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A receding contact plane problem between a functionally graded layer and a homogeneous substrate

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Abstract

In this paper, we consider the plane problem of a frictionless receding contact between an elastic functionally graded layer and a homogeneous half-space, when the two bodies are pressed together. The graded layer is modeled as a non-homogeneous medium with an isotropic stress–strain law and over a certain segment of its top surface is subjected to normal tractions while the rest of this surface is free of tractions. Since the contact between the two bodies is assumed to be frictionless, then only compressive normal tractions can be transmitted in the contact area. Using integral transforms, the plane elasticity equations are converted analytically into a singular integral equation in which the unknowns are the contact pressure and the receding contact half-length. The global equilibrium condition of the layer is supplemented to solve the problem. The singular integral equation is solved numerically using Chebychev polynomials and an iterative scheme is employed to obtain the correct receding contact half-length that satisfies the global equilibrium condition. The main objective of the paper is to study the effect of the material nonhomogeneity parameter and the thickness of the graded layer on the contact pressure and on the length of the receding contact.

Keywords: Functionally graded layer; Receding contact; Singular integral equation

1. Introduction

Problems involving the contact of two separate bodies pressed against each other has been widely studied in the literature. In these problems, the length of the contact zone and the contact pressure which is zero

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at the ends of the contact segment are the primary unknowns of the problem. If the contact zone shrinks as the bodies are deformed, then such contact is referred to as receding contact (Dundurs, 1975). In linear elastostatics problems in which the boundary conditions are satisfied on the original geometry, a receding contact is one where the contact surface in the loaded configuration is contained within the initial contact surface (Johnson, 1985).

The receding contact problem has been studied during the past three decades by several researchers both numerically and analytically. The latest numerical studies on this type of contact problems were either based on the finite element method (e.g., Jing and Liao, 1990; Satish Kumar et al., 1996) or on the boundary element method (Anderson, 1982; Garrido et al., 1991; Garrido and Lorenzana, 1998; París et al., 1992, 1995).

Among the analytical studies involving receding contact are those of Stippes et al. (1962) dealing with inclusions that can separate from the matrix and the works by Weitsman (1969) and Pu and Hussain (1970) who considered a layer pressed against a substrate. The results for various inclusions and more complete lists of references prior to 1970 on the receding contact problem can be found in the papers by Wilson and Goree (1967), Noble and Hussain (1967, 1969), Hussain et al. (1968) and Margetson and Morland (1970).

Keer et al. (1972) investigated the smooth receding contact problem between an elastic layer and a half-space formulated under the assumption of plane stress, plane strain and axisymmetric conditions. Gladwell (1976) solved the same problem by treating the layer as a simple beam in bending. Ratwani and Erdogan (1973) considered the plane smooth contact problem for an elastic layer lying on an elastic half-space with a compressive load applied to the layer through a frictionless rigid stamp. Civelek and Erdogan (1974) investigated the general axisymmetric double frictionless contact problem for an elastic layer pressed against a half-space by an elastic stamp under the assumptions that the three materials have different elastic properties.

Gecit (1986) studied the frictionless contact problem of a semi-infinite cylinder compressed against a half-space. Nowell and Hills (1988) considered the plane elastic contact problem between a thin strip and two symmetric rollers under the assumption of frictional sliding, frictionless and frictional indentation. Birinci and Erdol (1999) solved the frictionless contact problem between a flat-ended or rounded rigid stamp and two elastic layers. Birinci and Erdol (1999) also studied the continuous and discontinuous contact problem of a layered composite made of two elastic layers subject to a loaded rigid rectangular stamp. Comez et al. (2004) investigated the plane double receding frictionless contact problem for a loaded rigid stamp in contact with two different elastic layers.

The materials research community has recently been exploring the possibility of using new concepts in coating or layer design, such as functionally graded materials (FGMs), as an alternative to the conventional homogeneous coatings or layers. These can be two-phase inhomogeneous particulate composites synthesized in such a way that the volume fractions of the constituent materials, such as ceramic and metal, vary continuously along a spatial direction to give a predetermined composition profile resulting in a relatively smooth variation of the mechanical properties. FGMs appear to promise attractive applications in a wide variety of thermal shielding problems, such as high temperature chambers, furnace liners, turbines, microelectronics and space structures, as well as in various contact mechanics applications, such as gears and cams (Holt et al., 1992).

In this paper, we consider the plane problem of a frictionless receding contact between an elastic functionally graded layer and a homogeneous half-space when the two bodies are pressed together. The layer is subjected over a certain segment of its top surface to normal tractions while the rest of this surface is free of tractions. The mixed-boundary value problem is solved analytically using the singular integral equation method. The formulation and the solution of the contact problems are described, respectively, in Sections 2 and 3. The numerical solution of the resulting singular integral equation is summarized in Section 4. Finally, numerical results are discussed in Section 5.

2. Formulation of the contact problem

As shown in Fig. 1, the problem under consideration consists of an infinitely long functionally graded layer of thickness h in contact with a homogeneous semi-infinite medium. The graded layer and the half-space occupy, respectively, the domains $0 \leqslant y \leqslant h$ and $y \leqslant 0$. For the graded layer, the material is modeled as a nonhomogeneous isotropic material with a gradient oriented along the y-direction. The Poisson's ratio ν is assumed to be a constant and the shear modulus μ_1 depends on the y-coordinate only and is modeled by an exponential function expressed by

$$\mu_1(y) = \mu_0 \exp(\beta y), \quad 0 < y \leqslant h, \tag{1}$$

where μ_0 is the shear modulus in the homogeneous medium and β is the nonhomogeneity parameter controlling the variation of the shear modulus in the graded medium.

The graded medium is subjected to normal tractions $p(x)$ distributed over the segment $|x| \leqslant a$ of the top surface of the layer, while the rest of this surface is free of tractions. For simplicity, only the case when the applied normal tractions $p(x)$ are symmetric about the center of the loaded segment is considered. As the two bodies deform, contact between the graded layer and the homogeneous half-space is maintained over the segment $|x| \leqslant c$, while separation takes place outside this interval, where the contact is assumed frictionless. The contact normal tractions within this segment, denoted $q(x)$, and the variable c, called the receding contact half-length, are the main unknowns of this problem.

We denote by u and v, respectively, the x and y components of the displacement field and by σ_{xx}, σ_{yy} and σ_{xy} the components of the stress field in the same coordinate system. The corresponding components of the strain field are denoted ε_{xx}, ε_{yy} and ε_{xy}.

The receding contact problem may be solved by considering separately the graded layer and the homogeneous half-space. The equations of the plane problem in both domains are the equilibrium equations with body forces neglected, the strain-displacement relationships and the linear elastic stress–strain law which are, respectively, given by

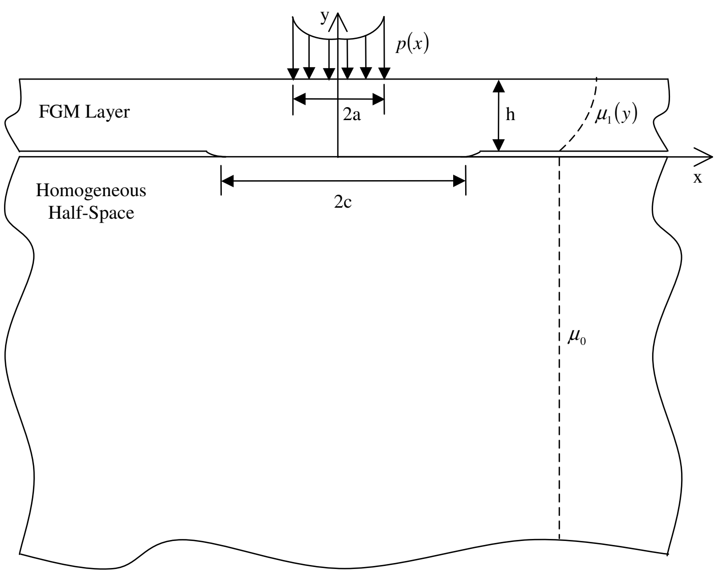

Fig. 1. Geometry and loading of the receding contact problem.

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=0,\quad \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0, \tag{2a, b}$$

$$\varepsilon_{xx}=\frac{\partial u}{\partial x},\quad \varepsilon_{yy}=\frac{\partial v}{\partial y},\quad \varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \tag{3a-c}$$

$$\sigma_{xx}=\frac{\mu_j}{\kappa-1}\left[(1+\kappa)\varepsilon_{xx}+(3-\kappa)\varepsilon_{yy}\right],\quad (j=0,1), \tag{4a, b}$$

$$\sigma_{yy}=\frac{\mu_j}{\kappa-1}\left[(3-\kappa)\varepsilon_{xx}+(1+\kappa)\varepsilon_{yy}\right],\quad (j=0,1), \tag{4c, d}$$

$$\sigma_{xy}=2\mu_j\varepsilon_{xy},\quad (j=0,1), \tag{4e, f}$$

where $\kappa = 3 - 4v$ for plane strain and $\kappa = (3 - v)/(1 + v)$ for generalized plane stress and μ_0 and μ_1 are defined in Eq. (1).

To solve the receding contact problem between the graded layer and the homogeneous half-space, we derive the following two-dimensional Navier's equations obtained by combining Eqs. (1)–(4):

$$(\kappa+1)\frac{\partial^2 u}{\partial x^2}+(\kappa-1)\frac{\partial^2 u}{\partial y^2}+2\frac{\partial^2 v}{\partial x\partial y}+\beta(\kappa-1)\frac{\partial u}{\partial y}+\beta(\kappa-1)\frac{\partial v}{\partial x}=0,\quad 0<y\leqslant h, \tag{5a}$$

$$(\kappa-1)\frac{\partial^2 v}{\partial x^2}+(\kappa+1)\frac{\partial^2 v}{\partial y^2}+2\frac{\partial^2 u}{\partial x\partial y}+\beta(3-\kappa)\frac{\partial u}{\partial x}+\beta(\kappa+1)\frac{\partial v}{\partial y}=0,\quad 0<y\leqslant h, \tag{5b}$$

$$(\kappa+1)\frac{\partial^2 u}{\partial x^2}+(\kappa-1)\frac{\partial^2 u}{\partial y^2}+2\frac{\partial^2 v}{\partial x\partial y}=0,\quad y\leqslant 0, \tag{5c}$$

$$(\kappa-1)\frac{\partial^2 v}{\partial x^2}+(\kappa+1)\frac{\partial^2 v}{\partial y^2}+2\frac{\partial^2 u}{\partial x\partial y}=0,\quad y\leqslant 0. \tag{5d}$$

For the graded layer, the plane elasticity Eqs. (5a,b) are subject to the following boundary conditions:

$$\sigma_{yy}(x,h)=-p(x)H(a-|x|),\quad \sigma_{xy}(x,h)=0,\quad |x|<+\infty, \tag{6, 7}$$

$$\sigma_{yy}(x,0^+)=-q(x)H(c-|x|),\quad \sigma_{xy}(x,0^+)=0,\quad |x|<+\infty, \tag{8, 9}$$

where H is the Heavyside function.

For the homogeneous medium, the plane elasticity Eqs. (5c,d) are subject to the following boundary conditions:

$$\sigma_{yy}(x,0^-)=-q(x)H(c-|x|),\quad \sigma_{xy}(x,0^-)=0,\quad |x|<+\infty. \tag{10, 11}$$

In addition, we assume that in both domains the stress field vanishes at infinity. In particular, we have

$$\sigma_{yy}(x,y)=0,\quad \sigma_{xy}(x,y)=0,\quad x^2+y^2\to\infty. \tag{12, 13}$$

By writing the global equilibrium of the FGM layer and by taking into account the fact that stresses vanish at infinity (i.e., Eqs. (12) and (13)), we deduce the following expression:

$$\int_{-c}^{+c} q(t)\,\mathrm{d}t=\int_{-a}^{+a} p(t)\,\mathrm{d}t. \tag{14}$$

Since the FGM layer and the homogeneous base are in smooth contact, then the vertical component of the displacement field across the contact segment is continuous:

$$v(x,0^+)=v(x,0^-),\quad |x|\leqslant c. \tag{15}$$

The above condition can be differentiated with respect to x which ensures continuity of the displacement along y and eliminates rigid-body displacements:

$$\frac{\partial}{\partial x}[v(x,0^+) - v(x,0^-)] = 0, \quad |x| \leqslant c. \tag{16}$$

3. Solution of the contact problem

The plane elasticity Eqs. (5a,b) and (5c,d) are solved separately using standard Fourier transforms with respect to the x-coordinate to yield the displacement field in both domains.

For the graded layer, the resulting expressions of the displacement field are given by

$$u(x,y) = \int_{-\infty}^{+\infty} \tilde{u}(\lambda,y)\mathrm{e}^{-\mathrm{i}x\lambda}\,\mathrm{d}\lambda, \quad v(x,y) = \int_{-\infty}^{+\infty} \tilde{v}(\lambda,y)\mathrm{e}^{-\mathrm{i}x\lambda}\,\mathrm{d}\lambda, \qquad 0 < y \leqslant h, \tag{17a, b}$$

where $\tilde{u}(\lambda,y)$ and $\tilde{v}(\lambda,y)$ are, respectively, the Fourier transforms of $u(x,y)$ and $v(x,y)$ which can be expressed as follows:

$$\tilde{u}(\lambda,y) = \sum_{k=1}^{4} C_k(\lambda)\mathrm{e}^{m_k y}, \quad \tilde{v}(\lambda,y) = \sum_{k=1}^{4} C_k(\lambda)s_k(\lambda)\mathrm{e}^{m_k y}, \quad 0 < y \leqslant h. \tag{18a, b}$$

In the above equations, the unknown functions $C_k(\lambda)$ $(k = 1,2,3,\ 4)$ are determined from the boundary conditions and $m_1,\dots,m_4$ are the four complex roots of the characteristic polynomial associated with the plane elasticity Eq. (5a,b), which may be written as

$$m^4 + 2\beta m^3 + (\beta^2 - 2\lambda^2)m^2 - 2\beta\lambda^2 m + \left(\lambda^4 + \lambda^2\beta^2\frac{3-\kappa}{1+\kappa}\right) = 0. \tag{19}$$

The roots of the above equation, denoted m_1, m_2, m_3 and m_4, are two by two complex conjugates where $m_3 = \bar{m}_1$ and $m_4 = \bar{m}_2$. The roots m_1 and m_2 are given by

$$m_1 = \frac{1}{2}\left(-\beta + \sqrt{\beta^2 + 4\lambda^2 + 4\mathrm{i}\lambda\beta\sqrt{\frac{3-\kappa}{1+\kappa}}}\right), \quad m_2 = \frac{1}{2}\left(-\beta - \sqrt{\beta^2 + 4\lambda^2 + 4\mathrm{i}\lambda\beta\sqrt{\frac{3-\kappa}{1+\kappa}}}\right). \tag{20a, b}$$

In Eq. (18b), the known functions $s_k(\lambda)(k = 1,2,3,4)$ may be expressed as follows:

$$s_k(\lambda) = \frac{(\kappa-1)m_k^2 + \beta(\kappa-1)m_k - \lambda^2(\kappa+1)}{\mathrm{i}\lambda(2m_k + \beta(\kappa-1))}, \quad (k = 1,\dots,4). \tag{21}$$

Substituting (17) into (3) and then the resulting expressions into (4) yields the stress field in the graded layer which are of interest and which may be written as follows:

$$\sigma_{yy}(x,y) = \int_{-\infty}^{+\infty} \tilde{\sigma}_{yy}(\lambda,y)\mathrm{e}^{-\mathrm{i}x\lambda}\,\mathrm{d}\lambda, \quad \sigma_{xy}(x,y) = \int_{-\infty}^{+\infty} \tilde{\sigma}_{xy}(\lambda,y)\mathrm{e}^{-\mathrm{i}x\lambda}\,\mathrm{d}\lambda, \quad 0 < y \leqslant h, \tag{22a, b}$$

where $\tilde{\sigma}_{yy}(\lambda,y)$ and $\tilde{\sigma}_{xy}(\lambda,y)$ are, respectively, the Fourier transforms of $\sigma_{yy}(x,y)$ and $\sigma_{xy}(x,y)$ which are given by

$$\tilde{\sigma}_{yy}(\lambda,y) = \frac{\mu_1(y)}{\kappa-1}\sum_{k=1}^{4} p_k\mathrm{e}^{m_k y}C_k(\lambda), \quad \tilde{\sigma}_{xy}(\lambda,y) = \mu_1(y)\sum_{k=1}^{4} q_k\mathrm{e}^{m_k y}C_k(\lambda), \quad 0 < y \leqslant h, \tag{23a, b}$$

in which the known functions $p_k(\lambda)$ and $q_k(\lambda)(k = 1,2,3,4)$ are given by

$$q_k(\lambda) = m_k - (i\lambda)s_k, \quad p_k(\lambda) = (1+\kappa)s_k m_k - (\mathrm{i}\lambda)(3-\kappa), \quad (k = 1,\dots,4). \tag{24a, b}$$

The plane elasticity Eqs. (5c,d) are solved in a similar manner to yield the expressions of the displacement in the homogeneous medium ($y \leqslant 0$) which have the same form as those for the FGM layer (Eqs. (17a,b)) but different expressions for the inverse Fourier transforms:

$$\tilde{u}(\lambda,y) = \tilde{u}_1(\lambda,y)e^{|\lambda|y} + \tilde{u}_2(\lambda,y)e^{-|\lambda|y}, \quad \tilde{v}(\lambda,y) = \tilde{v}_1(\lambda,y)e^{|\lambda|y} + \tilde{v}_2(\lambda,y)e^{-|\lambda|y}, \quad y \leqslant 0, \tag{25a, b}$$

where the functions $\tilde{u}_1(\lambda,y), \tilde{u}_2(\lambda,y), \tilde{v}_1(\lambda,y)$ and $\tilde{v}_2(\lambda,y)$ are given by

$$\tilde{u}_1(\lambda,y) = C_5(\lambda) + C_6(\lambda)y, \quad \tilde{u}_2(\lambda,y) = C_7(\lambda) + C_8(\lambda)y, \tag{26a, b}$$

$$\tilde{v}_1(\lambda,y) = C_5(\lambda)s_5 + C_6(\lambda)(s_5y + s_6), \quad \tilde{v}_2(\lambda,y) = -C_7(\lambda)s_5 + C_8(\lambda)(-s_5y + s_6), \tag{26c, d}$$

in which the known functions $s_5(\lambda)$ and $s_6(\lambda)$ are given by

$$s_5 = \frac{i\lambda}{|\lambda|}, \quad s_6 = -\frac{i\kappa}{\lambda}. \tag{27a, b}$$

The expressions of the stress field in the homogeneous medium which are of interest may be written as follows:

$$\sigma_{yy}(x,y) = \int_{-\infty}^{+\infty} \tilde{\sigma}_{yy}(\lambda,y)e^{-ix\lambda}\,d\lambda, \quad \sigma_{xy}(x,y) = \int_{-\infty}^{+\infty} \tilde{\sigma}_{xy}(\lambda,y)e^{-ix\lambda}\,d\lambda, \quad y \leqslant 0, \tag{28a, b}$$

where $\tilde{\sigma}_{yy}(\lambda,y)$ and $\tilde{\sigma}_{xy}(\lambda,y)$ are, respectively, the Fourier transforms of $\sigma_{yy}(x,y)$ and $\sigma_{xy}(x,y)$ which may be written, after applying the regularity conditions (12) and (13), as follows:

$$\tilde{\sigma}_{yy}(\lambda,y) = \frac{\mu_0}{\kappa - 1}(p_5C_5(\lambda) + (p_5y + p_6)C_6(\lambda))e^{|\lambda|y}, \quad y \leqslant 0, \tag{29a}$$

$$\tilde{\sigma}_{xy}(\lambda,y) = \mu_0(q_5C_5(\lambda) + (q_5y + q_6)C_6(\lambda))e^{|\lambda|y}, \quad y \leqslant 0, \tag{29b}$$

in which the known functions $p_k(\lambda)$ and $q_k(\lambda)(k = 5,6,7,8)$ are given by

$$p_5 = (2i\lambda)(\kappa - 1), \quad p_6 = (1 - \kappa^2)\frac{i\lambda}{|\lambda|}, \quad q_5 = 2|\lambda|, \quad q_6 = 1 - \kappa. \tag{30a-d}$$

Applying the boundary conditions (6)–(9) to the expressions of the stress field obtained for the FGM layer (22a,b) and using inverse Fourier transforms yields a linear algebraic system of equations in which the unknown functions $C_k(\lambda)(k = 1,2,3,4)$ are expressed in terms of the Fourier transforms of the known tractions p and the unknown contact stress q, as follows:

$$\begin{bmatrix} p_1e^{m_1h} & p_2e^{m_2h} & p_3e^{m_3h} & p_4e^{m_4h} \\ q_1e^{m_1h} & q_2e^{m_2h} & q_3e^{m_3h} & q_4e^{m_4h} \\ p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{Bmatrix} C_1(\lambda) \\ C_2(\lambda) \\ C_3(\lambda) \\ C_4(\lambda) \end{Bmatrix} = \begin{Bmatrix} -\tilde{p}(\lambda) \\ 0 \\ -\tilde{q}(\lambda) \\ 0 \end{Bmatrix} \tag{31}$$

where $\tilde{p}(\lambda)$ and $\tilde{q}(\lambda)$ are given by

$$\tilde{p}(\lambda) = \frac{\kappa - 1}{2\pi\mu_0e^{\beta h}}\int_{-a}^{+a} p(x)e^{ix\lambda}\,dx, \quad \tilde{q}(\lambda) = \frac{\kappa - 1}{2\pi\mu_0}\int_{-c}^{+c} q(x)e^{ix\lambda}\,dx. \tag{32a, b}$$

Eq. (31) is inverted analytically leading to the expressions of $C_k(\lambda)(k = 1,2,3,4)$ in terms of $\tilde{p}(\lambda)$ and $\tilde{q}(\lambda)$ as follows:

$$C_k(\lambda) = (-1)^k\left[\frac{D_{1k}}{D}\tilde{p}(\lambda) + \frac{D_{3k}}{D}\tilde{q}(\lambda)\right], \quad (k = 1,\ldots,4), \tag{33}$$

where D is the determinant and D_{jk} ($j = 1,3$; $k = 1,2,3,4$) is the subdeterminant (corresponding to the elimination of the jth row and kth column) of the coefficient matrix in the system of Eq. (31).

Applying the regularity conditions (12) and (13) to the expressions of the stress field obtained for the homogeneous medium (27a,b) yields $C_7(\lambda) = C_8(\lambda) = 0$. Furthermore, applying the boundary conditions (10) and (11) to this stress field produces the following expressions of $C_5(\lambda)$ and $C_6(\lambda)$ in terms of $\tilde{q}(\lambda)$:

$$C_5(\lambda) = \frac{-q_6}{p_5 q_6 - p_6 q_5}\tilde{q}(\lambda), \quad C_6(\lambda) = \frac{q_5}{p_5 q_6 - p_6 q_5}\tilde{q}(\lambda). \tag{34a, b}$$

Applying the remaining boundary condition (16) and using (31) and (32) yields the following singular integral equation, in which the unknowns are the contact pressure q and the receding contact half-length c

$$\int_{-c}^{+c} \overline{K}(x,t) q(t)\,\mathrm{d}t = \mathrm{e}^{-\beta h} \int_{-a}^{+a} p(t) f(x,t)\,\mathrm{d}t, \quad |x| \leqslant c, \tag{35}$$

where $f(x,t)$ is a known function and $\overline{K}(x,t)$ is the kernel of the integral equation whose expressions are given by

$$f(x,t) = \int_0^{\infty} A(\lambda) \sin[\lambda(t-x)]\,\mathrm{d}\lambda, \quad \overline{K}(x,t) = \lim_{y\to 0} \int_0^{+\infty} N(y,\lambda) \sin[\lambda(t-x)]\,\mathrm{d}\lambda, \tag{36a, b}$$

in which $A(\lambda)$ and $N(y,\lambda)$ are given in Appendix A.

It can be easily verified that $N(y,\lambda)$ in Eq. (36b) is bounded as λ goes to zero, but diverges as λ goes to infinity. The dominant part of the kernel may be separated by taking the asymptotic expansion of $N(y,\lambda)$ as λ goes to infinity. Using MAPLE, the asymptotic expansion of $N(y,\lambda)$ is given by the following expression:

$$N^{\infty}(y,\lambda) = \left(b_0 + \frac{b_1}{\lambda} + \frac{b_2}{\lambda^2} + \cdots + \frac{b_8}{\lambda^8}\right)\mathrm{e}^{\lambda y}, \tag{37}$$

where the coefficients b_0 to b_8 are given in Appendix A.

The Cauchy singularity can be extracted from the kernel given by Eq. (36b) as follows:

$$\overline{K}(x,t) = \lim_{y\to 0} \int_0^{+\infty} b_0 \mathrm{e}^{\lambda y} \sin[\lambda(t-x)]\,\mathrm{d}\lambda + \lim_{y\to 0} \int_0^{+\infty} \left[N(y,\lambda) - b_0 \mathrm{e}^{\lambda y}\right] \sin[\lambda(t-x)]\,\mathrm{d}\lambda, \tag{38}$$

which simplifies to

$$\overline{K}(x,t) = \frac{b_0}{(t-x)} + \int_0^{+\infty} [B(\lambda) - b_0] \sin[\lambda(t-x)]\,\mathrm{d}\lambda, \tag{39}$$

where $B(\lambda) = N(0,\lambda)$ and whose expression is given in Appendix A.

Substituting (39) into (35) and dividing the resulting equation by b_0 yields the following singular integral equation:

$$\int_{-c}^{+c} \left(\frac{1}{t-x} + k(x,t)\right) q(t)\,\mathrm{d}t = \frac{1}{b_0}\mathrm{e}^{-\beta h} \int_{-a}^{+a} p(t) f(x,t)\,\mathrm{d}t, \quad |x| \leqslant c, \tag{40}$$

in which $k(x,t)$ is a Fredholm kernel that depends on the nonhomogeneity parameter β and whose expression is given by

$$k(x,t) = \int_0^{+\infty} \left(\frac{1}{b_0} B(\lambda) - 1\right) \sin[\lambda(t-x)]\,\mathrm{d}\lambda, \tag{41}$$

where the evaluation of the above improper integral is shown in Appendix A.

In order to solve Eq. (40) for the contact pressure and the receding contact half-length, the global equilibrium condition of the graded layer (i.e., Eq. (14)) has to be supplemented.

4. Numerical solution of the singular integral equation

We apply the following normalizations and definitions to the singular integral Eq. (40) and to the additional condition (14):

$$s = \frac{t}{c}, \quad r = \frac{x}{c}, \quad q(t) = Q(s), \tag{42a-c}$$

$$k(x,t) = K(r,s), \quad f(x,t) = F(r,s), \quad p(t) = P(s). \tag{42d-f}$$

Eqs. (40) and (14) become

$$\int_{-1}^{+1} \left(\frac{1}{s-r} + cK(r,s)\right) Q(s)\,\mathrm{d}s = \frac{1}{b_0}\mathrm{e}^{-\beta h}\int_{-\frac{a}{c}}^{+\frac{a}{c}} cP(s)F(r,s)\,\mathrm{d}s, \quad |r| \leqslant 1, \tag{43}$$

$$\int_{-1}^{+1} Q(s)\,\mathrm{d}s = \int_{-\frac{a}{c}}^{+\frac{a}{c}} P(s)\,\mathrm{d}s. \tag{44}$$

It was shown in Erdogan and Gupta (1972) that the singular integral Eq. (43) has an index -1 because of the absence of singularities at the end points ± 1. Its solution may be expressed as $Q(s) = w(s)\phi(s)$ where $w(s) = \sqrt{1-s^2}$ is the weight function associated with the Chebyshev polynomials of the second kind $U_n(s) = \sin((n+1)\arccos(s))/\sqrt{1-s^2}$ and $\phi(s)$ is a continuous and bounded function in the interval $[-1,1]$ which may be expressed as a truncated series of Chebyshev polynomial of the second kind. Therefore, the solution of (43) may be expressed as

$$Q(s) = \sqrt{1-s^2}\sum_{n=0}^{N} a_n U_n(s), \quad |s| \leqslant 1. \tag{45}$$

The above solution is substituted in (43) resulting in the following equation which is linear in terms of the $(n+1)$ unknowns $a_0, \ldots, a_N$ but nonlinear in terms of last unknown being the receding contact half-length c:

$$\sum_{n=0}^{N}[-\pi T_{n+1}(r)]a_n + \sum_{n=0}^{N}[g(r)]ca_n = h(r), \quad |r| \leqslant 1, \tag{46}$$

where $T_{n+1}(r) = \cos((n+1)\arccos(r))$ is the Chebyshev polynomial of the first kind, $h(r)$ and $g(r)$, which are nonlinear functions of c, are both given by

$$h(r) = \frac{1-\kappa}{1+\kappa}\mathrm{e}^{-\beta h}\int_{-\frac{a}{c}}^{+\frac{a}{c}} cP(s)F(r,s)\,\mathrm{d}s, \quad g(r) = \int_{-1}^{+1}\sqrt{1-s^2}K(r,s)U_n(s)\,\mathrm{d}s. \tag{47a, b}$$

Eq. (46) may be solved by selecting a set of $N+1$ points using the collocation method proposed by Erdogan and Gupta (1972), as follows:

$$r_j = \cos\left(\frac{\pi}{2}\frac{2j-1}{N+1}\right), \quad (j = 1, \ldots, N+1), \tag{48}$$

Using the collocation points given by Eq. (48) to Eq. (46) yields a system of $(n+1)$ equations with $(n+2)$ unknowns, namely $a_0, \ldots, a_N$ and c which may be expressed as:

$$\sum_{n=0}^{N}[-\pi T_{n+1}(r_j)]a_n + \sum_{n=0}^{N}[g(r_j)]ca_n = h(r_j), \quad (j = 1, \ldots, N+1). \tag{49}$$

In order to solve for the $(n+2)$ unknowns, the system of equations given by (49) are supplemented by the global equilibrium condition (44) which after substituting the form of the solution given by (45) becomes:

$$\int_{-1}^{+1} \sqrt{1-s^2} \sum_{n=0}^{N} a_n U_n(s)\, \mathrm{d}s = \int_{-\frac{a}{c}}^{+\frac{a}{c}} P(s)\, \mathrm{d}s. \tag{50}$$

Since the system of equations given by (49) and (50) is nonlinear in terms of the variable c, an iterative procedure is used to solve for the unknowns. An initial estimate of the variable c is assumed, then the system of Eqs. (49) is solved for the $(n+1)$ unknowns $a_0, \ldots, a_N$. Eq. (50) is then used to verify if the global equilibrium of the graded layer is satisfied. Since the loading applied to the graded layer is known, then the right-hand term of (50) is always constant and the left-hand term of this equation varies from one iteration to another. Furthermore, based on the physics of the problem, if the right-hand term of (50) is larger in absolute value than the left-hand term, then the value of the variable c is increased or vice versa. An additional test of convergence of the variable c is also made by computing the relative error between the values of c obtained from two successive iterations.

5. Results and discussion

The geometry and coordinate system of the considered problem is shown in Fig. 1. We consider two cases of FGM materials, a compliant graded layer with $\beta h < 0$ and stiff graded layer with $\beta h > 0$. The loading applied on the graded layer, $p(t)$, can be either a concentrated force with $a/h = 0.01$ or a uniformly distributed loading with different values of $a/h (a/h = 1.0, 2.0, 4.0)$, such that the resultant force $\int_{-a/n}^{+a/h} p(t)\, \mathrm{d}t$ is always equal to -1.

Table 1 shows the number of iterations performed in order to reach the solution of the receding contact problem for the case of a concentrated load for different values of the stiffness parameter, βh $(=0.001, -1, +1)$. Since the applied load is independent of βh, then its resultant $\int_{-a/h}^{+a/h} p(t)\, \mathrm{d}t$ is also independent of βh and is equal to -1. Columns (2)–(5) of Table 1 correspond, respectively, to the iteration number, the resultant of the contact stress $\int_{-c/h}^{+c/h} q(t)\, \mathrm{d}t$, the normalized receding contact length c/h and the relative error of the normalized receding contact length between two successive iterations, for the case of a homogeneous layer (i.e., $\beta h = 0.001$). The same quantities are tabulated in columns (6)–(9) for the case of a compliant graded layer ($\beta h = -1$) and in columns (10)–(13) for the case of a stiff graded layer ($\beta h = +1$). The iterative solution is performed until (i) the relative error between the resultant contact stress $\int_{-c/h}^{+c/h} q(t)\, \mathrm{d}t$ and the resultant applied load $\int_{-a/h}^{+a/h} p(t)\, \mathrm{d}t$ becomes less than a tolerance of 10^{-6} which means verification of the global equilibrium of the graded layer (Eq. (50)); and (ii) the relative error of the normalized receding contact length between two successive iterations becomes less than 0.4%. As shown in Table 1, few iterations were required to solve the receding contact problem for the case of a homogeneous layer (i.e., $\beta h = 0.001$) and to show the fast convergence of the receding contact length. The corresponding normalized receding contact length is 1.3243 which is approximately the same value reported by Keer et al. (1972) (see Fig. 2 of this reference, case of $\alpha = 0$).

As an additional proof of the solution convergence, Erdogan and Gupta (1972) indicated that the unknown coefficients $a_0, \ldots, a_N$ of the truncated series of the solution $Q(s) = \sqrt{1-s^2}\sum_{n=0}^{N} a_n U_n(s)$, given by Eq. (45), must converge to a very small value as N increases. The results reported in this paper were obtained for $N = 10$. Table 2 shows the values of the coefficients $a_0, \ldots, a_N$ for the case where $\beta h = -1$ obtained for iterations 1, 4 and 7 (i.e., columns 6, 7 and 8 of Table 1). Table 2 clearly indicates that the coefficients $a_0, \ldots, a_N$ are converging.

Table 1
Solution iteration of the receding contact problem for different values of the stiffness parameter βh for the case of a concentrated load, $a/h = 0.01$

$\int_{-a/h}^{+a/h} p(t)\,dt$	$\beta h = 0.001$				$\beta h = -1$				$\beta h = +1$			
	Iteration no.	$\int_{-c/h}^{+c/h} q(t)\,dt$	c/h	E (%)[a]	Iteration no.	$\int_{-c/h}^{+c/h} q(t)dt$	c/h	E (%)[a]	Iteration no.	$\int_{-c/h}^{+c/h} q(t)dt$	c/h	E (%)[a]
	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
−1.0	1	−0.0481347	0.0100	...	1	−0.0774339	0.0100	...	1	−0.0299870	0.0100	...
	2	−0.9726017	0.2563	96.10	2	−0.9820959	0.2563	96.10	2	−0.9579621	0.2563	96.10
	3	−0.9999675	1.2975	80.25	3	−1.0001099	1.2975	80.25	3	−0.9995316	1.2975	80.25
	4	−1.0002957	1.9462	33.33	4	−1.0002560	1.7637	26.43	4	−1.0002184	1.9462	33.33
	5	−0.9999985	1.3228	−47.13	5	−0.9999785	1.1597	−52.08	5	−0.9998836	1.5015	−29.62
	6	−1.0000019	1.3257	0.22	6	−0.9999968	1.1752	1.32	6	−1.0000050	1.6079	6.62
	7	−1.0000002	1.3243	−0.11	7	−0.9999998	1.1778	0.22	7	−0.9999998	1.6026	−0.33

[a] E is the relative error of the receding contact length between two successive iterations given in percentage as follows: $E(\%) = \frac{(c/h)_i - (c/h)_{i-1}}{(c/h)_i} \times 100$ in which i is the iteration number.

Table 2
Convergence of the solution coefficients $a_0, \ldots, a_N$ given by Eq. (45) for the case where $\beta h = -1$ obtained for iterations 1, 4 and 7 (i.e., columns 6, 7 and 8 of Table 1)

Coefficient	Iteration no.		
	1	4	7
a_0	−4.92959801159705	−0.361049353779599	−0.540511414555690
a_1	−3.706837947265695E−04	0.214221243002300	0.165091250112821
a_2	7.707426909428907E−06	−0.102869936240056	−5.56937154087725E−02
a_3	−6.927363560588361E−08	4.550703611497985E−02	1.540345394723657E−02
a_4	−2.484711231619304E−06	−1.921238613754444E−02	−4.285582745191018E−03
a_5	3.590462272732949E−06	7.833850352914508E−03	1.075885334947709E−03
a_6	−2.259096455700216E−06	−3.121185077901784E−03	−3.287428981549372E−04
a_7	−1.117207576293463E−07	1.196122864467772E−03	5.708941240064792E−05
a_8	2.145955187670967E−06	−4.251373089290544E−04	1.731609295548627E−05
a_9	−3.870726907698476E−06	1.092087941965897E−04	−5.767047445102213E−05
a_{10}	5.425832001806739E−06	1.777047378899843E−06	7.761764153912499E−05
Verification of layer global equilibrium (Eq. (50))			
$\int_{-a/h}^{+a/h} p(t)\,\mathrm{d}t$	−1.0		
$\int_{-c/h}^{+c/h} q(t)\,\mathrm{d}t$	−0.0774339	−1.0002560	−0.9999998

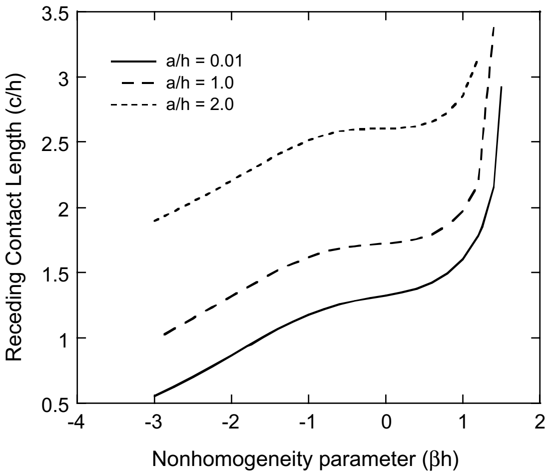

Fig. 2. Effect of the stiffness parameter βh [in $\mu_1(y) = \mu_0 \exp(\beta y)$] on the normalized receding contact length for a concentrated load and for various uniformly distributed loads.

Fig. 2 shows the normalized receding contact length, c/h, as a function of the stiffness parameter βh for a concentrated loading and various types of uniformly distributed loads ($a/h = 0.01, 1.0, 2.0, 4.0$). It can be seen that for a fixed value of βh, the normalized receding contact length, c/h, increases for increasing values of a/h. Therefore, the contact length corresponding to the applied concentrated force ($a/h = 0.01$) is smaller than that associated with the distributed loads. Examination of Fig. 2 indicates also that the contact length, c/h, increases for increasing values of the stiffness parameter βh. On the other hand, if the value of $\beta\ h$ becomes large in the positive sense, then the contact zone becomes large.

To further understand this point, we introduce the following quantity $D = \int_0^h y^2 \mu_1(y)\mathrm{d}y$ which represents the flexural rigidity of the graded layer because the layer can be assumed as a beam since its thickness is negligible compared to its length. Based on Eq. (1), the expression of layer's flexural rigidity becomes

$D = \int_0^h y^2 \mu_0 e^{\beta h(y/h)} dy$ which indicates that the rigidity is an exponential function of βh. As a result, when βh becomes large in the positive sense, both the flexural rigidity of the graded layer and the receding contact length increase. On the other hand, when βh is negative, the FGM layer's flexural rigidity is reduced leading to smaller contact zone with the homogeneous medium.

Figs. 3–6 illustrate the effect of the stiffness parameter $\beta h[\text{in}\mu_1(y) = \mu_0 \exp(\beta y)]$ on the normalized contact pressure for, respectively, the case of a concentrated load ($a/h = 0.01$) and the cases of various uniformly distributed loads ($a/h = 1.0,2.0,4.0$). From these figures, it can be concluded that for a fixed value of a/h, increasing the value of βh in the negative sense results in a reduction of the contact zone in addition to an increase of the peak of the contact pressure. The opposite effect can be observed when increasing the value of βh in the positive sense.

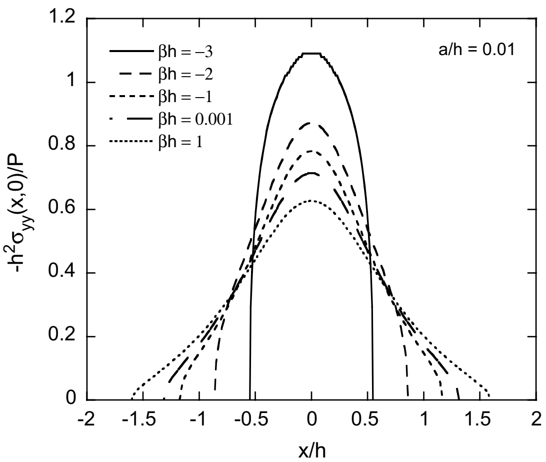

Fig. 3. Effect of the stiffness parameter βh [inμ_1 $(y) = \mu_0 \exp(\beta y)$] on the normalized contact pressure for the case of a concentrated load, $a/h = 0.01$.

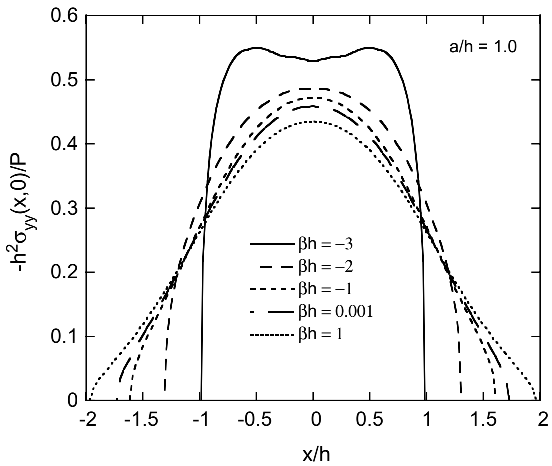

Fig. 4. Effect of the stiffness parameter βh [in μ_1 $(y) = \mu_0 \exp(\beta y)$] on the normalized contact pressure for the case of a uniformly distributed load, $a/h = 1.0$.

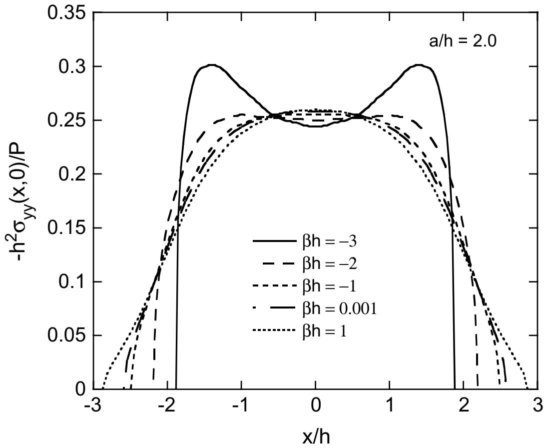

Fig. 5. Effect of the stiffness parameter βh [inμ_1 $(y) = \mu_0 \exp(\beta y)$] on the normalized contact pressure for the case of a uniformly distributed load, $a/h = 2.0$.

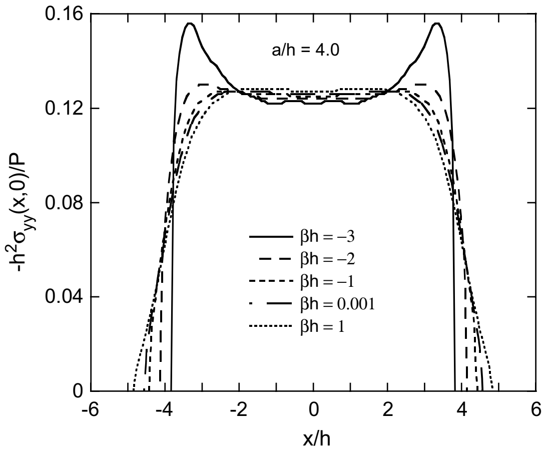

Fig. 6. Effect of the stiffness parameter βh [inμ_1 $(y) = \mu_0 \exp(\beta y)$] on the normalized contact pressure for the case of a uniformly distributed load, $a/h = 4.0$.

6. Conclusion

In this paper, the plane problem of a frictionless receding contact of a functionally graded layer pressed against a homogeneous half-space was considered. The layer was subjected over a certain segment of its top surface to normal tractions while the rest of this surface was free of tractions. Using standard Fourier transforms, the plane elasticity equations were converted analytically into a singular integral equation in which the unknowns are the contact pressure and the receding contact half-length. The global equilibrium condition of the layer was supplemented to solve the problem. The singular integral equation was solved numerically using Chebychev polynomials and an iterative scheme was employed to obtain the correct receding contact half-length that satisfies the global equilibrium condition. A detailed parametric study was conducted to investigate the effect of the material nonhomogeneity parameter of the graded layer βh on the contact pressure and on the length of the receding contact for different thicknesses of the layer.

Different conclusions were reached when the graded layer is either compliant (i.e., $\beta h < 0$) or stiff (i.e., $\beta h > 0$). When βh is negative and its absolute value is increased, an the peak of the contact pressure is increased and the layer's flexural rigidity is reduced leading to smaller contact zone with the homogeneous medium. For large negative values of βh, the receding contact zone becomes small. On the other hand, when βh is positive and is increased, the peak of the contact pressure is decreased and the layer's flexural rigidity is increased resulting in a larger contact zone.

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Appendix A

A.1. Expressions of quantities appearing in Eq. (36a,b)

$$A(\lambda) = (2\lambda)\left[\sum_{k=1}^{4}(-1)^k \frac{D_{1k}}{D} s_k\right], \tag{A.1}$$

$$N(y,\lambda) = (2\lambda)\left[\left(\sum_{k=1}^{4}(-1)^k \frac{D_{3k}}{D} s_k e^{m_k y}\right) + \frac{q_6 s_5 - q_5 s_6}{p_5 q_6 - p_6 q_5} e^{\lambda y}\right]. \tag{A.2}$$

where D is the determinant and D_{jk} ($j = 1,3$; $k = 1,2,3,4$) is the subdeterminant (corresponding to the elimination of the jth row and kth column) of the coefficient matrix in the system of Eqs. (31).

A.2. Expressions of quantities appearing in Eq. (37)

$$b_0 = \frac{1+\kappa}{1-\kappa}, \quad b_1 = \frac{\beta(\kappa+5)}{2(\kappa-1)}, \quad b_2 = -\frac{5\beta^2}{2(\kappa-1)}, \tag{A.3–5}$$

$$b_3 = \frac{\beta^3}{\kappa-1}, \quad b_4 = -\frac{\beta^4(5\kappa-8)}{8(\kappa^2-1)}, \quad b_5 = \frac{\beta^5(\kappa-3)}{4(\kappa^2-1)}, \tag{A.6–8}$$

$$\begin{aligned} b_6 &= -\frac{\beta^6(5\kappa^2-36\kappa+57)}{32(\kappa-1)(\kappa+1)^2}, \\ b_7 &= -\frac{\beta^7(\kappa^{10}-2\kappa^9-13\kappa^8+8\kappa^7+76\kappa^6+48\kappa^5)}{128(\kappa-1)(\kappa+1)^2} \\ &\quad -\frac{\beta^7(-138\kappa^4-256\kappa^3-129\kappa^2+138\kappa-117)}{128(\kappa-1)(\kappa+1)^2}, \end{aligned} \tag{A.9–10}$$

$$\begin{aligned} b_8 &= \frac{\beta^8(\kappa^{12}-2\kappa^{11}-6\kappa^{10}+2\kappa^9-25\kappa^8+8\kappa^7+364\kappa^6)}{256(\kappa-1)(\kappa+1)^3} \\ &\quad +\frac{\beta^8(522\kappa^5-457\kappa^4-1800\kappa^3-1854\kappa^2-1538\kappa-1205)}{256(\kappa-1)(\kappa+1)^3}, \end{aligned} \tag{A.11}$$

A.3. Expressions of quantities appearing in Eq. (39)

$$B(\lambda) = (2\lambda)\left[\left(\sum_{k=1}^{4}(-1)^k \frac{D_{3k}}{D}s_k\right) + \frac{q_6 s_5 - q_5 s_6}{p_5 q_6 - p_6 q_5}\right]. \tag{A.12}$$

A.4. Evaluation of the Fredholm kernel $k(x, t)$ given by Eq. (41)

The Freholm kernel $k(x,t)$ given by Eq. (41) is evaluated by dividing the integral from 0 to A and from A to ∞ as follows:

$$k(x,t) = \int_0^A \left(\frac{1}{b_0}B(\lambda) - 1\right)\sin[\lambda(t-x)]\,\mathrm{d}\lambda + \int_A^{+\infty}\left(\frac{1}{b_0}B(\lambda) - 1\right)\sin[\lambda(t-x)]\,\mathrm{d}\lambda, \tag{A.13}$$

where A is an integration cut-off point.

The second integral is further divided by adding and subtracting the asymptotic development of the function $(\frac{1}{b_0}B(\lambda) - 1)$ (see Delale and Erdogan, 1983). The above equation becomes

$$\begin{aligned} k(x,t) = &\int_0^A \left(\frac{1}{b_0}B(\lambda) - 1\right)\sin[\lambda(t-x)]\,\mathrm{d}\lambda \\ &+ \int_A^{+\infty}\left\{\left(\frac{1}{b_0}B(\lambda) - 1\right) - \left(\frac{1}{b_0}B(\lambda) - 1\right)^{\infty}\right\}\sin[\lambda(t-x)]\,\mathrm{d}\lambda \\ &+ \int_A^{+\infty}\left(\frac{1}{b_0}B(\lambda) - 1\right)^{\infty}\sin[\lambda(t-x)]\,\mathrm{d}\lambda, \end{aligned} \tag{A.14}$$

which can be rewritten as follows:

$$\begin{aligned} k(x,t) = &\int_0^A \left(\frac{1}{b_0}B(\lambda) - 1\right)\sin[\lambda(t-x)]\,\mathrm{d}\lambda \\ &+ \int_A^{+\infty}\left\{\left(\frac{1}{b_0}B(\lambda) - 1\right) - \left(\frac{b_1/b_0}{\lambda} + \frac{b_2/b_0}{\lambda^2} + \cdots + \frac{b_8/b_0}{\lambda^8}\right)\right\}\sin[\lambda(t-x)]\,\mathrm{d}\lambda \\ &+ \int_A^{+\infty}\left(\frac{b_1/b_0}{\lambda} + \frac{b_2/b_0}{\lambda^2} + \cdots + \frac{b_8/b_0}{\lambda^8}\right)\sin[\lambda(t-x)]\,\mathrm{d}\lambda. \end{aligned} \tag{A.15}$$

The first integral can be computed numerically using Gauss–Quadrature technique and the second integral becomes negligible for sufficiently large values of A (see Delale and Erdogan, 1983; Erdogan and Gupta, 1972). The third integral is computed in closed-form using the following expressions (Chen, 1990):

$$\begin{aligned} \int_A^{+\infty}\frac{\sin[\lambda(t-x)]}{\lambda^{2n-1}}\,\mathrm{d}\lambda = &\cos[A(t-x)]\sum_{i=1}^{n-1}\frac{(-1)^{i+1}(t-x)^{2(i-1)}(2n-2i-2)!}{(2n-2)!A^{2n-2i-1}} + \sin[A(t-x)] \\ &\times\sum_{i=1}^{n}\frac{(-1)^{i+1}(t-x)^{2(i-1)}(2n-2i-1)!}{(2n-2)!A^{2n-2i}} + (-1)^{n+1}\frac{(t-x)^{2n-2}}{(2n-2)!}\mathrm{si}[A(t-x)], \end{aligned} \tag{A.16}$$

$$\int_{A}^{+\infty}\frac{\sin[\lambda(t-x)]}{\lambda^{2n}}\,d\lambda=\cos[A(t-x)]\sum_{i=1}^{n-1}\frac{(-1)^{i+1}(t-x)^{2i-1}(2n-2i-1)!}{(2n-1)!A^{2n-2i}}+\sin[A(t-x)]\times\sum_{i=1}^{n}\frac{(-1)^{i+1}(t-x)^{2(i-1)}(2n-2i)!}{(2n-1)!A^{2n-2i+1}}+(-1)^{n}\frac{(t-x)^{2n-1}}{(2n-1)!}\mathrm{Ci}[A(t-x)],\tag{A.17}$$

where n is a positive integer ($n = 1,2,\ldots$), $\mathrm{Ci}(z)$ is the cosine integral and $\mathrm{si}(z)$ is a function of the sine integral $\mathrm{Si}(z)$ and whose expressions are given by

$$\mathrm{Ci}(z)=-\int_{z}^{+\infty}\frac{\cos t}{t}\,dt=\gamma_0+\ln|z|+\int_{0}^{|z|}\frac{\cos t-1}{t}\,dt,\quad \mathrm{si}(z)=\mathrm{Si}(z)-\mathrm{sign}(z)\frac{\pi}{2},\tag{A.18}$$

in which $\mathrm{Si}(z)=\int_{0}^{z}\frac{\sin t}{t}\,dt$ and $\gamma_0 = 0.57721566490$ is the Euler's constant.

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