mathbb{Z}[[t]],$$

where $b_i(M) = \operatorname{rank}_k \operatorname{Tor}_i^R(M, k)$ are the *Betti numbers* of M.

Since some of the results existing in the literature are stated in terms of Tor rather than Ext, we remind the reader of the following:

1.2. Assume that R is artinian and let E denote the injective hull of k. For an R-module M we set $M^\vee = \operatorname{Hom}_R(M, E)$. By Matlis duality, for all finitely generated R-modules M and N and all i we have:

$$\operatorname{Tor}_i^R(M, N^\vee) \cong \operatorname{Ext}_R^i(M, N)^\vee.$$

In some cases one can actually prove a property much stronger than (**uac**). We call it *trivial vanishing* (tv) and it states:

(**tv**) *For any pair* (M, N) *of finitely generated* R*-modules, if* $\operatorname{Ext}_R^i(M, N) = 0$ *for all* $i \gg 0$, *then either* M *has finite projective dimension or* N *has finite injective dimension.*

Proof of Proposition 1.1. Let M and N be finitely generated R-modules such that $\operatorname{Ext}_R^i(M, N) = 0$ for all $i \gg 0$ and assume that M is not free.

(1) The first syzygy $\operatorname{Syz}_1(M)$ in a minimal free resolution of M is annihilated by $\mathfrak{m}$, hence it is a finite sum of copies of k. Since $\operatorname{Ext}_R^i(M, N) = 0$ for some $i > 1$ implies $\operatorname{Ext}_R^{i-1}(\operatorname{Syz}_1(M), N) = 0$, we conclude that N has finite injective dimension. The ring therefore satisfies (**tv**), and hence (**uac**).

(2) By Scheja [22], R is either a complete intersection or a Golod ring (see 1.3). If it is a complete intersection, apply [3, 4.7] (cf. also the last section). If it is Golod, then apply Proposition 1.4 below.

(3) If N is injective, then $\operatorname{Ext}_R^i(M, N) = 0$ for all $i > 0$. Therefore assume that N is not injective, hence $N^\vee$ is nonfree. From 1.2 we have $\operatorname{Tor}_i^R(M, N^\vee) = 0$ for all $i \gg 0$.

By taking syzygies, we may assume that there exist finitely generated nonzero R-modules X and Y such that $\mathrm{Tor}_i^R(X,Y)=0$ for all $i>0$ and $\mathfrak{m}^2X=\mathfrak{m}^2Y=0$. We conclude from [15, 2.5] that there exist positive integers u, v such that $u+v=3$ and $b_{i+1}(X)=ub_i(X)$, $b_{i+1}(Y)=vb_i(Y)$ for all $i\geqslant 0$. It follows that one of the modules X or Y has constant Betti numbers (because either $u=1$ or $v=1$), hence one of the modules M or $N^\vee$ has eventually constant Betti numbers. Using a result of Avramov [2, 1.6] we conclude that one of the modules $\mathrm{Syz}_1(M)$, $\mathrm{Syz}_1(N^\vee)$ has a periodic resolution of period 2. The hypothesis then implies $\mathrm{Tor}_i^R(M,N^\vee)=0$ for all $i>1$, hence $\mathrm{Ext}_R^i(M,N)=0$ for all $i>1$.

(4) By (3), we may assume that edim $R\geqslant 4$. The ring R then satisfies the condition in the hypothesis of [15, 3.1], hence, in view also of 1.2, it satisfies (**tv**). $\square$

1.3. Serre proved a coefficientwise inequality

$$P_k^R(t)\preccurlyeq\frac{(1+t)^{\operatorname{edim} R}}{1-\sum_{j=1}^{\infty}\operatorname{rank}\mathrm{H}_j(K^R)t^{j+1}}$$

of formal power series, where K^R denotes the Koszul complex on a minimal set of generators of $\mathfrak{m}$. If equality holds, then R is said to be a *Golod ring*.

1.4. Proposition. *If R is a Golod ring, then it satisfies* (**tv**), *and hence* (**uac**).

For the proof we need some considerations on complexes. We refer to [3, Appendix A] for the basic notions. The Poincaré series of a complex is the extension of the corresponding notion for modules, cf. [3, Section 7], for example.

Vanishing of homology over Golod rings was studied by Jorgensen [17, 3.1]. His result was extended in [4, 8.3] to complexes with finite homology as follows:

1.5. Let R be a Golod ring and M and N complexes with finite homology.

If $\mathrm{Tor}_i^R(M,N)=0$ for all $i\gg 0$, then $P_M^R(t)$ or $P_N^R(t)$ is a Laurent polynomial.

1.6. A complex of R-modules D is said to be *dualizing* if it has finite homology and there is an integer d such that $\mathrm{Ext}_R^d(k,D)\cong k$ and $\mathrm{Ext}_R^i(k,D)=0$ for $i\neq d$. (By [12, V, 3.4], this definition agrees with the one given by Hartshorne in [12].)

Any quotient of a local Gorenstein ring has a dualizing complex. In particular, a complete local ring has a dualizing complex.

1.6.1. For a complex G we set $G^\dagger=\mathbf{R}\operatorname{Hom}_R(G,D)$. As noted in [12, V, Section 2], if G has finite homology, then so does $G^\dagger$.

Proof of Proposition 1.4. We may assume that R is complete, hence it has a dualizing complex D. Let M and N be finitely generated R-modules such that $\mathrm{Ext}_R^i(M,N)=0$

for all $i \gg 0$. By [12, V, 2.6(b)] we have:

$$\mathbf{R}\operatorname{Hom}_R(M,N)^\dagger \simeq M \otimes^{\mathbf{L}}_R N^\dagger.$$

By hypothesis, $\mathbf{R}\operatorname{Hom}_R(M,N)$ has finite homology, and by 1.6.1 so does $M \otimes^{\mathbf{L}}_R N^\dagger$. This means that $\operatorname{Tor}_i^R(M,N^\dagger) = 0$ for all $i \gg 0$, hence $P_M^R(t)$ or $P_{N^\dagger}^R(t)$ is a Laurent polynomial, from 1.5. It follows that M has finite projective dimension or N has finite injective dimension (see [12, V, 2.6(a)]). $\square$

Several classes of local Gorenstein rings are known to satisfy (**uac**). They will be discussed in Section 4.

2. Constructions

Let k be a field and let $\alpha \in k$ be a nonzero element. In this section we describe the rings A_α and B_α constructed in [10] by Gasharov and Peeva, we define the module M of our main theorem, and we discuss our method for computing homology and cohomology.

The ring $A = A_\alpha$: Consider the polynomial ring $k[X_1, X_2, X_3, X_4, X_5]$ in five (commuting) variables and set $A = k[X_1, X_2, X_3, X_4, X_5]/I_\alpha$, where I_α is the ideal generated by the following quadric relations:

$$\alpha X_1X_3 + X_2X_3,\ \ X_1X_4 + X_2X_4,\ \ X_3^2 + \alpha X_1X_5 - X_2X_5,$$

$$X_4^2 + X_1X_5 - X_2X_5,\ \ X_1^2,\ \ X_2^2,\ \ X_3X_4,\ \ X_3X_5,\ \ X_4X_5,\ \ X_5^2.$$

By [10], A_α is a local Gorenstein ring with Hilbert series $\operatorname{Hilb}_{A_\alpha}(t) = 1 + 5t + 5t^2 + t^3$. As a k-vector space, it has a basis consisting of the 12 elements

$$1,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_1x_2,\ x_1x_3,\ x_1x_4,\ x_1x_5,\ x_2x_5,\ x_1x_2x_5,$$

where x_i denotes the residue class of X_i modulo I_α.

The ring $B = B_\alpha$: Set $B_\alpha = A_\alpha/(x_5)$. As noted in [10], B_α is a local ring with Hilbert series $\operatorname{Hilb}_{B_\alpha}(t) = 1 + 4t + 3t^2$. As a k-vector space, it has a basis formed by the images in B_α of the following 8 elements in A_α:

$$1,\ x_1,\ x_2,\ x_3\ x_4,\ x_1x_2,\ x_1x_3,\ x_1x_4.$$

When there is no danger of confusion we will suppress α from the notation and simply write A or B for A_α or B_α.

2.1. One may check that the set of generators of I_α listed above is itself a Gröbner basis for I_α with respect to the reverse lexicographic term order. Since all of these generators are quadrics, by [9, Section 4] we have that $A = A_\alpha$ is Koszul. Similarly, the generators

$$\alpha x_1x_3 + x_2x_3,\ x_1x_4 + x_2x_4,\ x_3^2,\ x_4^2,\ x_1^2,\ x_2^2,\ x_3x_4$$

of the ideal defining B_α form a Gröbner basis of their ideal with respect to the reverse lexicographic term order, and so $B = B_\alpha$ is also Koszul.

Modules with nonperiodic (or periodic of period $\neq 2$) minimal resolutions having constant Betti numbers equal to 2 were given in [10] over the rings A_α and B_α with $\alpha \neq \pm 1$. We wanted a module with nonperiodic resolution and constant Betti numbers, but we found that the modules in [10] did not provide counterexamples using our technique.

The modules M and L: Let M be the image of the map $d_0 : A^2 \to A^2$ given in the standard basis of A^2 as a free A-module by the matrix

$$\begin{pmatrix} x_1 & x_3 \\ x_4 & x_2 \end{pmatrix}.$$

Set $L = M \otimes_A B$.

For any ring R we let $-^*$ denote the R-module $\mathrm{Hom}_R(-, R)$.

2.2. Lemma. *Consider the sequence of homomorphisms*

$$\boldsymbol{C} : \cdots \to A^2 \xrightarrow{d_{i+1}} A^2 \xrightarrow{d_i} A^2 \xrightarrow{d_{i-1}} A^2 \to \cdots,$$

where each map d_i is given in the standard basis of A^2 over A by the matrix

$$\begin{pmatrix} x_1 & \alpha^i x_3 \\ x_4 & x_2 \end{pmatrix}.$$

Then $\boldsymbol{C}$ is an exact complex. Moreover, the complexes $\boldsymbol{C}^*$, $\boldsymbol{C} \otimes_A B$, and $\boldsymbol{C}^* \otimes_A B$ are also exact.

Note that $\boldsymbol{C}^* \otimes_A B$ and $\mathrm{Hom}_B(\boldsymbol{C} \otimes_A B, B)$ are isomorphic complexes. We use the subscript of the differential to keep track of degrees within our complexes.

2.3. Remark. Let W be a finitely generated module over a Noetherian ring R. A *complete resolution* of W is a complex $\boldsymbol{T}$ of finite projective R-modules such that $\mathrm{H}_i(\boldsymbol{T}) = \mathrm{H}_i(\boldsymbol{T}^*) = 0$ for all $i \in \mathbb{Z}$, and $\boldsymbol{T}_{\geqslant r} = \boldsymbol{P}_{\geqslant r}$ for some projective resolution $\boldsymbol{P}$ of W and some r.

If $W \neq 0$, then its *G-dimension* is the shortest length of a resolution by modules G with $G \cong G^{**}$ and $\mathrm{Ext}^i(G, R) = \mathrm{Ext}^i_R(G^*, R) = 0$ for all $i > 0$; it is denoted $\mathrm{G\text{-}dim}_R\, W$. By a basic result of Auslander and Bridger [5], the ring R is Gorenstein if and only if every finite R-module W has $\mathrm{G\text{-}dim}_R\, W < \infty$.

By [3, 4.4.4], W has a complete resolution if and only if $\text{G-dim}_R W < \infty$. Lemma 2.2 shows that $\boldsymbol{C}$ is a complete resolution of the A-module M and $\boldsymbol{C} \otimes_A B$ is a complete resolution of the B-module L. In particular, L has finite G-dimension. (Since G-dimension is bounded by the dimension of the ring, we actually have $\text{G-dim}_B L = 0$.)

Proof of Lemma 2.2. It is immediate to check from the defining equations of A that $d_i d_{i+1} = 0$.

As a k-vector space, A^2 is 24 dimensional. We let (a, b) denote an element of A^2 written in the standard basis of A^2 as a free A-module. It is easy to see that for each i, the following elements in $\operatorname{Im} d_i$ are linearly independent over k:

$$
\begin{aligned}
d_i(1,0) &= (x_1, x_4), & d_i(x_5, 0) &= (x_1x_5, 0),\\
d_i(0,1) &= (\alpha^i x_3, x_2), & d_i(0, x_1) &= (\alpha^i x_1x_3, x_1x_2),\\
d_i(x_1,0) &= (0, x_1x_4), & d_i(0, x_3) &= (0, -\alpha x_1x_3),\\
d_i(x_2,0) &= (x_1x_2, -x_1x_4), & d_i(0, x_5) &= (0, x_2x_5),\\
d_i(x_3,0) &= (x_1x_3, 0), & d_i(x_2x_5, 0) &= (x_1x_2x_5, 0),\\
d_i(x_4,0) &= (x_1x_4, x_2x_5 - x_1x_5), & d_i(0, x_1x_5) &= (0, x_1x_2x_5).
\end{aligned}
$$

As a consequence, $\operatorname{rank}_k(\operatorname{Im} d_i) \geqslant 12$ for each i. On the other hand, we have

$$\operatorname{rank}_k(\operatorname{Ker} d_i) = \operatorname{rank}_k(A^2) - \operatorname{rank}_k(\operatorname{Im} d_i) \leqslant 12$$

for each i. It follows that $\operatorname{rank}_k(\operatorname{Im} d_{i+1}) = \operatorname{rank}_k(\operatorname{Ker} d_i) = 12$ for each i, hence the complex $\boldsymbol{C}$ is exact. Since A is self-injective, the complex $\boldsymbol{C}^*$ is exact as well.

Similar computations over the ring B show that the complexes $\boldsymbol{C} \otimes_A B$ and $\boldsymbol{C}^* \otimes_A B$ are exact. $\square$

2.4. The proof of the lemma shows that

$$\operatorname{Hilb}_M(t) = \operatorname{Hilb}_{M^*}(t) = 2t + 8t^2 + 2t^3$$

and one may verify that

$$\operatorname{Hilb}_L(t) = \operatorname{Hilb}_{L^*}(t) = 2t + 6t^2.$$

Every finitely generated graded module W over a standard graded local ring R with $R_0 = k$ has a minimal graded free resolution. Consequently, the modules $\operatorname{Tor}_i^R(W, k)$ inherit a structure of graded R-modules. The bigraded Poincaré series of W is the formal power series in two variables:

$$P_W^R(t, z) = \sum_{i,j} \operatorname{rank}_k \operatorname{Tor}_i^R(W, k)_j \, t^i z^j,$$

where j is the index of degree. The usual Poincaré series is obtained by letting $z = 1$.

The module W is said to have a *linear resolution* if all its minimal generators are in the same degree p and W has a minimal graded free resolution in which all the entries of the matrices defining the differentials have degree 1. This is equivalent to $\mathrm{Tor}_i(W,k)_j = 0$ for all $j \neq i+p$.

By definition, the k-algebra R is Koszul if the R-module k has a linear resolution. In this case, it is known that

$$P_k^R(t,z) = \frac{1}{\mathrm{Hilb}_R(-tz)}.$$

2.5. It is clear from Lemma 2.2 that the A-modules M and M^*, as well as the B-modules N and N^*, have a linear resolution, and

$$P_M^A(t,z) = P_{M^*}^A(t,z) = P_L^B(t,z) = P_{L^*}^B(t,z) = \frac{2z}{1-tz}.$$

Note that all the syzygies of these modules have the same Poincaré series. Since the rings A and B are Koszul, the Poincaré series of k over A and B are

$$P_k^A(t,z) = \frac{1}{1-5tz+5t^2z^2-t^3z^3} \quad \text{and} \quad P_k^B(t,z) = \frac{1}{1-4tz+3t^2z^2}.$$

Next, we describe our approach to calculating (co)homology over the rings A, respectively, B when one of the modules is M, respectively, L.

2.6. Computing Ext and Tor. We set $\boldsymbol{F} = \boldsymbol{C}_{\geqslant 0}$ and $\boldsymbol{G} = \boldsymbol{C}_{<0}$. Lemma 2.2 shows that $\boldsymbol{F}$ is a minimal free resolution of M over A and $\boldsymbol{F} \otimes_A B$ is a minimal free resolution of L over B. Also, $\boldsymbol{G}^*$ is a minimal free resolution of M^* over A and $\boldsymbol{G}^* \otimes_A B$ is a minimal free resolution of L^* over B.

Let N be a finitely generated A-module and let P be a finitely generated B-module.

2.6.1. The module $\mathrm{Tor}_i^A(M,N)$ is the ith homology of the complex

$$\boldsymbol{F} \otimes_A N: \quad \cdots \to N^2 \xrightarrow{d_i \otimes_A N} N^2 \xrightarrow{d_{i-1} \otimes_A N} N^2 \to \cdots \to N^2 \xrightarrow{d_1 \otimes_A N} N^2 \to 0,$$

that is,

$$\mathrm{Tor}_i^A(M,N) = \mathrm{Ker}(d_i \otimes_A N)/\mathrm{Im}(d_{i+1} \otimes_A N).$$

Similarly, $\mathrm{Tor}_i^B(L,P)$ is the ith homology of the complex $(\boldsymbol{F} \otimes_A B) \otimes_B P = \boldsymbol{F} \otimes_A P$.

2.6.2. The module $\mathrm{Ext}_A^i(M,N)$ is the $(-i)$th homology of the complex $\mathrm{Hom}_A(\boldsymbol{F},N)$ and this complex can be identified with

$$\boldsymbol{F}^* \otimes_A N: 0 \to N^2 \xrightarrow{d_1^* \otimes_A N} N^2 \to \cdots \to N^2 \xrightarrow{d_i^* \otimes_A N} N^2 \xrightarrow{d_{i+1}^* \otimes_A N} N^2 \to \cdots,$$

where $\boldsymbol{F}^* = \mathrm{Hom}_A(\boldsymbol{F}, A)$, and for each i the map d_i^* is given in the standard basis of A^2 by the transpose of the matrix corresponding to d_i. Thus

$$\mathrm{Ext}_A^i(M,N) = \mathrm{Ker}(d_{i+1}^* \otimes_A N)/\mathrm{Im}(d_i^* \otimes_A N).$$

Similarly, $\mathrm{Ext}_B^i(L,P)$ is the $(-i)$th homology of the complex $\boldsymbol{F}^* \otimes_A P$.

2.6.3. The module $\mathrm{Ext}_A^i(N,M)$ is isomorphic to $\mathrm{Tor}_i^A(M^*,N)^*$, and $\mathrm{Tor}_i^A(M^*,N)$ is the ith homology of the complex

$$\boldsymbol{G}^* \otimes_A N : \cdots \to N^2 \xrightarrow{d_{-i}^* \otimes_A N} N^2 \xrightarrow{d_{-i+1}^* \otimes_A N} N^2 \to \cdots \to N^2 \xrightarrow{d_{-1}^* \otimes_A N} N^2 \to 0.$$

Thus

$$\mathrm{Ext}_A^i(N,M) = (\mathrm{Ker}(d_{-i}^* \otimes_A N)/\mathrm{Im}(d_{-i-1}^* \otimes_A N))^*.$$

Similarly, $\mathrm{Tor}_i^B(L^*,P)$ is the ith homology of the complex $\boldsymbol{G}^* \otimes_A P$.

3. Results on vanishing

In this section, we prove the main results stated in the introduction. Our method and constructions were inspired by the paper [13] of Heitmann.

We fix $\alpha \in k$ and use the notation introduced in Section 2.

For each integer q we set $T_q = A/J_q$, where J_q is the ideal of $A = A_\alpha$ generated by the linear relations

$$x_1 - x_2, \ x_1 - \alpha^q x_3, \ x_1 - x_4, \ x_5.$$

Note that T_q is also a B-module.

We let $o(\alpha)$ denote the order of α in the group of units of k, and set

$$s = \begin{cases} 0 & \text{if } o(\alpha) = \infty, \\ o(\alpha) & \text{otherwise.} \end{cases}$$

Note that $a \equiv b \bmod (0)$ if and only if $a = b$.

3.1. Proposition. *The following hold for every integer q and every $i > 0$:*

(a) $\mathrm{Tor}_i^A(M, T_q) \neq 0$ *if and only if* $i \equiv q-1, q \bmod (s)$.
(b) $\mathrm{Ext}_A^i(M, T_q) \neq 0$ *if and only if* $i \equiv q-1, q \bmod (s)$.
(c) $\mathrm{Ext}_A^i(T_q, M) \neq 0$ *if and only if* $i \equiv -q, -q-1 \bmod (s)$.

3.2. Proposition. *The following hold for every integer q and every $i > 0$:*

(a) $\mathrm{Tor}_i^B(L, T_q) \neq 0$ *if and only if* $i \equiv q-1, q \bmod (s)$.

(b) $\operatorname{Ext}^i_B(L, T_q) \neq 0$ *if and only if* $i \equiv q-1, q \bmod (s)$.
(c) $\operatorname{Tor}^B_i(L^*, T_q) \neq 0$ *if and only if* $i \equiv -q, -q-1 \bmod (s)$.

3.3. Corollary. *If* $o(\alpha) = \infty$, *then the following hold for any integer* $q > 0$:

(1) $\operatorname{Ext}^i_A(M, T_q) \neq 0$ *if and only if* $i = 0,\ q-1,\ q$.
(2) $\operatorname{Ext}^i_B(L, T_q) \neq 0$ *if and only if* $i = 0,\ q-1,\ q$.

3.4. Remark. The corollary shows that, when $o(\alpha) = \infty$, the rings $A = A_\alpha$ and $B = B_\alpha$ provide counterexamples to Auslander's Conjecture. In view also of 3.8 below, the first part gives part (1) of the main theorem in the introduction.

3.5. Let $\mathfrak{m}$ denote the maximal ideal of B. The expression for $\operatorname{Hilb}_B(t)$ given above indicates that $\mathfrak{m}^3 = 0$, $\operatorname{edim} B = 4$ and $\lambda(B) = 8$. Comparing these numerical data with the results stated in Proposition 1.1, we see that our examples are minimal primarily with respect to the invariant $\inf\{n \mid \mathfrak{m}^n = 0\}$ and secondarily with respect to the invariants edim and length.

3.6. Remark. Let R be a Noetherian ring and let W and N be finitely generated R-modules. Assume that W has a complete resolution $\boldsymbol{T}$, as defined in 2.3. For each i the *Tate* (*co*)*homology* groups are defined by

$$\widehat{\operatorname{Ext}}^i_R(W, N) = \operatorname{H}_{-i} \operatorname{Hom}(\boldsymbol{T}, N) \quad \text{and} \quad \widehat{\operatorname{Tor}}^R_i(W, N) = \operatorname{H}_i(\boldsymbol{T} \otimes_R N).$$

If r is as in 2.3, then it is clear that for all $i > r$ one has

$$\widehat{\operatorname{Ext}}^i_R(W, N) \cong \operatorname{Ext}^i_R(W, N) \quad \text{and} \quad \widehat{\operatorname{Tor}}^R_i(W, N) \cong \operatorname{Tor}^R_i(W, N).$$

Also, when $\text{G-dim}_R\, W = 0$, one has

$$\widehat{\operatorname{Ext}}^{-i-1}_R(W, N) \cong \widehat{\operatorname{Tor}}^R_i(W^*, N).$$

In terms of Tate (co)homology, the propositions above can be formulated as follows. Let q be an integer. Then for all i we have:

(1) $\widehat{\operatorname{Ext}}^i_A(M, T_q) \neq 0$ if and only if $i \equiv q-1, q \bmod (s)$.
(2) $\widehat{\operatorname{Tor}}^A_i(M, T_q) \neq 0$ if and only if $i \equiv q-1, q \bmod (s)$.
(3) $\widehat{\operatorname{Ext}}^i_B(L, T_q) \neq 0$ if and only if $i \equiv q-1, q \bmod (s)$.
(4) $\widehat{\operatorname{Tor}}^B_i(L, T_q) \neq 0$ if and only if $i \equiv q-1, q \bmod (s)$.

3.7. Remark. It is now clear that, using the modules in the propositions, one can obtain a wide variety of distributions of nonzero (co)homology. In particular, when $s = 0$ one can construct arbitrarily large intervals of either vanishing or nonvanishing (co)homology: for integers a, b satisfying $0 \leqslant a < b$ there exist finitely generated A-modules $N_{a,b}$ and $Z_{a,b}$ such that

(1) $\mathrm{Ext}^i_A(M, N_{a,b}) \neq 0$ if and only if $a \leqslant i \leqslant b$ (and $i = 0$), and
(2) $\mathrm{Ext}^i_A(M, Z_{a,b}) = 0$ for all $a < i < b$ and $\mathrm{Ext}^i_A(M, Z_{a,b}) \neq 0$ for $i = a, b$ (and $i = 0$).

Indeed, one may take $N_{a,b} = \oplus_{q=a+1}^{b} T_q$ and $Z_{a,b} = T_a \oplus T_{b+1}$.

When s is positive, we obtain recurring intervals of vanishing/nonvanishing cohomology. For example, assume that $s = 4$ and set $T = T_0$. We have then

$$\mathrm{Ext}^i_A(M, T) = 0 \quad \text{for all } i \equiv 1, 2 \bmod (4),$$

$$\mathrm{Ext}^i_A(M, T) \neq 0 \quad \text{for all } i \equiv 0, 3 \bmod (4).$$

We give only the proof of Proposition 3.1. The proof of Proposition 3.2 is similar.

Proof of Proposition 3.1. We let overbars denote residue classes modulo J_q and we perform computations of Ext and Tor as explained in 2.6. We only give the proof of (1); the other arguments are similar.

(1) The differential $\bar{d}_i = d_i \otimes_A T_q$ of the complex $\boldsymbol{F} \otimes_A T_q$ is given in the standard basis of T_q^2 over T_q by the matrix

$$\begin{pmatrix} \bar{x}_1 & \alpha^{i-q}\bar{x}_1 \\ \bar{x}_1 & \bar{x}_1 \end{pmatrix}.$$

As a k-vector space T_q has a basis consisting of $1, \bar{x}_1$, and for each $i \geqslant 0$

$$\mathrm{rank}_k(\mathrm{Im}\, \bar{d}_i) = \begin{cases} 1 & \text{if } i \equiv q \bmod (s), \\ 2 & \text{otherwise.} \end{cases}$$

Since $\dim_k(\mathrm{Ker}\, \bar{d}_i) = \dim_k(T_q^2) - \dim_k(\mathrm{Im}\, \bar{d}_i)$, we then have

$$\mathrm{rank}_k(\mathrm{Ker}\, \bar{d}_i) = \begin{cases} 3 & \text{if } i \equiv q \bmod (s), \\ 2 & \text{otherwise.} \end{cases}$$

By 2.6.1 we have $\mathrm{Tor}_i^A(M, T_q) = \mathrm{H}_i(\boldsymbol{F} \otimes_A T_q)$ and the conclusion follows from the above computations. $\square$

3.8. For each q the module T_q has Hilbert series

$$\mathrm{Hilb}_{T_q}(t) = 1 + t.$$

Assume that $o(\alpha) = \infty$. By Propositions 3.1 and 2.5 there exists an A-module W with $P_W^A(t, z) = 2z(1 - tz)^{-1}$ such that $\mathrm{Tor}_i^A(W, T_q) = 0$ for all $i > 0$ and $W \otimes_R T_q$ is isomorphic to a sum of 2 copies of k each generated in degree 1. Indeed, if $q \leqslant 0$ then take W to be a first syzygy of M, and if $q > 0$ then take W to be a first syzygy of M^*, and use for example [15, 1.4(2)] to see that $W\otimes_R T_q$ is annihilated by the maximal ideal of A.

The bigraded version of a usual computation, cf. [19, 1.1] for example, gives

$$P^A_{W\otimes_A T_q}(t,z) = P^A_W(t,z)P^A_{T_q}(t,z).$$

Since $W\otimes_R T_q$ is a sum of copies of k, we may use the formula for $P^A_k(t,z)$ given in 2.5 to conclude that

$$P^A_{T_q}(t,z) = \frac{1}{1-4tz+t^2z^2}.$$

The expansion of this fraction as a power series shows that the A-module T_q has a linear resolution. Similar computations show that T_q has a linear resolution as a B-module as well.

Now let U be the cokernel of the map $A^6\to A^2$ given in the standard bases of A^6, respectively, A^2 over A by the matrix

$$\begin{pmatrix} x_3 & 0 & x_1 & x_4 & x_2 & 0 \\ -x_2 & x_3 & -x_4 & 0 & 0 & x_1 \end{pmatrix}$$

and set $V = U\otimes_A A/(\mathfrak{m}^2, x_5)$. Note that V is also a B-module.

The next proposition gives part (2) of the main theorem in the introduction. (See also 3.12 below.)

3.9. Proposition. *With the notation above, the following hold*:

(a) $\operatorname{Tor}^A_i(M,V)=0$ *for all* $i>0$.

(b) $\operatorname{Ext}^i_A(M,V)\neq 0$ *for all* $i>0$.

3.10. Remark. We can obtain an example of vanishing Exts and nonvanishing Tors just by replacing V with V^*. (Since A is zero-dimensional and Gorenstein, one has $-^* = -^\vee$. Recall then from 1.2 that $\operatorname{Tor}^A_i(M,V^*)\cong \operatorname{Ext}^i_A(M,V)^*$ and $\operatorname{Ext}^i_A(M,V^*)\cong \operatorname{Tor}^A_i(M,V)^*$.)

One can replace A with B and M with L in the statement of Proposition 3.9. The proof is similar.

Proof of Proposition 3.9. (1) Recall from 2.6.1 that $\operatorname{Tor}^A_i(M,V)$ is the ith homology of the complex $\boldsymbol{F}\otimes_A V$, where the differential $d_i: A^2\to A^2$ of $\boldsymbol{F}$ is given in the standard basis of A^2 by the matrix

$$\begin{pmatrix} x_1 & \alpha^i x_3 \\ x_4 & x_2 \end{pmatrix}.$$

The module V is the quotient of A^2 by $(\mathfrak{m}^2, x_5)A^2$ and the relations $(x_3, -x_2)$, $(0, x_3)$, $(x_1, -x_4)$, $(x_4, 0)$, $(x_2, 0)$, $(0, x_1)$. We let $\operatorname{cls} a$ denote the residue class in V of $a \in A^2$. As a k-vector space, V has a basis formed by the 4 elements

$$v_1 := \operatorname{cls}(1,0), \quad v_2 := \operatorname{cls}(0,1), \quad v_3 := \operatorname{cls}(x_1, 0), \quad v_4 := \operatorname{cls}(0, x_2).$$

Given this basis for V, we then make the obvious choice for a basis of V^2.

We set $\delta_i = d_i \otimes_A V : V^2 \to V^2$. For each i, the following elements in $\operatorname{Im}(\delta_i)$ can be easily seen to be linearly independent over k:

$$\delta_i(v_1, 0) = (v_3, 0),$$

$$\delta_i(v_2, 0) = (0, v_3),$$

$$\delta_i(0, v_1) = \alpha^i(v_4, 0),$$

$$\delta_i(0, v_2) = (0, v_4).$$

As a consequence, $\operatorname{rank}_k(\operatorname{Im} \delta_i) \geqslant 4$. On the other hand,

$$\operatorname{rank}_k(\operatorname{Ker} \delta_i) = \operatorname{rank}_k(V^2) - \operatorname{rank}_k(\operatorname{Im} \delta_i) \leqslant 8 - 4 = 4.$$

It follows that $\operatorname{Ker}(\delta_i) = \operatorname{Im}(\delta_{i+1})$ for each $i > 0$, hence $\mathrm{H}_i(\boldsymbol{F} \otimes_A V) = 0$.

(2) Recall from 2.6.2 that $\operatorname{Ext}_i^A(M, V)$ is the $(-i)$th homology of the complex $\boldsymbol{F}^* \otimes_A V$, where the differential $d_i^* : A^2 \to A^2$ of $\boldsymbol{F}^*$ is given in the standard basis of A^2 by the matrix

$$\begin{pmatrix} x_1 & x_4 \\ \alpha^i x_3 & x_2 \end{pmatrix}.$$

Set $\delta_i^* = d_i^* \otimes_A V$. Similar computations as above show that only two elements of the basis of V^2 over k are not in $\operatorname{Ker}(\delta_i^*)$. Their images are

$$\delta_i^*(v_1, 0) = (v_3, \alpha^i v_4),$$

$$\delta_i^*(0, v_2) = (v_3, v_4)$$

We conclude

$$\operatorname{rank}_k(\operatorname{Im} \delta_i) \leqslant 2 \quad \text{and} \quad \operatorname{rank}_k(\operatorname{Ker} \delta_i) \geqslant 8 - 2 = 6 \quad \text{for all } i.$$

We thus have $\mathrm{H}_i(\boldsymbol{F}^* \otimes_A V) \neq 0$ for all $i > 0$. $\square$

3.11. Remark. Formulated in terms of Tate (co)homology, the proof of Proposition 3.9 shows that

(a) $\widehat{\operatorname{Tor}}_i^A(M, V) = 0$ for all i.
(b) $\widehat{\operatorname{Ext}}_A^i(M, V) \neq 0$ for all i.

3.12. The proof of the proposition shows that the module V has

$$\mathrm{Hilb}_V(t) = 2 + 2t.$$

Similar computations as in 3.8 show that V has a linear resolution, both as an A-module, and as a B-module.

For each finitely generated A-module N we set

$$c(N) = \mathrm{rank}_k(N) - \mathrm{rank}_k(\mathrm{Socle}\, N).$$

The next proposition shows that when $o(\alpha) = \infty$, M is rather rigid.

3.13. Proposition. *When $o(\alpha) = \infty$ the following hold for any finitely generated A-module N:*

(1) *If $\mathrm{Tor}_j^A(M,N) = 0$ for some $j>0$, then $\mathrm{Tor}_i^A(M,N) \neq 0$ for at most $2c(N)$ values of $i>0$.*
(2) *If $\mathrm{Ext}_A^j(M,N) = 0$ for some $j>0$, then $\mathrm{Ext}_A^i(M,N) \neq 0$ for at most $2c(N)$ values of $i>0$.*
(3) *If $\mathrm{Ext}_A^j(N,M) = 0$ for some $j>0$ then $\mathrm{Ext}_A^i(N,M) \neq 0$ for at most $2c(N)$ values of $i>0$.*
(4) *$\mathrm{Ext}_A^i(M,N) = 0$ for all $i \gg 0$ if and only if $\mathrm{Ext}_A^i(N,M) = 0$ for all $i \gg 0$.*

3.14. Remark. Statements (1) and (2) remain valid, with similar proofs, when replacing A by B, and M by L.

Proof. We will show that if $\mathrm{H}_j(\boldsymbol{C}^* \otimes_A N) = 0$ for some j, then $\mathrm{H}_i(\boldsymbol{C}^* \otimes_A N) \neq 0$ for at most $2c(N)$ values of i. In view of 2.6.2 and 2.6.3, this proves (2), (3) and (4). The proof of (1) is along similar lines, using the complex $\boldsymbol{F} \otimes_A N$ and 2.6.1, and it is omitted.

For every i set

$$u_i = \mathrm{rank}_k(\mathrm{Im}(d_{i+1}^* \otimes_A N)) \quad \text{and} \quad v_i = \mathrm{rank}_k(\mathrm{Ker}(d_i^* \otimes_A N)).$$

Since $\boldsymbol{C}^* \otimes_A N$ is a complex, we have $u_i \leqslant v_i$ for all i, with equality if and only if $\mathrm{H}_i(\boldsymbol{C}^* \otimes_A N) = 0$.

Assume that $u_j = v_j$ for some $j \in \mathbb{Z}$. We need to show that $u_i \neq v_i$ for at most $2c(N)$ values of $i \in \mathbb{Z}$. Set

$$r = \max\{u_i \mid i \in \mathbb{Z}\}.$$

Since $u_i + v_{i+1} = 2\,\mathrm{rank}_k N$ and $u_i \leqslant v_i$ for all i, we conclude that $u_i + u_{i-1} \leqslant 2\,\mathrm{rank}_k N$, with equality if and only if $u_i = v_i$. Taking $i = j$ we obtain that $u_j + u_{j-1} = 2\,\mathrm{rank}_k N$. Since $u_i \leqslant r$ for all i, we have $\mathrm{rank}_k N \leqslant r$.

Claim. *$u_i \neq r$ for at most $c(N)$ values of $i \in \mathbb{Z}$.*

Assuming the claim for the moment, we finish the proof.

As noted above, we need to show that $u_i + u_{i-1} \neq 2\,\mathrm{rank}_k\, N$ for at most $2c(N)$ values of $i \in \mathbb{Z}$. We have

$$\{i \in \mathbb{Z} \mid u_i + u_{i-1} \neq 2\,\mathrm{rank}_k\, N\} = \{i \in \mathbb{Z} \mid u_i \neq r\} \cup \{i \in \mathbb{Z} \mid u_{i-1} \neq r\},$$

and the claim shows that this set has at most $2c(N)$ elements.

Proof of claim. Let $\mathscr{E}$ be a basis of N over k and set $e = \mathrm{rank}_k\, N$. Let χ_i be the $e \times e$ matrix which represents in the basis $\mathscr{E}$ the map $N \to N$ given by multiplication by x_i, for $i = 1, \ldots, 4$.

With the obvious choice for the basis of N^2 over k, the map $d_i \otimes_A N : N^2 \to N^2$ is represented by the $2e \times 2e$ matrix

$$\Omega_i = \begin{pmatrix} \chi_1 & \alpha^i \chi_3 \\ \chi_4 & \chi_2 \end{pmatrix}.$$

Let $k[y]$ be the polynomial ring over k in a single variable y. We consider the following $2e \times 2e$ matrix with elements in $k[y]$:

$$\Omega(y) = \begin{pmatrix} \chi_1 & y\chi_3 \\ \chi_4 & \chi_2 \end{pmatrix}.$$

Since r is the maximum of $\{\mathrm{rank}\, \Omega_i\}_{i \in \mathbb{Z}}$, there exists a nonzero $r \times r$ minor Δ_ℓ of Ω_ℓ for some ℓ. Let $\Delta(y)$ denote the $r \times r$ minor of $\Omega(y)$ corresponding to Δ_ℓ. Then $\Delta(y)$ is a polynomial in y. Since $\Delta(\alpha^\ell) = \Delta_\ell$ is nonzero, $\Delta(y)$ is a nonzero polynomial. Note that it has degree at most $c(N)$, and therefore it has at most $c(N)$ roots in k. In conclusion, the $r \times r$ minor $\Delta_i = \Delta(\alpha^i)$ of Ω_i is zero for at most $c(N)$ values of i. $\square$

3.15. Remark. Formulated in terms of Tate (co)homology, cf. 3.6, the proofs of parts (1) and (2) of Proposition 3.13 actually show the following:

(1) If $\widehat{\mathrm{Tor}}_j^A(M,N) = 0$ for some j, then $\widehat{\mathrm{Tor}}_i^A(M,N) \neq 0$ for at most $2c(N)$ values of i.

(2) If $\widehat{\mathrm{Ext}}_A^j(M,N) = 0$ for some j, then $\widehat{\mathrm{Ext}}_A^i(M,N) \neq 0$ for at most $2c(N)$ values of i.

Similar statements can be given for the ring B and the module L.

4. Classes of Gorenstein rings

In this section we discuss homologically defined classes of local Gorenstein rings, introduced in [3,14], and show using the examples in the previous section that these classes of local rings lie properly between the class of local complete intersections and the class of local Gorenstein rings.

Throughout this section R is a local ring with maximal ideal $\mathfrak{m}$. Let (**ci**) denote the condition that R is a local complete intersection, and (**gor**) the condition that R is a local Gorenstein ring. We further consider the following properties (cf. [3, 6.3]):

(**te**) $\operatorname{Tor}_i^R(M,N)=0$ *for all* $i \gg 0$ *implies* $\operatorname{Ext}_R^i(M,N)=0$ *for all* $i \gg 0$,
(**et**) $\operatorname{Ext}_R^i(M,N)=0$ *for all* $i \gg 0$ *implies* $\operatorname{Tor}_i^R(M,N)=0$ *for all* $i \gg 0$,
(**ee**) $\operatorname{Ext}_R^i(M,N)=0$ *for all* $i \gg 0$ *implies* $\operatorname{Ext}_R^i(N,M)=0$ *for all* $i \gg 0$,

where M and N range over all finitely generated R-modules.

4.1. Note that by taking $N=R$, the property (**te**) implies R is Gorenstein; by taking $M=R$, the property (**ee**) implies R is Gorenstein.

4.2. Avramov and Buchweitz prove in [3, 6.1] that if R is a local complete intersection, then it satisfies both (**et**) and (**te**). This gives implication (1) in the following diagram, reproduced from [3, 6.3]; the remaining implications are clear.

$$\begin{tikzcd}
 & & (\mathbf{ee}) \arrow[dr, Rightarrow] & \\
(\mathbf{ci}) \arrow[r, Rightarrow, "(1)"] & (\mathbf{et})\ \&\ (\mathbf{te}) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow, "(2)"'] & & (\mathbf{gor}) \\
 & & (\mathbf{te}) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

In [3, 6.4] the question is raised whether any of these implications can be reversed. We first note that (2) is reversible:

4.3. Proposition. *The following statements are equivalent*:

(1) *R satisfies* (**te**).
(2) *R is Gorenstein and satisfies* (**et**).

Proof. We give only the proof of (1) $\Rightarrow$ (2). The reverse implication can be proved similarly.

Assume that R satisfies (**te**). By 4.1, the ring R is Gorenstein. By taking syzygies, it suffices to prove that (**et**) holds for maximal Cohen–Macaulay R-modules M and N. Note that such modules are in particular reflexive, that is they are isomorphic to their double dual.

Let M and N be maximal Cohen–Macaulay R-modules such that $\mathrm{Ext}_R^i(M,N)=0$ for all $i \gg 0$. By [14, 2.1] we have then $\mathrm{Tor}_i^R(M,N^*)=0$ for all $i \gg 0$. Since **(te)** holds, this implies $\mathrm{Ext}_R^i(M,N^*)=0$ for all $i \gg 0$. Using again [14, 2.1] we conclude that $\mathrm{Tor}_i^R(M,N)=0$ for all $i \gg 0$. $\square$

In [14], R is said to be an *AB ring* whenever it satisfies the following condition:

(ab) *R is Gorenstein and there exists an integer n such that for all pairs (M,N) of finitely generated R-modules*

$$\mathrm{Ext}_R^i(M,N)=0 \textit{ for all } i \gg 0 \textit{ implies } \mathrm{Ext}_R^i(M,N)=0 \textit{ for all } i>n.$$

It is shown in [14] that if R is an AB ring, then the integer n above can be taken to be $\dim R$, but not less. Note that the condition **(ab)** is the uniform Auslander condition **(uac)** from Section 1 together with the requirement that R is Gorenstein.

Another property of a local Gorenstein ring R is defined in [14]. We say that $\mathrm{Ext}_R(M,N)$ has a *gap of length* g if there exists an $n>0$ such that $\mathrm{Ext}_R^i(M,N)\neq 0$ for $i=n-1$ and $i=n+g$ and $\mathrm{Ext}_R^i(M,N)=0$ for all $n \leqslant i \leqslant n+g-1$. Set

$$\text{Ext-gap}(R) \coloneqq \sup\{g \mid \mathrm{Ext}_R(M,N) \text{ has a gap of length } g\},$$

where M and N range over all finitely generated R-modules. Similarly, one can define the notion of Tor-gap. It is proved in [14, 3.3(2)] that over a Gorenstein ring Ext-gap is finite if and only if Tor-gap is finite.

We define the property *finite gap* as follows:

(gap) *R is Gorenstein and* Ext-gap(R) *is finite.*

4.4. The following implications are known to hold:

$$(\mathbf{ci}) \overset{(3)}{\Rightarrow} (\mathbf{gap}) \overset{(4)}{\Rightarrow} (\mathbf{ab}) \overset{(5)}{\Rightarrow} (\mathbf{ee})$$

Implication (3) is proved in [20, 1.6], cf. also [16, 2.3] for a more precise version. Implication (4) is given by [14, 3.3(3)] and (5) by [14, 4.1].

In [14], it is also proved that implication (3) above and (1) in 4.2 are not reversible. The details of this are as follows.

Every local Gorenstein ring R (which is not a complete intersection) has multiplicity at least $\mathrm{edim}\, R - \dim R + 2$. A local Gorenstein ring R is said to have *minimal multiplicity* if its multiplicity is equal to $\mathrm{edim}\, R - \dim R + 2$. When R is artinian, minimal multiplicity just means $\mathfrak{m}^3=0$. A local Gorenstein ring R of minimal multiplicity is not a complete intersection precisely when $\mathrm{edim}\, R - \dim R \geqslant 3$.

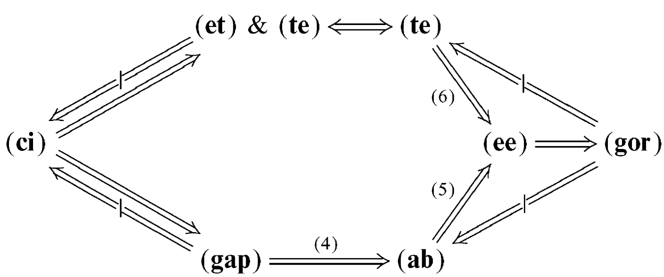
4.5. Let R be a local Gorenstein ring of minimal multiplicity. By [14, 3.6] and [14, 3.2(3)], R satisfies (gap), and hence (**ab**).

By [14, 3.6], if R is a Gorenstein ring of minimal multiplicity, but not a complete intersection, then R satisfies the property (**tv**) introduced in Section 1. Thus all Gorenstein rings of minimal multiplicity also satisfy (**te**).

The main theorem stated in the introduction and proved in the previous section, shows that:

(a) there exist local Gorenstein rings which are not AB rings.
(b) there exist local Gorenstein rings which do not satisfy (**te**).

4.6. The facts discussed above are summarized in the following refinement of the diagrams in 4.2 and 4.4.



In conclusion, the following classes of local rings lie strictly between the class of the local complete intersections and that of local Gorenstein rings:

(a) the local rings satisfying (**te**).
(b) the AB rings.
(c) the local rings satisfying (**gap**).

4.7. Our examples in the previous section are local Gorenstein rings R with $\mathfrak{m}^4 = 0$, $\operatorname{edim} R = 5$, and $\lambda(R) = 12$ which satisfy neither (**ab**) nor (**te**). By 4.5, these examples are minimal with respect to the invariant $\inf\{n \mid \mathfrak{m}^n = 0\}$. Other aspects of the minimality of these examples can be deduced from [21] as we now describe.

Let R be a Gorenstein local ring. By [21, 3.4] and 4.2(1), if $\operatorname{edim} R - \dim R \leqslant 4$, then R satisfies (**ab**). Our examples of Gorenstein rings not satisfying (**ab**) are thus minimal with respect to embedding dimension.

If R is standard graded with $\lambda(R) < 12$, then it follows that $\mathfrak{m}^3 = 0$ or $\operatorname{edim} R \leqslant 4$, hence 4.5 or the above considerations apply. In particular, any such ring satisfies (**ab**), and so in the standard graded case our examples of rings not satisfying (**ab**) are minimal with respect to length.

4.8. There is a question remaining:

Are there further implications between the properties displayed in 4.6? That is, are any of the implications (4), (5), or (6) reversible?

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