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Berge's conjecture on directed path partitions—a survey

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Abstract

Berge's conjecture from 1982 on path partitions in directed graphs generalizes and extends Dilworth's theorem and the Greene–Kleitman theorem which are well known for partially ordered sets. The conjecture relates path partitions to a collection of k independent sets, for each $k \geqslant 1$. The conjecture is still open and intriguing for all $k > 1$.[1] In this paper, we will survey partial results on the conjecture, look into different proof techniques for these results, and relate the conjecture to other theorems, conjectures and open problems of Berge and other mathematicians.

Keywords: Path partitions; Partial colourings; Conjugate partitions; Strongly connected digraphs; Diperfect graphs

1. General introduction and motivation

Claude Berge is famous for his Perfect Graph Conjecture [9] which was recently proved [27]. In this survey we discuss another, less famous, but no less elegant conjecture related to path partitions in directed graphs. The conjecture is still open, and still intrigues many mathematicians. In this paper, we will survey partial results on the conjecture, some of which are fascinating and deep theorems by themselves. We will look into different proof techniques for these theorems, and relate the conjecture to other conjectures of Berge and others. A precise statement of the conjecture will be given in Section 2.

We begin with definitions and notations which hold throughout this paper: let $G = (V, E)$ be a loopless directed graph (or, for brevity, digraph) defined by a set V of vertices and a set $E \subseteq V \times V$ of directed edges. A *path* P in G is a sequence of distinct vertices $(v_1, v_2, \ldots, v_l)$ such that $(v_i, v_{i+1}) \in E$, for $i = 1, 2, \ldots, l - 1$. The set of vertices $\{v_1, v_2, \ldots, v_l\}$ of a path $P = (v_1, v_2, \ldots, v_l)$ is denoted by $V(P)$. The *cardinality* of P, denoted by $|P|$, is $|V(P)|$. We let $|V(G)| = n$. For other concepts, we adopt the terminology of [19].

A family $\mathscr{P}$ of paths is called a *path partition* of G if its members are vertex disjoint and $\cup\{V(P); P \in \mathscr{P}\} = V$. A directed graph may have many path partitions. The *trivial* path partition, where every path is a single vertex, is an example of a path partition. Let $\pi(G)$ denote the minimum number of paths in any path partition of G. In other words, $\pi(G) = \min |\mathscr{P}|$, where the minimum is taken over all path partitions $\mathscr{P}$. A path partition with a minimum number of paths is called *optimal*. Denote by $\alpha(G)$ the *independence number* (or *stability number*, in Berge's terminology),

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[1] Only recently it was proved Berger and Ben-Arroyo Hartman [56] for $k = 2$ (added in proof).

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that is, the maximum number of pairwise non-adjacent vertices in G. The following is a classical result of Gallai and Milgram:

Theorem 1.1 (*Gallai and Milgram [36]*). *Every directed graph satisfies* $\pi(G) \leqslant \alpha(G)$.

If G is acyclic and transitive, i.e., the digraph of a partially ordered set (poset), then every path in a path partition is, in fact, a clique which can meet an independent set at most once. Hence, the size of a maximum independent set is at most the number of paths in any path partition, and $\pi(G) \geqslant \alpha(G)$. Together with Theorem 1.1 we get the following well known theorem:

Theorem 1.2 (*Dilworth [30]*). *If G is a digraph of a poset then* $\pi(G) = \alpha(G)$. *Furthermore, every path in an optimal path partition in G meets every maximum independent set exactly once.*

Definition 1.3. We say that an independent set S is *orthogonal* to a path partition $\mathscr{P}$ if S meets every path in $\mathscr{P}$ at least once.

The Gallai–Milgram theorem can be proved by induction on the number of vertices. From the proof, a stronger result follows. Denote by $\mathrm{in}(\mathscr{P})$ and $\mathrm{ter}(\mathscr{P})$ the sets of initial and terminal vertices, respectively, of paths in $\mathscr{P}$.

Theorem 1.4 (*Linial [46]*). *Let $\mathscr{P}$ be a path partition of a digraph G. If there exists no independent set S orthogonal to $\mathscr{P}$, then there exists a path partition $\mathscr{Q}$ containing fewer paths than $\mathscr{P}$ and with* $\mathrm{in}(\mathscr{Q}) \subsetneq \mathrm{in}(\mathscr{P})$.

Proof. By induction on $n = |V|$. For $n = 2$ the theorem obviously holds. Let $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ be a path partition of G. By assumption, the set of initial vertices of $\mathscr{P}$ is not an independent set, therefore, there exists some edge (a_i, a_j) where vertices a_i and a_j are initial vertices of some paths P_i and P_j, respectively, in $\mathscr{P}$. If (a_i) is a path (of cardinality one) in $\mathscr{P}$, then we can attach the edge (a_i, a_j) to the path beginning at a_j and we are done. We assume, therefore, that a_i has a successor in P_i, which we call b_i. Consider now the digraph $G' = G \backslash a_i$. The path partition $\mathscr{P}' = \{P_1, P_2, \ldots, P_i \backslash a_i, \ldots, P_m\}$ does not have an independent orthogonal set, because if it did, then $\mathscr{P}$ would have one, contrary to the assumption in the theorem. Therefore, by the induction hypothesis, there exists a path partition $\mathscr{Q}'$ of G' satisfying $\mathrm{in}(\mathscr{Q}') \subsetneq \mathrm{in}(\mathscr{P}')$. There are three cases to be considered. In each case we shall show how to construct a new path partition $\mathscr{Q}'$ from $\mathscr{Q}$ satisfying the conditions of the theorem.

1. If $b_i \in \mathrm{in}(\mathscr{Q}')$ we attach a_i to the path beginning with b_i.
2. If $b_i \notin \mathrm{in}(\mathscr{Q}')$ and $a_j \in \mathrm{in}(\mathscr{Q}')$ we attach a_i to the path beginning with a_j.
3. If $b_i, a_j \notin \mathrm{in}(\mathscr{Q}')$ then $\mathscr{Q}'$ contains at least two fewer paths than $\mathscr{Q}$, and we can add the path (a_i) to $\mathscr{Q}'$.

In all three cases, we have constructed a path partition $\mathscr{Q}$ of G with $\mathrm{in}(\mathscr{Q}) \subsetneq \mathrm{in}(\mathscr{P})$, as required. □

In the above proof, a stronger induction hypothesis can be assumed, where the path partition $\mathscr{Q}$ satisfies $\mathrm{ter}(\mathscr{Q}) \subsetneq \mathrm{ter}(\mathscr{P})$, as well as $\mathrm{in}(\mathscr{Q}) \subsetneq \mathrm{in}(\mathscr{P})$. Using this hypothesis, the Proof of Theorems 1.1 and 1.4 demonstrate an efficient algorithm where, for any path partition $\mathscr{P}$, we can either find another path partition $\mathscr{Q}$ with fewer paths satisfying $\mathrm{in}(\mathscr{Q}) \subsetneq \mathrm{in}(\mathscr{P})$ and $\mathrm{ter}(\mathscr{Q}) \subsetneq \mathrm{ter}(\mathscr{P})$, or find an independent set orthogonal to $\mathscr{P}$ (see Lemma in [18]).

Berge [13] extended Theorem 1.4 by considering an *arborescence* (i.e. a directed rooted tree, where all edges are directed away from the root) instead of a path. Thus, the equivalent notion of a path partition would be an *arborescence forest*, i.e. a spanning subgraph of G where each component is an arborescence. Denote the set of roots of an arborescence $\mathscr{H}$ by $\mathrm{in}(\mathscr{H})$ and the set of sinks by $\mathrm{ter}(\mathscr{H})$. A maximal path in $\mathscr{H}$ ending in a sink with no vertex x with $\deg^+_{\mathscr{H}}(x) \geqslant 2$ is called a *terminal branch* of $\mathscr{H}$.

Theorem 1.5 (*Berge [13]*). *Let $\mathscr{H}$ be an arborescence forest of G such that there exists no other arborescence forest $\mathscr{H}'$ with* $\mathrm{in}(\mathscr{H}') \subsetneq \mathrm{in}(\mathscr{H})$ *and* $\mathrm{ter}(\mathscr{H}') \subsetneq \mathrm{ter}(\mathscr{H})$. *Then there exists an independent set which meets every terminal branch of $\mathscr{H}$.*

The proof of Theorem 1.5 is by induction, similar to the proof of Theorem 1.4.

We shall now consider extensions of the theorems and proofs presented in this section along the lines of the famous Greene–Kleitman theorem.

2. Extending the Greene–Kleitman theorem

2.1. Introduction and definitions

Dilworth's theorem (Theorem 1.2) was generalized by Greene and Kleitman [38] by considering a set of k disjoint independent sets ($1 \leqslant k \leqslant n$), (or *antichains*) instead of one independent set (*antichain*).

Definition 2.1 (*k-colouring*). A *k-colouring* is a family $\mathscr{C}^k = \{C_1, C_2, \dots, C_k\}$ of k disjoint independent sets called *colour classes*. (Some of the colour classes may be empty). The cardinality of a k-colouring $\mathscr{C}^k = \{C_1, C_2, \dots, C_k\}$ is $|\mathscr{C}^k| = \sum_{i=1}^{k} |C_i|$ and $\mathscr{C}^k$ is said to be *optimal* if $|\mathscr{C}^k|$ is as large as possible. Denote by $\alpha_k(G)$ the cardinality of an optimal k-colouring in G.

Berge [10] called a k-colouring a *partial k-colouring*, because it is, in fact, a partial colouring of the vertices of a graph using k colours. We prefer the shorter name *k-colouring*. In the cases of Theorems 1.1 and 1.2 we considered path partitions with a minimum number of paths. We need here an extension of this norm.

Definition 2.2 (*k-norm of a path partition*). For each positive integer k, the k-norm $|\mathscr{P}|_k$ of a path partition $\mathscr{P} = \{P_1, P_2, \dots, P_m\}$ is defined by

$$|\mathscr{P}|_k = \sum_{i=1}^{m} \min\{|P_i|, k\}.$$

A partition which minimizes $|\mathscr{P}|_k$ is called *k-optimal*. Denote by $\pi_k(G)$ the k-norm of a k-optimal path partition in G.

Note that a 1-optimal path partition is a partition that contains a minimum number of paths, and $\pi_1(G) = \pi(G)$. Denote by $\mathscr{P}^+$ the set of paths in $\mathscr{P}$ of cardinality more than k, (which we also call *long* paths), and by $\mathscr{P}^0$ the set of paths in $\mathscr{P}$ of cardinality at most k, (called *short* paths). For a set of vertex disjoint paths $\mathscr{P}$, we write $V[\mathscr{P}] = \cup\{V(P);\ P \in \mathscr{P}\}$. Now Definition 2.2 can be alternatively written as

$$|\mathscr{P}|_k := \sum_{i=1}^{m} \min\{|P_i|, k\} = k|\mathscr{P}^+| + |V[\mathscr{P}^0]|.$$

We are now able to state the Greene–Kleitman theorem:

Theorem 2.3 (*Greene–Kleitman theorem [38]*). *Let G be a digraph of a poset, and let k be a positive integer. Then* $\alpha_k(G) = \pi_k(G)$.

Observation 2.4. (1) *For* $k = 1$, *Theorem* 2.3 *is identical to Dilworth's theorem* (*Theorem* 1.2).

(2) *If G is transitive and, hence, acyclic (i.e. a digraph of a poset), then it is trivial to prove that* $\alpha_k(G) \leqslant \pi_k(G)$.

Observation 2.4(2) follows from the fact that each path P_i meets any k-colouring in at most $\min\{|P_i|, k\}$ vertices (since a colour class and a path (i.e. a clique), can meet by at most one vertex). Hence, for any k-colouring $\mathscr{C}^k = \{C_1, C_2, \dots, C_k\}$ and any path partition $\mathscr{P} = \{P_1, P_2, \dots, P_m\}$ we have

$$|\mathscr{C}^k| = \sum_{i=1}^{m} |V[\mathscr{C}^k] \cap V(P_i)| \leqslant \sum_{i=1}^{m} \min\{|P_i|, k\} = |\mathscr{P}|_k \quad (1)$$

(The first equality follows from the fact that a path partition partitions the vertex set of a graph into disjoint sets.)

In particular, if we take an optimal k-colouring and a k-optimal path partition then we obtain $\alpha_k(G) \leqslant \pi_k(G)$, for transitive digraphs.

From Observation (2), together with Theorem 2.3, we conclude:

Corollary 2.5. *Let G be a digraph of a poset, k a positive integer, $\mathscr{P}$ a k-optimal path partition, and $\mathscr{C}^k$ an optimal k-colouring. Then $\mathscr{C}^k$ meets every path in $\mathscr{P}$ in exactly* $\min\{|P_i|, k\}$ *vertices.*

If G is not a graph of a poset, then a colour class (or a k-colouring) may meet a path P more than once (more than k times), thus, it is possible that the opposite inequality holds in Eq. (1).

Linial [47] conjectured the following extension of the Greene–Kleitman theorem to all digraphs:

Conjecture 2.6 (*Linial [47]*). *Let G be a digraph and k a positive integer. Then $\alpha_k(G) \geqslant \pi_k(G)$.*

Conjecture 2.6 can be referred to as the "*weak path partition conjecture*". Berge was intrigued by the relationship between k-optimal path partitions and optimal k-colourings as reflected in Corollary 2.5. The following definitions captures this special relationship:

Definition 2.7 (*Orthogonality of path partitions and k-colourings*). A k-colouring $\mathscr{C}^k$ is *orthogonal* to a path partition $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ if $\mathscr{C}^k$ meets every path in $\mathscr{P}$ in $\min\{|P_i|, k\}$ different colour classes.

Clearly, the maximum number of colours that a path P can meet a k-colouring is $\min\{|P|, k\}$. Berge [10] used a different term for orthogonality, defining a k-colouring $\mathscr{C}^k$ to be *strong for a path P* if $\mathscr{C}^k$ meets P in exactly $\min\{|P|, k\}$ different colour classes. We prefer the concept of *orthogonality* as used in [33,4].

We are now ready to state Berge's conjecture extending the Greene–Kleitman theorem, a conjecture which is still open today.

Conjecture 2.8 (*Berge's strong path partition conjecture [10]*). *Let G be a digraph and let k be a positive integer. Then for every k-optimal path partition $\mathscr{P}$ there exists a k-colouring orthogonal to it.*

Conjecture 2.8 implies Conjecture 2.6 as follows: let $\mathscr{P}$ be a k-optimal path partition and let $\mathscr{C}^k$ be a k-colouring orthogonal to it. Then

$$\alpha_k(G) \geqslant |\mathscr{C}^k| = \sum_{P \in \mathscr{P}} |V[\mathscr{C}^k] \cap V(P)| \geqslant \sum_{P \in \mathscr{P}} \min\{|P|, k\} = |\mathscr{P}|_k = \pi_k(G).$$

Berge observed in [10] that Conjecture 2.8 holds in the following special cases:

1. For $k = 1$, by Theorem 1.4.
2. In the case that the k-optimal path partition contains only short paths, i.e. paths of cardinality at most k. This implies that the conjecture also holds for $k \geqslant \lambda$.
3. For digraphs containing a Hamilton path P_0. In this case $\mathscr{P} = \{P_0\}$ is a k-optimal path partition with $\mathscr{P}|_k = k$, and any k-colouring with no empty colour classes is orthogonal to it.
4. For bipartite digraphs.

If G is acyclic then Conjecture 2.6 was shown to be true in [47] and independently in [20]. Moreover, Conjecture 2.8 was proved for acyclic digraphs in [50,21,4,57]. In fact, a stronger result holds for acyclic digraphs:

Theorem 2.9 (*Saks [50], Cameron [21], Aharoni et al. [4]*). *Let G be an acyclic digraph, and let k be a positive integer. Then there exists a k-colouring $\mathscr{C}^k$ which is orthogonal to every k-optimal path partition $\mathscr{P}$ of G.*

Obviously, Theorem 2.9 implies Berge's conjecture (Conjecture 2.8) in the case that G is acyclic. Theorem 2.9 is not true for all digraphs, as was shown by a counter-example in [4].

Aharoni and Hartman [3] proved that Berge's conjecture holds also in the case that the k-optimal path partition has only *long* paths, i.e. paths of cardinality at least k. Only recently, Berger and Ben-Arroyo Hartman [56] proved Berge's conjecture for $k=2$ (added in proof).

Another possible extension of Theorem 2.3 is by considering a partition of V into disjoint paths and cycles, called *path-cycle* partition. The k-norm of a path-cycle partition with m paths and an arbitrary number of cycles is defined as

$$|\mathscr{P}^c|_k = \sum_{i=1}^{m} \min\{|P_i|, k\} = k|\mathscr{P}^+| + |V[\mathscr{P}^0]|.$$

Note that by this definition cycles do not count in the k-norm. Similarly, a path-cycle partition $\mathscr{P}^c$ and a k-colouring $\mathscr{C}^k$ are *orthogonal* if each path $P_i \in \mathscr{P}^c$ meets $\min\{|P|, k\}$ different colour classes of $\mathscr{C}^k$, and each cycle $C_i \in \mathscr{P}^c$ can meet any arbitrary number of vertices in $\mathscr{C}^k$. With this new definition of orthogonality, Hartman [7] proved, using the proof technique described in Section 2.2.1, the following:

Theorem 2.10 (*Hartman [7]*). *Let G be any digraph, and let k be a positive integer. Then there exists a k-colouring $\mathscr{C}^k$ which is orthogonal to every k-optimal path-cycle partition $\mathscr{P}^c$ of G.*

Another partial result on Berge's conjecture is by Sridharan [53,54], who proved it in the case that the underlying undirected graph has no two cycles intersecting in a unique vertex.

In the following section we shall discuss various proof techniques for proving Berge's conjecture (Conjecture 2.8) for acyclic digraphs.

2.2. *Proof techniques for extending the Greene–Kleitman theorem*

As we mentioned in the previous section, Berge's conjecture (Conjecture 2.8) holds for acyclic digraphs, and for general digraphs only few specific cases are known. We shall discuss here some proof techniques used for proving it in the acyclic case. Perhaps one of these techniques can be adapted to prove the conjecture for all digraphs.

2.2.1. *Linear programming*

Several researchers (see [4,21,23,29]) used LP techniques to extend the theorem of Greene and Kleitman to acyclic digraphs. We shall describe here the recent proof suggested by Pierre Charbit [25].

Let $G=(V,E)$ be an acyclic digraph where $|V|=n$.

First, we modify the graph G in order to construct G' in the following way: add a new vertex v_0 to G, and add an edge in each direction from v_0 to every vertex of G and finally we add a loop for every vertex of G. Now, since G is acyclic, a path partition of G is a cycle cover of G', where all cycles greater than one go through v_0, and no two cycles have any other common vertex. We define the classic linear program of finding an integer flow of G' such that all vertices v_i, $i \geqslant 1$, have out flow of value one.

Define the following matrix:

$$\mathbf{A} = \begin{bmatrix} \mathbf{M} \\ \mathbf{M}_+ \end{bmatrix},$$

where $\mathbf{M}=(m_{ij})$ is the $|V(G)| \times |E(G')|$ incidence matrix of G', where the row corresponding to v_0 is not included:

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the head of edge } a_j, \\ -1 & \text{if } v_i \text{ is the tail of edge } a_j, \\ 0 & \text{otherwise} \end{cases}$$

and $\mathbf{M}_+ = (n_{ij})$ is the $|V(G)| \times |E(G')|$ incidence matrix representing the outdegree of G':

$$n_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the tail of } a_j, \\ 0 & \text{otherwise.} \end{cases}$$

We now define vectors $\mathbf{b} = (b_1, \dots, b_{2n})$ and $\mathbf{c} = (c_1, \dots, c_{|E(G')|})$ as follows:

$$b_i = \begin{cases} 0 & \text{if } 1 \leqslant i \leqslant n, \\ 1 & \text{if } 1+n \leqslant i \leqslant 2n \end{cases}$$

and

$$c_j = \begin{cases} k & \text{if } v_0 \text{ is the tail of } a_j, \\ 1 & \text{if } a_j \text{ is a loop}, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the linear program (P):

$$\begin{array}{rl} \text{minimize} & \mathbf{cx} \\ \text{subject to} & \mathbf{Ax} = \mathbf{b}, \\ & \mathbf{x} \geqslant \mathbf{0}. \end{array}$$

Since the matrix $\mathbf{A}$ is totally unimodular, there exist integral optimal solutions to (P). The system of constraints $\mathbf{Mx} = \mathbf{0}$ expresses the fact that these solutions represent integral flows on the vertices of G (and thus of G'), in other words, a family of cycles. Moreover, the system $\mathbf{M}_+\mathbf{x} = \mathbf{1}$ implies that these cycles go exactly once through each vertex of G. Since G is acyclic, all the cycles in G' meet v_0 and, therefore, represent paths of G. This proves that feasible solutions of (P) are exactly path partitions of G. Furthermore, the objective function counts k for each cycle that goes through x_0 and one for each loop. So, for the path partition, it counts k for each path of cardinality more than one, and one for each path of cardinality one. It is then clear that a solution that minimizes this objective function is a k-optimal path partition and the value of the objective function is its k-norm.

Consider now the dual problem. The variables of the problem are two vectors $\mathbf{y}$ and $\mathbf{z}$ corresponding to the vertices of $V(G)$ and we want to maximize

$$\sum_{i=1}^{n} y_i$$

subject to the constraints $[\mathbf{z}, \mathbf{y}]A \leqslant \mathbf{c}$. In other words, we have a set of constraints, each corresponding to an edge of G'.

$$\begin{cases} y_i + z_j - z_i \leqslant 0 & \text{for each edge } (v_i, v_j) \in E, \\ y_i - z_i \leqslant 0 & \text{for each } i \text{ (these constraints represent all the } (v_i, v_0) \text{ edges in } G'), \\ z_i \leqslant k & \text{for each } i \text{ (these constraints represent all the } (v_0, v_i) \text{ edges in } G'), \\ y_i \leqslant 1 & \text{for each } i \text{ (these constraints represent loops).} \end{cases} \tag{2}$$

We associate a k-colouring $\mathscr{C}^k = \{C_1, C_2, \dots, C_k\}$ to such a solution in the following way: Let

$$C_r := \{v_i \geqslant 1; \mid y_i = 1 \text{ and } z_i = r\}, \quad r = 1, 2, \dots, k.$$

Note that if vertices v_i, v_j belong to C_r then $y_i + z_j - z_i = 1$ and by (2), that means that C_r is an independent set, and $\mathscr{C}^k$ is indeed a k-colouring.

We shall now show that $\mathscr{C}^k$ is orthogonal to every k-optimal path partition $\mathscr{P}$ in G. Being a k-optimal path partition, $\mathscr{P}$ is represented by a vector $\mathbf{x} \in \mathbb{N}^{E(G')}$ which is a solution of the primal problem. Complementary slackness conditions mean that if an entry of $\mathbf{x}$ is non-zero then the corresponding inequality constraint in the dual is an equality.

Let P be a short path, i.e., a path in $\mathscr{P}^0$. We have seen that such paths are represented in $\mathbf{x}$ by the loops on vertices $v_i \in V(P)$. So if $v_i \in V(P)$, by complementary slackness, we get $y_i = 1$, and since by (2) we have $y_i \leqslant z_i \leqslant k$, we know that v_i belongs to some C_r. Furthermore, if $(v_i, v_j) \in E$ and $v_i \in C_r$, then by (2), $z_j - z_i \leqslant -y_i = -1$, implying that the colours in P are strictly decreasing.

Assume now that P is a long path. To simplify the notations, assume that $P = (v_1, \dots, v_l)$. We have seen that in the primal problem P is represented by a cycle going through v_0. Thus, again, by complementary slackness, we know that equality is satisfied in all the constraints of (2) corresponding to edges of this cycle. Then

$$\begin{array}{rr} & z_1 = k, \\ \forall\, 1 \leqslant i \leqslant l-1 & z_{i+1} - z_i = -y_i, \\ & z_l = y_l. \end{array}$$

But y_i is at most 1. Since the variable z_i starts at k and finishes at 1, it has to take all values between these two. Thus for each p, $1 \leqslant p \leqslant k$, we can define $i_p = \max\{1 \leqslant i \leqslant l;\ z_i = p\}$. But this implies that $z_{i_p+1} = p-1$ (if $z_{i_p+1} \geqslant p$ then it means that there exists $j > i_p$ such that $z_j = p$, which contradicts the maximality of i_p). So $y_{i_p} = 1$, implying that for each p, $v_{i_p} \in C_p$. In other words, P meets all k colours in $\mathscr{C}^k$.

This completes the proof that $\mathscr{C}^k$ is orthogonal to every k-optimal path partition $\mathscr{P}$.

Cameron [20] gave a common extension of the Greene–Kleitman theorem and Greene's theorem (Theorem 3.6 in Section 3) using a different linear program. More related work appears in [21,23,22].

2.2.2. *Cartesian products*

This elegant technique uses the Gallai–Milgram theorem to prove Conjectures 2.6 and 2.8 for acyclic digraphs. It was first discovered by Saks [50] to prove Greene–Kleitman's theorem using Dilworth's theorem. Later, Linial used the same technique to prove Conjecture 2.6 for acyclic digraphs, using the Gallai–Milgram theorem. Finally, Aharoni and Hartman [1] used it to prove Berge's conjecture for acyclic digraphs. We will briefly describe the general technique. The interested reader can find the details in the relevant papers.

Let G be an acyclic digraph, and let k be a positive integer. Following Saks [50], we define a digraph $G_k = (V_k, E_k)$ which consists of k copies of G and is defined as follows: (see Fig. 1)

$$V_k := V \times \{1, 2, \ldots, k\}$$

and

$$E_k := \{((x,i),(x,j)) | x \in V, 1 \leqslant i < j \leqslant k\} \cup \{((x,i),(y,i)) | (x,y) \in E, 1 \leqslant i \leqslant k\}.$$

Saks [50] proved that for an acyclic digraph G

$$\pi_1(G_k) \geqslant \pi_k(G).$$

It is easy to see that

$$\alpha_k(G) = \alpha_1(G_k)$$

and thus we get

$$\alpha_k(G) = \alpha_1(G_k) \geqslant \pi_1(G_k) \geqslant \pi_k(G)$$

and Conjecture 2.6 is proved for acyclic digraphs.

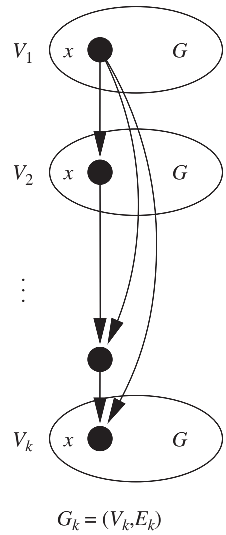


Fig. 1. Cartesian product.

In [1] Aharoni and Hartman give an algorithm that for any k-optimal path partition $\mathscr{P}$ of G, the algorithm finds a k-colouring orthogonal to $\mathscr{P}$, using the digraph G_k and a constructive proof of the Gallai–Milgram theorem, thus proving Conjecture 2.8 for acyclic digraphs.

2.2.3. Network flows and bipartite graphs

In [33], Frank proves the Greene–Kleitman theorem [38] and Greene's theorem [37] (see Section 3) using network flows. Frank uses an algorithm of Ford and Fulkerson [32] for the following problem:

Let $N=(V,E)$ be a network with two specified vertices: a source s and a sink t. Let $a(x,y)$ be a non-negative integral cost, and let $c(x,y)$ be a positive integral capacity assigned to each edge (x,y). The algorithm of Ford and Fulkerson [32] finds a minimum cost flow from s to t, having a value v given in advance. The algorithm invokes dual variables $\pi(x)$ assigned to the vertices of G. This so-called potential function π is non-negative integer valued, and $\pi(s)=0$ throughout the process. The current $\pi(t)=p$ is called the *potential value*. We do not go into the details here of how the network is constructed using the given digraph. However, we will state that the flow from s to t corresponds to a path partition in G, and, similar to the LP technique, the orthogonal k-colouring is constructed using the potential function $\pi(x)$, with potential value $p=k$.

The algorithm proceeds in stages. Initially, $v=1$ and $p=k=1$. At each stage, either the value of the flow v is increased, in which case the p-norm of the current path partition improves, or the potential value p (i.e. k) increases by one, thus allowing more edges in the current network to be admissible for the next stage.

Frank's algorithm is applied only to digraphs of posets, and it supplies a constructive proof of the theorems by Greene and Kleitman (Theorem 2.3) and Greene (Theorem 3.6 in Section 3).

Aharoni and Hartman [3] used Frank's ideas to generalize his algorithm to acyclic digraphs, and in the case that $\mathscr{P}=\mathscr{P}^+$, for all digraphs. They preferred to use standard techniques in bipartite graphs, rather than Ford and Fulkerson's algorithm for minimum cost flow of a given value. We will demonstrate below some of their ideas and proof techniques, which are very intuitive, and use basic notions of matchings and alternating paths in bipartite graphs.

Let $G=(V,E)$ be a directed graph. We associate with G an undirected bipartite graph $\overline{G}=(\overline{V},\overline{E})$ as follows:

$$\overline{V}:=V'\cup V'',$$

where $V':=\{v';\,v\in V\}$ and $V'':=\{v'';\,v\in V\}$.

The edge set $\overline{E}$ is defined by

$$\overline{E}:=\widehat{E}\cup K,$$

where $\widehat{E}:=\{(u',v'');\,(u,v)\in E\}$ and $K:=\{(v',v'');\,v\in V\}$ (see Fig. 2, for an example).

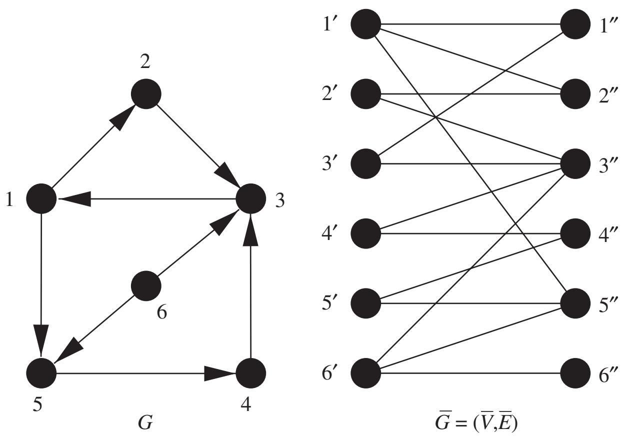


Fig. 2. The bipartite graph corresponding to G.

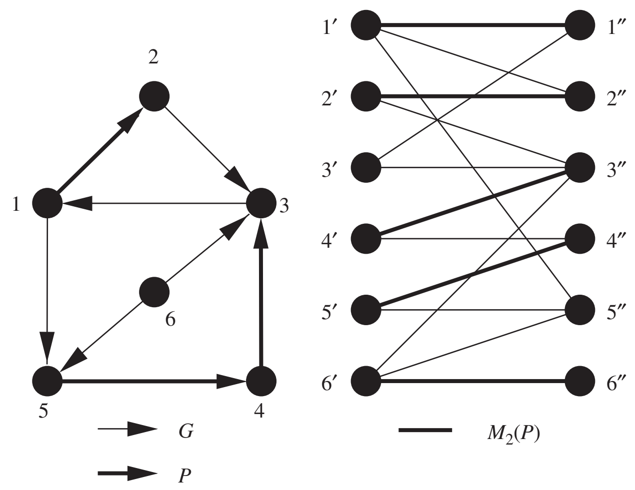


Fig. 3. A path partition and the corresponding matching.

A weight function w is defined on $\overline{E}$ by

$$w(e) := 1 \quad \text{if } e \in K; \quad w(e) := 0 \quad \text{if } e \in \widehat{E}.$$

For a subset $F \subseteq \overline{E}$ we write $w(F) := \sum_{e \in F} w(e)$, or equivalently, $w(F) = |F \cap K|$.

A *matching* in a graph is a set of vertex disjoint edges. With any path partition $\mathscr{P}$ of G, and an integer k, we associate a matching $M = M_k(\mathscr{P})$ in $\overline{G}$ defined by

$$M_k(\mathscr{P}) := \left\{(u', v''); (u, v) \in E[\mathscr{P}^+]\right\} \cup \left\{(v', v''); v \in V[\mathscr{P}^0]\right\}$$

(see Fig. 3), where $E[\mathscr{P}] := \cup\{E(P); P \in \mathscr{P}\}$, and $V[\mathscr{P}] := \cup\{V(P); P \in \mathscr{P}\}$.

Since every path $P \in \mathscr{P}^+$ contributes $|P| - 1$ edges to M and every path $P \in \mathscr{P}^0$, contributes $|P|$ edges, we have $|M| = n - |\mathscr{P}^+|$. Hence

$$|\mathscr{P}|_k = k|\mathscr{P}^+| + |V[\mathscr{P}^0]| = k(n - |M|) + w(M) = kn - (k|M| - w(M)). \tag{3}$$

Eq. (3) defines the relationship between the size and weight of the matching, and the k-norm of a path partition. In order to reduce the k-norm of a path partition, we need to increase the amount $k|M| - w(M)$, in other words, find another matching N such that $k|N| - w(N) > k|M| - w(M)$. Moving to another matching is done using alternating and augmenting paths (as in the *Hungarian Algorithm*). Using the alternating paths, a potential function is defined on the vertex set $\overline{V}$, which defines the orthogonal k-colouring. For acyclic digraphs, any matching M (provided that for each $v \in G$ either $v' \in \overline{V}$ or $v'' \in \overline{V}$, or both, is met by M) corresponds to a path partition in G. Hence, a proof of Berge's strong path partition conjecture for acyclic digraphs, is quite straightforward, in this case. The real (yet to be solved) challenge, is to prove it for general graphs.

For further details, we refer the reader to [3,2,57].

3. Extending the theorems of Gallai–Roy and Greene

One can "dualise" the concepts mentioned in Sections 1 and 2 by replacing the notion of a *path* by that of an *independent set*, and vice versa. Thus, the equivalent notions of *path partition* and *k-colouring* would be *colouring* and *k-path system*, respectively:

Definition 3.1 (*Colouring*). A *colouring*, $\mathscr{C} := \{C_1, C_2, \ldots, C_m\}$, is a partition of $V(G)$ into disjoint independent sets C_i.

Definition 3.2 (*k-norm of a colouring*). For each positive integer k, the k-norm $|\mathscr{C}|_k$ of a colouring $\mathscr{C}=\{C_1, C_2, \ldots, C_m\}$ is defined by

$$|\mathscr{C}|_k := \sum_{i=1}^{m} \min\{|C_i|, k\}.$$

A colouring which minimizes $|\mathscr{C}|_k$ is called *k-optimal*. Denote by $\chi_k(G)$ the k-norm of a k-optimal colouring in G.

Note that a 1-optimal colouring is a colouring that contains a minimum number of colours, so $\chi_1(G) = \chi(G)$, the chromatic number of G.

Definition 3.3 (*k-path system*). A *k-path system* is a family $\mathscr{P}^k := \{P_1, P_2, \ldots, P_t\}$ of at most k disjoint paths. The cardinality of a k-path-system $\mathscr{P}^k = \{P_1, P_2, \ldots, P_t\}$ is $|\mathscr{P}^k| := \sum_{i=1}^{t} |P_i|$ and $\mathscr{P}^k$ is said to be *optimal* if $|\mathscr{P}^k|$ is as large as possible. Denote by $\lambda_k(G)$ the cardinality of an optimal k-path system in G.

An optimal 1-path system is a longest path. The following is a well known theorem:

Theorem 3.4 (*Gallai–Roy theorem [35,49]*). *In any digraph G, there exists a path of cardinality at least $\chi(G)$, i.e. $\lambda_1(G) \geqslant \chi_1(G)$.*

A special case of Theorem 3.4 is when the graph is a tournament, i.e. $\chi(G) = n$:

Corollary 3.5 (*Rédei's theorem [48]*). *Any tournament contains a Hamilton path.*

We are now able to state Greene's theorem:

Theorem 3.6 (*Greene's theorem [37]*). *Let G be a digraph of a poset, and let k be a positive integer. Then $\lambda_k(G) = \chi_k(G)$.*

We now make a few observations similar to Observation 2.4.

Observation 3.7. (1) *For $k = 1$, Theorem* 3.6 *states that in a digraph of a poset the cardinality of the longest path in G equals the chromatic number of G.*

(2) *It is easy to prove that the inequality $\lambda_k(G) \leqslant \chi_k(G)$ holds for G transitive and acyclic (i.e. the digraph of a poset). We leave the proof to the reader.*

Conjecture 3.8 (*Linial [47]*). *Let G be a digraph and k a positive integer. Then $\lambda_k(G) \geqslant \chi_k(G)$.*

For $k = 1$ Conjecture 3.8 holds by the Gallai–Roy theorem [35,49]. For $k \geqslant \pi_1(G)$ the conjecture also trivially holds since $\lambda_k(G) = n$ in this case. If G is acyclic, Hoffman [41] and Saks [50] showed independently that $\lambda_k(G) \geqslant \chi_k(G)$.

Definition 3.9 (*orthogonality of colourings and k-path systems*). A colouring $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ and a k-path system $\mathscr{P}^k$ are *orthogonal* if every colour class C_i in $\mathscr{C}$ meets $\min\{|C_i|, k\}$ different paths in $\mathscr{P}^k$.

The natural conjecture extending Berge's conjecture (Conjecture 2.8) would be:

Conjecture 3.10 (*false*). *Let G be a digraph and let k be a positive integer. Then for every k-optimal colouring $\mathscr{C}$ there exists a k-path system orthogonal to it.*

This conjecture holds for digraphs of posets. However, it is false for general digraphs by the counter-example in Fig. 4 for $k = 1$ ([4]). Here no path meets all colours in $\mathscr{C}$.

Aharoni et al. [4] suggested the following alternative to Conjecture 3.10:

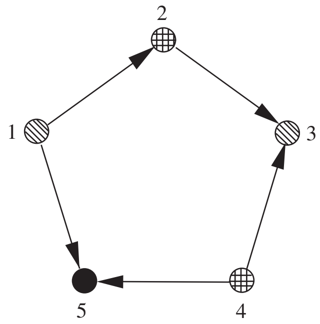


Fig. 4. Counter-example to Conjecture 3.10.

Conjecture 3.11 (*Aharoni, Hartman, Hoffman's conjecture extending Greene's theorem [4]*). *Let G be a digraph and let k be a positive integer. Then for every optimal k-path system $\mathscr{P}^k$ there exists a colouring $\mathscr{C}$ orthogonal to it.*

Conjecture 3.11 holds in the case that $k=1$ (by the proof of the Gallai–Roy theorem), $k \geqslant \pi_1(G)$, and when G is the digraph of a poset. For acyclic digraphs, Aharoni et al. [4] proved the following stronger result, the "dual" analogue of Theorem 2.9.

Theorem 3.12 (*Aharoni et al. [4]*). *Let G be an acyclic digraph, and let k be a positive integer. Then there exists a colouring $\mathscr{C}$ orthogonal to every optimal k-path system $\mathscr{P}^k$.*

The proof of Theorem 3.12 uses a linear programming technique. An alternative constructive proof is described in [5].

In [8], Conjecture 3.11 was proved in the special case where an optimal k-path system contains some path of cardinality one. This result was used to prove Conjecture 3.11 for all bipartite digraphs. In addition, Hartman et al. [8] proved Conjecture 3.11 for all orientations of split[2] graphs.

We have seen, so far, various concepts, theorems, and conjectures related to two different worlds: on the one hand, the world of path partitions and k-colourings described in Section 2, and on the other hand, the "dual" world of colourings and k-path systems, described in Section 3. There seems to be little connection between these worlds, just as there is no apparent connection between the Gallai–Milgram theorem (Theorem 1.1) and the Gallai–Roy theorem (Theorem 3.4), (except for the common author *Gallai*). However, there is a tight connection between a path partition and an orthogonal k-colouring on the one hand, and a k-path system and an orthogonal colouring on the other hand, as is evident in the next theorem:

Theorem 3.13 (*Hartman [5]*). *Let $\mathscr{P}$ be a path partition, and $\mathscr{C}^k$ a k-colouring. Then $\mathscr{P}$ and $\mathscr{C}^k$ are orthogonal if and only if the h-path system defined by $\mathscr{P}^h = \mathscr{P}^+$, and the associated colouring $\mathscr{C} = \mathscr{C}^k \cup \{\{x\} : x \notin V[\mathscr{C}^k]\}$ are orthogonal.*

For posets, Greene [37] proved a tight connection between the parameters λ_i and α_i. Let $\Delta_i = \lambda_i - \lambda_{i-1}$, and $\delta_j = \alpha_j - \alpha_{j-1}$ for $i = 1, 2, \ldots, \alpha$ and $j = 1, 2, \ldots, \lambda$, where $\lambda_0 = \alpha_0 = 0$.

Theorem 3.14 (*Greene's theorem on conjugate partitions [37]*). *In digraphs of posets, the sequences $\Delta = \{\Delta_i\}$ and $\delta = \{\delta_i\}$ are monotone decreasing and form conjugate partitions[3] of the number n, for each $h = 1, 2, \ldots, \lambda$.*

Theorem 3.14 does not extend to acyclic digraphs. However, if we denote $\Delta_j^* = \pi_j - \pi_{j-1}$, where $\pi_0 = 0$, $j = 1, 2, \ldots, \lambda$, then we have:

[2] A graph G is a *split graph* if $V(G)$ is the union of two disjoint sets X and Y, where X is a clique and Y is an independent set.

[3] That is, δ_h equals the number of parts in Δ of size at least h.

Theorem 3.15 (*Hartman [5]*). *In acyclic digraphs the sequences* $\{\Delta_i\},\ i=1,\ldots,\pi_1$, *and* $\{\Delta_j^*\},\ j=1,\ldots,\lambda$, *are monotone decreasing and form conjugate partitions of the number n.*

Frank [33] presented an elegant way to derive together both the Greene–Kleitman theorem 2.3 and Greene's theorem 3.6, using network flow techniques, as discussed in Section 2.2.3. Aharoni and Hartman generalized Frank's theorem to all acyclic digraphs in the following way:

Theorem 3.16 (*Hartman [5]*). *Let G be an acyclic digraph. Then there exists a sequence*

$$\mathscr{P}_0|\mathscr{C}^1\mathscr{C}^2\ldots\mathscr{C}^{i_1}|\mathscr{P}_1\mathscr{P}_2\ldots\mathscr{P}_{j_1}|\mathscr{C}^{i_1+1}\mathscr{C}^{i_1+2}\ldots\mathscr{C}^{i_2}|\mathscr{P}_{j_1+1}\ldots\mathscr{P}_{j_2}|\ldots$$

where each $\mathscr{P}_j$ *is a path partition of G and each* $\mathscr{C}^i$ *is an i-colouring satisfying*:

1. *any member of the sequence* (*whether* $\mathscr{P}_j$ *or* $\mathscr{C}^i$) *is orthogonal to the last member of other type preceding it*;
2. *if* $\mathscr{P}_j$ *and* $\mathscr{C}^i$ *are orthogonal* (*where* $\mathscr{P}_j$ *and* $\mathscr{C}^i$ *are in the sequence*), *then* $\mathscr{P}_j$ *is i-optimal*; *and*
3. $|\mathscr{P}_j^+|=|\mathscr{P}_{j-1}^+|-1$ *for each* $j=1,2,\ldots$.

The following corollaries are extensions of other theorems of Greene [37] to acyclic digraphs. (See also Saks [50], Groflin and Hoffman [39] and Hoffman and Schwartz [42] for other extensions.)

Corollary 3.17 (*Hartman [5]*). *For each* $k\geqslant 1$, *there exists a path partition which is simultaneously k-optimal and* $(k+1)$-*optimal.*

The dual statement of Corollary 3.17 is not necessarily correct. However, the following variation extends Greene's theorem [37] to all acyclic digraphs.

Corollary 3.18 (*Hartman [5]*). *For each* $h\geqslant 1$ *there exists a colouring which is simultaneously orthogonal to some optimal h-path system* $\mathscr{P}^h$ *and some optimal* $(h+1)$-*path system* $\mathscr{P}^{h+1}$.

The proofs of Corollaries 3.17 and 3.18 follow from Theorems 3.16 and 3.13.

4. Strongly connected graphs

A digraph G is *strongly connected*, or for brevity, *strong*, if for any ordered pair of vertices (x, y), there exists a path from x to y in G. The following well known theorem by Camion [24] strengthens the Gallai–Milgram theorem (Theorem 1.1) in the case that $\alpha(G)=1$ and G is strong.

Theorem 4.1 (*Camion [24]*). *A strongly connected tournament contains a Hamilton cycle.*

In view of Camion's theorem it is natural to attempt to strengthen the Gallai–Milgram theorem for all α, in the case of strong digraphs.

Chen and Manalastas [26] have done so for $\alpha=2$.

Theorem 4.2 (*Chen and Manalastas [26]*). *Every strongly connected digraph G with* $\alpha(G)=2$ *is spanned by two consistent cycles.*[4]

An alternative proof of Theorem 4.2 can be found in [18].

An immediate result of Theorem 4.2 is that every strong digraph with $\alpha=2$ is spanned by a Hamilton path.

Thomassé proved the following theorem which was originally conjectured by Las-Vergnas [11].

Theorem 4.3 (*Thomassé [55]*). *Every strongly connected digraph G with* $\alpha(G)>1$ *has a spanning arborescence H with* $|\mathrm{ter}(H)|\leqslant\alpha(G)-1$.

[4] Two cycles C_1 and C_2 are *consistent* if their intersection is empty, or a single vertex, or a subpath of C_1 and C_2.

Corollary 4.4. *For every strong digraph G with $\alpha(G) > 1$, $\pi(G) \leqslant \alpha(G) - 1$.*

Corollary 4.4 follows from Theorem 4.3 since any spanning arborescence H contains a (spanning) path partition $\mathscr{P}$ satisfying $\mathrm{ter}(\mathscr{P}) = \mathrm{ter}(H)$.

Another attempt to strengthen the Gallai–Milgram theorem is Gallai's conjecture [34] from 1963, only recently proved by Bessy and Thomassé [16]:

Theorem 4.5 (*Bessy and Thomassé [16]*). *Every strongly connected digraph G is spanned by the union of $\alpha(G)$ cycles.*

As a corollary, we get for $\alpha = 1$ Camion's theorem. The case $\alpha = 2$ is a corollary of the Chen–Manalastas theorem (Theorem 4.2).

If G is an undirected graph with connectivity $\kappa(G)$ and independence number $\alpha(G) \leqslant k$, then by the Chvátal–Erdős theorem [28] G contains a Hamilton cycle. As a corollary, it follows that if $\kappa(G) < \alpha(G)$, then $\pi(G) \leqslant \alpha(G) - \kappa$. This can be seen by constructing a new graph G^* from G by adding $\alpha(G) - \kappa$ new vertices to G and joining them to each vertex in $V(G)$. The resultant graph G^* has connectivity $\kappa = \alpha(G^*)$, and hence contains a Hamilton cycle, implying that $V(G)$ can be partitioned into $\alpha(G) - \kappa$ paths. This is clearly a strengthening of the Gallai–Milgram theorem for undirected graphs.

Can we have such a strengthening for directed graphs?

Definition 4.6. A directed graph $G = (V, E)$ is *k-connected* if $G[V - V']$ is strong for every $V' \subseteq V$, $|V'| < k$.

The graph in Fig. 5 has $k = \alpha = 2$ and contains no Hamilton cycle, thus showing that the Chvátal–Erdős theorem cannot necessarily be extended to directed graphs. Also the example given by Bondy [18], the digraph obtained from the composition of $K_{1,3}$ with K_2, is 2-connected, has $\alpha = 3$, but contains no Hamilton path. Other counterexamples for $k = 2$ and $k = 3$ also exist (see [45,6]).

The following is a more reasonable extension of the Chvátal–Erdős theorem to directed graphs.

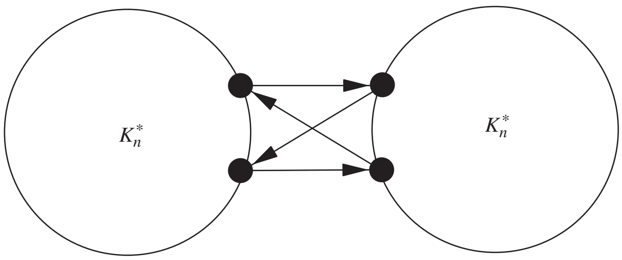
Conjecture 4.7 (*Bessy and Thomassé [15]*). *Let G be a k-connected directed graph with $\alpha \geqslant k$. Then $V(G)$ can be partitioned into at most α disjoint cycles.*

Conjecture 4.7 is true for $\alpha = 1$ by Camion's theorem 4.1, but is open for all $\alpha > 1$.

For other possible extensions of the Chvátal–Erdős theorem to directed graphs see [43,44,15].

Now that the Gallai–Milgram theorem has been strengthened for strong digraphs, it is reasonable to ask whether the Gallai–Roy theorem can be similarly strengthened. If G is a tournament (i.e. $\chi(G) = n$) then G contains a Hamilton path. This special case of the Gallai–Roy theorem is known as Rédei's theorem [48]. If G is a strong tournament, then, as we have already seen, Camion's theorem (Theorem 4.1) implies the existence of a Hamilton cycle. The following theorem by Bondy [17], originally conjectured by Las-Vergnas, strengthens the Gallai–Roy theorem for strong digraphs.

Theorem 4.8 (*Bondy [17]*). *Every strongly connected digraph contains a directed cycle of length at least χ.*



K_n^* denotes the complete symmetric digraph

Fig. 5. G contains no Hamilton cycle.

Corollary 4.9. *Let G be strongly connected digraph with* $n \geqslant \chi(G)+1$. *Then* $\lambda(G) \geqslant \chi(G)+1$.

5. α- and χ-diperfect graphs

5.1. α-Diperfect graphs

Berge [11] suggested the following definition:

Definition 5.1 (*Berge [11]*). A digraph G is *α-diperfect* if for every maximum independent set S, there exists a path partition $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ such that $|S \cap P_i| = 1$ for all i, $1 \leqslant i \leqslant m$, and if every induced subgraph of G has the same property.

We remark that if G is an α-diperfect graph then it has a path partition with exactly α paths, implying the Gallai–Milgram theorem in this case. Note also that every orientation of a perfect graph is α-diperfect, as was shown in [11]. This follows from the fact that in a perfect graph there is a partition of the vertex set into $\alpha(G)$ cliques. By Rédei's theorem (Corollary 3.5), each clique is spanned by a path P_i and $|S \cap P_i| = 1$ for all i.

It was also shown in [11] that all symmetric digraphs are α-diperfect.

Definition 5.2 (*Berge [11]*). (see Fig. 6). An odd cycle $C = (x_1, x_2, \ldots, x_{2k+1})$ is *anti-directed* if its length is at least three, the longest path in C is of length two, and each of the vertices $x_1, x_2, x_3, x_4, x_6, x_8, \ldots, x_{2k}$ is either a source or a sink.

It is not difficult to see that an anti-directed odd cycle without chords is not α-diperfect. This motivated Berge to the following:

Conjecture 5.3 (*Berge [11]*). *A graph G is α-diperfect if and only if G does not contain any chordless anti-directed odd cycle*.

We can attempt to extend the concept of α-diperfect by considering a k-colouring instead of an independent set. Thus, we extend the concept in a similar way that the Greene–Kleitman theorem extends Dilworth's theorem (Theorems 2.3 and 1.2, respectively).

Definition 5.4 (*α_k-diperfect*). Given a positive integer k, a digraph G is *α_k-diperfect* if for every optimal k-colouring $\mathscr{C}^k$, there exists a path partition $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ orthogonal to $\mathscr{C}^k$, and if every induced subgraph of G has the same property.

For $k = 1$, this definition coincides with the definition of an α-diperfect graph.

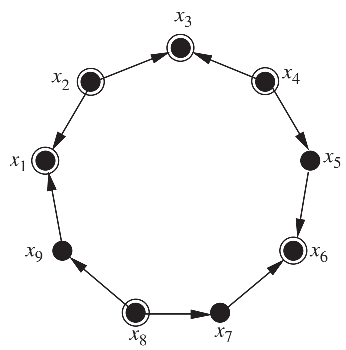


Fig. 6. Antidirected odd cycle.

Lemma 5.5. *If G is α_k-diperfect then Conjecture* 2.6 *holds for G.*

Proof. Let $\mathscr{C}^k$ be an optimal k-colouring and let $\mathscr{P}=\{P_1, P_2, \ldots, P_m\}$ be a path partition orthogonal to it. Then

$$\alpha_k = |\mathscr{C}^k| = \sum_{i=1}^{m} |V[\mathscr{C}^k] \cap V(P_i)| = \sum_{i=1}^{m} \min\{|P_i|, k\} = |\mathscr{P}|_k \geqslant \pi_k \tag{4}$$

implying Conjecture 2.6 for G. □

Theorem 5.6. *Every transitive acyclic digraph is α_k-diperfect for each positive integer k.*

Proof. Let $\mathscr{C}^k$ be an optimal k-colouring, and let $\mathscr{P}=\{P_1, P_2, \ldots, P_m\}$ be a k-optimal path partition. By the Greene–Kleitman theorem 2.3

$$\alpha_k = |\mathscr{C}^k| = \sum_{i=1}^{m} |V[\mathscr{C}^k] \cap V(P_i)| = \sum_{i=1}^{m} \min\{|P_i|, k\} = |\mathscr{P}|_k = \pi_k \tag{5}$$

and $\mathscr{P}$ is orthogonal to $\mathscr{C}^k$. The same holds for every induced subgraph H of G which is also transitive and acyclic. □

5.2. χ-Diperfect graphs

Definition 5.7 (*Berge [11]*). A digraph G is *χ-diperfect* if for every optimal colouring $\mathscr{C}=\{C_1, C_2, \ldots, C_m\}$, there exists a path P such that $|P \cap C_i| = 1$ for all $1 \leqslant i \leqslant m$, and if every induced subgraph of G, has the same property.

We remark that if G is χ-diperfect then both the Gallai–Roy and Gallai–Milgram theorems (Theorems 3.4 and 1.1) are implied for G. This follows from the fact that P is necessarily of cardinality $\chi(G)$ (implying the Gallai–Roy theorem), and if we repeatedly remove the vertices of P, we get a path partition of G with at most t paths, where t is the size of the largest colour class in $\mathscr{C}$. Clearly, $t \leqslant \alpha$, and the Gallai–Milgram theorem follows.

It is easy to see that every perfect graph is χ-diperfect, as well as every symmetric graph (see [11]). Berge also found that a chordless anti-directed odd cycle as well as an orientation of the complement of a chordless odd cycle are not χ-diperfect. Thus, using the strong perfect graph theorem [27], we have:

Theorem 5.8. *A simple graph is χ-diperfect for each orientation of its edges if and only if G is perfect.*

We can now extend the concept of a χ-diperfect graph by considering a k-path system instead of a single path:

Definition 5.9 (χ_k-*diperfect*). For a given integer k, a digraph G is χ_k-*diperfect* if for every k-optimal colouring $\mathscr{C}=\{C_1, C_2, \ldots, C_m\}$, there exists a k-path system $\mathscr{P}^k$ orthogonal to $\mathscr{C}$, and if every induced subgraph of G has the same property.

Lemma 5.10. *If G is χ_k-diperfect, then Conjecture* 3.8 *holds for G.*

Proof. Let $\mathscr{C}=\{C_1, C_2, \ldots, C_m\}$ be a k-optimal colouring, and let $\mathscr{P}^k$ be a k-path system orthogonal to it. Then

$$\lambda_k \geqslant |\mathscr{P}^k| = \sum_{i=1}^{m} |V[\mathscr{P}^k \cap V(C_i)]| = \sum_{i=1}^{m} \min\{|C_i|, k\} = |\mathscr{C}|_k = \chi_k \tag{6}$$

implying Conjecture 3.8 for G. □

Theorem 5.11. *Every transitive acyclic digraph G is χ_k-diperfect for each positive integer k.*

Proof. Let $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be a k-optimal colouring, and let $\mathscr{P}^k$ be a k-optimal path partition. By Greene's theorem 3.6

$$\lambda_k = |\mathscr{P}^k| = \sum_{i=1}^{m} |V[\mathscr{P}^k \cap V(C_i)]| = \sum_{i=1}^{m} \min\{|C_i|, k\} = |\mathscr{C}|_k = \chi_k \tag{7}$$

and each colour class C_i meets exactly $\min\{|C_i|, k\}$ different paths in G, implying that $\mathscr{C}$ and $\mathscr{P}^k$ are orthogonal. The same holds for every induced subgraph of G. □

Theorem 5.12. *Every symmetric digraph G is χ_k-diperfect for each positive integer k.*

Proof. Let $G = (V, E)$ be a symmetric digraph, i.e. $(u, v) \in E \Rightarrow (v, u) \in E$. Let $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be a k-optimal colouring. Consider the graph G' obtained from G by removing the edges going from C_j to C_i for $j > i$. Let $\mathscr{P}^k = \{P_1, P_2, \ldots, P_k\}$ be an optimal k-path system in G'. Note that G' is acyclic, therefore, Conjecture 3.8 holds, and

$$\lambda_k(G) \geqslant \chi_k(G). \tag{8}$$

On the other hand,

$$\lambda_k = |\mathscr{P}^k| = \sum_{i=1}^{m} |V[\mathscr{P}^k \cap V(C_i)]| \leqslant \sum_{i=1}^{m} \min\{|C_i|, k\} = |\mathscr{C}|_k = \chi_k. \tag{9}$$

The inequality in (9) holds since, by the structure of G', every colour class C_i in G' can meet the k-path system in at most $\min\{|C_i|, k\}$ vertices. Together with Eq. (8) we have equality, i.e. every colour class C_i meets the k-path system in exactly $\min\{|C_i|, k\}$ vertices, implying that $\mathscr{C}$ and $\mathscr{P}^k$ are orthogonal. □

It is an open question to find other classes of χ_k-diperfect graphs, or characterize them.

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