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Multiplicativity Factors for Function Norms

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Let $(\mathbf{T}, \Omega, m)$ be a measure space; let ρ be a function norm on $\mathcal{M} = \mathcal{M}(\mathbf{T}, \Omega, m)$, the algebra of measurable functions on $\mathbf{T}$; and let L_ρ be the space $\{f \in \mathcal{M}: \rho(f) < \infty\}$ modulo the null functions. If L_ρ is an algebra, then we call a constant $\mu > 0$ a multiplicativity factor for ρ if $\rho(fg) \leqslant \mu\rho(f)\,\rho(g)$ for all $f, g \in L_\rho$. Similarly, $\lambda > 0$ is a quadrativity factor if $\rho(f^2) \leqslant \lambda\rho(f)^2$ for all f. The main purpose of this paper is to give conditions under which L_ρ is indeed an algebra, and to obtain in this case the best (least) multiplicativity and quadrativity factors for ρ. The first of our two principal results is that if ρ is σ-subadditive, then L_ρ is an algebra if and only if L_ρ is contained in L^∞. Our second main result is that if $(\mathbf{T}, \Omega, m)$ is free of infinite atoms, ρ is σ-subadditive and saturated, and L_ρ is an algebra, then the multiplicativity and quadrativity factors for ρ coincide, and the best such factor is determined by $\sup\{\|f\|_\infty : f \in L_\rho, \rho(f) \leqslant 1\}$. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let $\mathcal{A}$ be a *function algebra* over a field $\mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$; that is, $\mathcal{A}$ is an algebra of $\mathbb{F}$-valued functions

$$f: \mathbf{T} \to \mathbb{F}$$

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defined on a given set T with the usual pointwise multiplication

$$(fg)(t)=f(t)\,g(t), \qquad f, g \in \mathscr{A},\ t \in \mathrm{T}. \tag{1.1}$$

As usual, a mapping

$$S\colon \mathscr{A} \to \mathbb{R}$$

is called a *seminorm* if for all $f, g \in \mathscr{A}$ and $\alpha \in \mathbb{F}$,

$$S(f) \geqslant 0,$$

$$S(\alpha f) = |\alpha|\, S(f),$$

$$S(f+g) \leqslant S(f) + S(g).$$

If in addition,

$$S(f) \neq 0 \qquad \forall f \neq 0,$$

then S is a *norm*. We call a seminorm S *proper* if S does not vanish identically and $S(f)=0$ for some $f \neq 0$. Finally, we say that S is *submultiplicative* if

$$S(fg) \leqslant S(f)\, S(g) \qquad \forall f, g \in \mathscr{A};$$

and *subquadrative* if

$$S(f^2) \leqslant S(f)^2 \qquad \forall f \in \mathscr{A}.$$

Given a seminorm S on $\mathscr{A}$ and a constant $\mu > 0$, then surely $S_\mu \equiv \mu S$ is a seminorm too. Obviously, S_μ may or may not be multiplicative. If it is, we call μ a *multiplicativity factor* or simply an *M-factor* for S. Similarly, if $S_\lambda \equiv \lambda S$ is quadrative on $\mathscr{A}$ for $\lambda > 0$, we call λ a *quadrativity factor* or a *Q-factor* for S.

We see at once that $\mu > 0$ *is an M-factor for S if and only if*

$$S(fg) \leqslant \mu S(f)\, S(g) \qquad \forall f, g \in \mathscr{A}.$$

Analogously, $\lambda > 0$ *is a Q-factor for S if and only if*

$$S(f^2) \leqslant \lambda S(f)^2 \qquad \forall f \in \mathscr{A}.$$

If α_0 is an M- or a Q-factor for S, then so is every $\alpha \geqslant \alpha_0$. Thus having a seminorm on $\mathscr{A}$, the question is whether it has M- or Q-factors; and if so, are there best (least) such factors? We answer this question by the following two theorems:

THEOREM 1.1 [AG1, Theorem 2.4]. *Let S be a seminorm on an algebra $\mathscr{A}$. Then*

(a) *S has M-factors if and only if* Ker S, *the kernel of S, is a (two-sided) ideal in $\mathscr{A}$ and*

$$\mu_{\inf} \equiv \sup\{S(fg): f, g \in \mathscr{A}, S(f) \leqslant 1, S(g) \leqslant 1\} < \infty. \tag{1.2}$$

(b) *If S has M-factors and $\mu_{\inf} > 0$, then $\mu_{\inf}$ is the best (least) M-factor for S.*

(c) *If S has M-factors and $\mu_{\inf} = 0$, then μ is an M-factor if and only if $\mu > 0$.*

THEOREM 1.2 [AG2, Theorem 1.2]. *Let $\mathscr{A}$ and S be as in Theorem* 1.1. *Then*:

(a) *S has Q-factors if and only if $\mathscr{K} =$* Ker S *is closed under squaring (i.e., $f^2 \in \mathscr{K}$ if $f \in \mathscr{K}$) and*

$$\lambda_{\inf} \equiv \sup\{S(f^2): f \in \mathscr{A}, S(f) \leqslant 1\} < \infty.$$

(b) *If S has Q-factors and $\lambda_{\inf} > 0$, then $\lambda_{\inf}$ is the best (least) Q-factor for S.*

(c) *If S has Q-factors and $\lambda_{\inf} = 0$, then λ is a Q-factor if and only if $\lambda > 0$.*

Our purpose is to study M- and Q-factors for certain monotonic seminorms on classes of function algebras. Here, as usual, we call a seminorm S on $\mathscr{A}$ *monotonic* if for $f, g \in \mathscr{A}$,

$$|f| \leqslant |g| \text{ implies } S(f) \leqslant S(g).$$

Evidently, if S is monotonic, it is also *absolute*, i.e.,

$$S(f) = S(|f|), \qquad \forall f \in \mathscr{A}.$$

Indeed, for $f \in \mathscr{A}$ set $g = |f|$. Then $|f| = |g|$, so $S(f) \leqslant S(g)$ and $S(g) \leqslant S(f)$; hence $S(f) = S(g) = S(|f|)$.

In [AG1, AG2] we discussed function algebras of bounded functions, and monotonic seminorms of the form

$$S_c(f) = \sup_{t \in \mathbf{T}} |c(t) f(t)|, \tag{1.3}$$

$0 \neq c \in \mathscr{A}$ being a fixed element. Clearly, S_c is a norm if and only if c is not a zero-divisor. Otherwise S_c is a proper seminorm.

We proved, for example:

THEOREM 1.3 [AG2, Theorem 3.2]. *Let $\mathscr{A}$ be the function algebra of $\mathbb{F}$-valued, bounded, continuous functions on a topological space* $\mathbf{T}$, *and let* S_c *be the seminorm in* (1.3). *Then*

(a) *The following are equivalent*:

(i) S_c *has M-factors.*

(ii) S_c *has Q-factors.*

(iii) $\varepsilon_c \equiv \inf\{|c(t)| : t \in \mathbf{T}, c(t) \neq 0\} > 0.$

(b) *If* $\varepsilon_c > 0$, *then the best* (*least*) *M- and Q-factors for* S_c *are both given by* ε_c^{-1}.

Theorem 1.3 implies that the (possibly empty) sets of M- and Q-factors for S_c coincide. Under a mild assumption on $\mathscr{A}$, the following theorem and its corollary provide the same result for arbitrary monotonic seminorms:

THEOREM 1.4. *Let S be a monotonic seminorm on a function algebra* $\mathscr{A}$, *and let* $\mathscr{A}$ *be closed under absolute values, i.e.*,

$$f \in \mathscr{A} \text{ implies } |f| \in \mathscr{A}.$$

Then S is multiplicative if and only if it is quadrative on $\mathscr{A}$.

Proof. If S is multiplicative, it is quadrative. Conversely, suppose S is quadrative. As S is monotonic and $\mathscr{A}$ is closed under absolute values, S is also absolute. Hence, for $f, g \in \mathscr{A}$ with $S(f) \leqslant 1$, $S(g) \leqslant 1$, we exploit the properties of S to obtain

$$\begin{aligned} S(fg) = S(|fg|) &\leqslant S\left(\frac{|f|^2 + |g|^2}{2}\right) \leqslant \frac{1}{2}\,[S(|f|^2) + S(|g|^2)] \\ &\leqslant \frac{1}{2}\,[S(f^2) + S(g^2)] \leqslant \frac{1}{2}\,[S(f)^2 + S(g)^2] \leqslant 1. \end{aligned} \tag{1.4}$$

Now, for arbitrary $f, g \in \mathscr{A}$ and $\varepsilon > 0$,

$$S\left(\frac{f}{S(f)+\varepsilon}\right) \leqslant 1, \qquad S\left(\frac{g}{S(g)+\varepsilon}\right) \leqslant 1;$$

so by (1.4)

$$S\left(\frac{f}{S(f)+\varepsilon}\,\frac{g}{S(g)+\varepsilon}\right) \leqslant 1.$$

Hence

$$S(fg) \leqslant [S(f)+\varepsilon][S(g)+\varepsilon] \xrightarrow[\varepsilon \to 0]{} S(f)\, S(g),$$

and the theorem is at hand. ∎

We immediately deduce

COROLLARY 1.1. *Let* $\mathbf{T}$, $\mathscr{A}$, *and* S *be as in Theorem* 1.4. *Then*

(a) *S has M-factors if and only if it has Q-factors.*

(b) *The (possibly empty) sets of M- and Q-factors for S coincide.*

For example, let S_1 and S_2 be monotonic seminorms on a function algebra $\mathscr{A}$, and let $\mathscr{A}$ be closed under absolute values. Obviously, $S \equiv S_1 + S_2$ is again a monotonic seminorm on $\mathscr{A}$. So by Corollary 1.1, $\mu > 0$ is an M-factor for S if and only if it is a Q-factor. For instance, take l^∞, the algebra of bounded sequences $a = \{\alpha_j\}_{j=1}^{\infty}$ over $\mathbb{F}$, with the usual multiplication

$$ab = \{\alpha_j \beta_j\}, \qquad a = \{\alpha_j\},\ b = \{\beta_j\} \in l^\infty.$$

Fix an element $0 \neq c = \{\gamma_j\} \in l^\infty$ and set

$$S_c(a) = \sup_j |\gamma_j \alpha_j|, \qquad a = \{\alpha_j\} \in l^\infty.$$

Since l^∞ is an algebra of bounded continuous functions on the (discrete) topological space $\mathbf{T} = \mathbb{Z}^+ = \{1, 2, 3, ...\}$, Theorem 1.3 applies; so S_c has M-factors if and only if

$$\varepsilon_c \equiv \inf_{\gamma_j \neq 0} |\gamma_j| > 0. \tag{1.5}$$

Now let c satisfy (1.5) and let $0 \neq d = \{\delta_j\} \in l^\infty$ satisfy

$$\varepsilon_d \equiv \inf_{\delta_j \neq 0} |\delta_j| > 0.$$

Then by Theorem 1.3, S_c and S_d have M- hence Q-factors; and so it is not hard to see that $S = S_c + S_d$ has such factors. Without going into what these factors actually are, Corollary 1.1 implies that the M- and Q-factors for S coincide.

2. Function Norms

Throughout the rest of the paper let $(\mathbf{T}, \Omega, m)$ be a measure space, where $\mathbf{T}$ is a nonempty set, Ω a σ-algebra of subsets of $\mathbf{T}$, and m a nontrivial, countably additive, nonnegative measure. Let $\mathcal{M} = \mathcal{M}(\mathbf{T}, \Omega, m)$ denote the class of $\mathbb{F}$-valued, Ω-measurable functions on $\mathbf{T}$. Then (e.g., [Z]) $\mathcal{M}$ is a function algebra with respect to the usual pointwise operations.

Following [LZ1] we call a mapping

$$\rho\colon \mathcal{M} \to [0, \infty]$$

a *function norm* on $\mathcal{M}$ if for $f, g \in \mathcal{M}$ and $\alpha \in \mathbb{F}$,

$$\rho(f) = 0 \qquad \text{if and only if} \quad f = 0 \text{ a.e.}, \tag{2.1a}$$

$$\rho(\alpha f) = |\alpha|\, \rho(f), \tag{2.1b}$$

$$\rho(f + g) \leqslant \rho(f) + \rho(g), \tag{2.1c}$$

$$|f(t)| \leqslant |g(t)| \text{ a.e. implies } \rho(f) \leqslant \rho(g) \text{ (monotonicity)}. \tag{2.1d}$$

As $\rho(f)$ is not necessarily finite for every $f \in \mathcal{M}$, we consider the class

$$\mathcal{M}_\rho = \mathcal{M}_\rho(\mathbf{T}, \Omega, m) \equiv \{ f \in \mathcal{M} : \rho(f) < \infty \}$$

of functions on which ρ is finite.

As is custumary, we identify in $\mathcal{M}$ the equivalence classes of functions equal a.e. on $\mathbf{T}$. If $[f]$ is the equivalence class to which f belongs, then obviously

$$g \in [f] \text{ implies } \rho(g) = \rho(f),$$

so we define

$$\rho([f]) = \rho(f), \qquad f \in \mathcal{M}.$$

The partitioning of $\mathcal{M}$ into classes partitions $\mathcal{M}_\rho$ as well, so we set

$$L_\rho = L_\rho(\mathbf{T}, \Omega, m) \equiv \{ [f] : f \in \mathcal{M}_\rho \} = \{ [f] : f \in \mathcal{M}, \rho(f) < \infty \}.$$

While the results in the remainder of the paper are stated in terms of spaces of equivalence classes like L_ρ, the proofs are invariably given in terms of individual functions. This situation is common in measure theoretic contexts, and requires no further comment.

We readily obtain:

THEOREM 2.1 [LZ1, Theorem 3.5]. *L_ρ is a linear space over $\mathbb{F}$, closed under absolute values; and ρ is an absolute, monotonic norm on L_ρ.*

We recall that a function $f \in \mathcal{M}$ is called *m-essentially bounded* if for some constant $\gamma > 0$,

$$m\{t \in \mathbf{T} : |f(t)| > \gamma\} = 0.$$

As usual, we denote by $L^\infty = L^\infty(\mathbf{T}, \Omega, m)$ the algebra of equivalence classes of all $\mathbb{F}$-valued, m-essentially bounded functions on T and let

$$\|f\|_\infty \equiv \inf\{\gamma > 0 : m\{t \in \mathbf{T} : |f(t)| > \gamma\} = 0\}$$

be the norm on L^∞.

With this standard definition of L^∞ we now prove:

THEOREM 2.2. *Let ρ be a function norm. Then*

(a) *The following are equivalent:*

(i) $L^\infty \subseteq L_\rho$.

(ii) *L_ρ contains the constant functions.*

(iii) *L_ρ contains e, the function of constant value* 1.

(b) *If $L_\rho \subseteq L^\infty$, then L_ρ is a (two-sided) ideal in L^∞, hence L_ρ is a subalgebra of L^∞.*

Proof. Obviously (a)(i) $\Rightarrow$ (a)(ii) $\Rightarrow$ (a)(iii). Now, if $e \in L_\rho$ then for every $f \in L^\infty$,

$$|f(t)| \leqslant \|f\|_\infty \, e(t) \qquad \text{a.e.};$$

hence

$$\rho(f) \leqslant \|f\|_\infty \, \rho(e) < \infty.$$

Thus, $f \in L_\rho$, and (a)(iii) implies (a)(i).

To prove (b), let $L_\rho \subseteq L^\infty$. If $f \in L^\infty$, $g \in L_\rho$, then

$$|f(t)\, g(t)| \leqslant \|f\|_\infty \, |g(t)| \qquad \text{a.e.};$$

so $fg \in L_\rho$ since

$$\rho(fg) = \rho(|fg|) \leqslant \|f\|_\infty \, \rho(|g|) \leqslant \|f\|_\infty \, \rho(g) < \infty,$$

and the assertion follows. ∎

Theorem 2.2, which shows that L_ρ is often large enough to contain L^∞, immediately yields:

COROLLARY 2.1. *If $L_\rho \subseteq L^\infty$ then $L_\rho = L^\infty$ if and only if L_ρ contains the constant functions on* $\mathbf{T}$.

We call our function norm ρ *σ-subadditive* if

$$\{f_n\}_1^\infty \subset \mathcal{M},\ f_n \geqslant 0, \text{ implies } \rho\left(\sum_1^\infty f_n\right) \leqslant \sum_1^\infty \rho(f_n).$$

Further, we say that ρ has the *Riesz–Fischer property* if

$$\{f_n\}_1^\infty \subset \mathcal{M},\ f_n \geqslant 0,\ \sum_1^\infty \rho(f_n) < \infty, \text{ implies } \rho\left(\sum_1^\infty f_n\right) < \infty.$$

With these definitions and with Theorems 4.2 and 4.8 in [LZ1] we get at once:

THEOREM 2.3 [LZ1]. *L_ρ is complete (with respect to ρ) if and only if ρ is σ-subadditive.*

We next prove:

THEOREM 2.4. *If ρ is σ-subadditive and $L_\rho \subseteq L^\infty$, then there exists a constant $C > 0$ such that*

$$\|f\|_\infty \leqslant C\rho(f) \qquad \forall f \in L_\rho.$$

Proof. Suppose the theorem is false. Then we can find a sequence $\{g_n\}_1^\infty \subset L_\rho$ so that

$$\rho(g_n) = 1, \qquad \|g_n\|_\infty \geqslant n^3, \qquad n = 1, 2, 3, ... \tag{2.2}$$

Set

$$g = \sum_1^\infty \frac{|g_n|}{n^2}.$$

As ρ is σ-subadditive and

$$\sum_1^\infty \rho\left(\frac{|g_n|}{n^2}\right) = \sum_1^\infty \frac{1}{n^2} < \infty,$$

we have $\rho(g) < \infty$. Thus $g \in L_\rho$, and so

$$g \in L^\infty. \tag{2.3}$$

On the other hand,

$$g \geqslant \frac{|g_n|}{n^2}, \qquad n = 1, 2, 3, ...;$$

so by the monotonicity of $\|\cdot\|_\infty$ and (2.2),

$$\|g\|_\infty \geqslant \frac{1}{n^2} \|g_n\|_\infty \geqslant n, \qquad n = 1, 2, 3, ...,$$

a contradiction to (2.3). ∎

Under the assumption of σ-subadditivity, part (b) of Theorem 2.2 can be strengthened as follows:

THEOREM 2.5 (First Main Theorem). *Let ρ be a σ-subadditive function norm. Then the following are equivalent:*

(a) *L_ρ is closed under squaring.*

(b) *L_ρ is closed under multiplication, hence an algebra.*

(c) $L_\rho \subseteq L^\infty$.

Proof. Suppose L_ρ is closed under squaring and take f, $g \in L_\rho$. Since ρ is absolute, monotonic, homogeneous, and subadditive, we have

$$\rho(fg) = \rho(|fg|) \leqslant \rho\left(\frac{|f|^2 + |g|^2}{2}\right)$$

$$\leqslant \frac{1}{2}\rho(|f|^2) + \frac{1}{2}\rho(|g|^2) = \frac{1}{2}\rho(f^2) + \frac{1}{2}\rho(g^2) < \infty.$$

Hence $fg \in L^\rho$, so (a) implies (b).

Next assume (b), choose $f \in L_\rho$, and let us prove that $f \in L^\infty$. If $f \notin L^\infty$ then there exist sets $\{\mathbf{A}_n\}_1^\infty \subset \Omega$ such that $m(\mathbf{A}_n) > 0$ and

$$|f(t)| \geqslant n^3 \qquad \text{on} \quad \mathbf{A}_n, \quad n = 1, 2, 3, ... \tag{2.4}$$

Let χ_n be the characteristic function of $\mathbf{A}_n$. As $\chi_n \neq 0$ and $|f| \geqslant n^3 \chi_n$, we have

$$0 \neq \chi_n \leqslant n^{-3}|f|.$$

So

$$0 < \rho(\chi_n) \leqslant n^{-3}\rho(f) < \infty,$$

and we can define

$$g_n = \frac{\chi_n}{\rho(\chi_n)} \in L_\rho, \qquad n = 1, 2, 3, \ldots .$$

Put

$$g = \sum_1^\infty \frac{g_n}{n^2}.$$

Since

$$\sum_1^\infty \rho\left(\frac{g_n}{n^2}\right) = \sum_1^\infty \frac{1}{n^2}\,\rho(g_n) = \sum_1^\infty \frac{1}{n^2} < \infty,$$

the σ-subadditivity of ρ implies that $\rho(g) < \infty$. Hence $g \in L^\rho$; so $fg \in L_\rho$ by hypothesis, and

$$\rho(fg) < \infty. \tag{2.5}$$

At the same time,

$$g \geqslant \frac{g_n}{n^2}, \qquad n = 1, 2, 3, \ldots;$$

so by (2.4)

$$|fg| \geqslant \frac{|f|\, g_n}{n^2} \geqslant \frac{n\chi_n}{\rho(\chi_n)}.$$

Thus

$$\rho(fg) = \rho(|fg|) \geqslant \rho\left(\frac{n\chi_n}{\rho(\chi_n)}\right) = n, \qquad n = 1, 2, 3, \ldots;$$

a contradiction to (2.5), so (c) follows.

Finally, (c) implies (b) by Theorem 2.2(b); and (a) is implied by (b). ∎

With Theorems 2.4 and 2.5 we proceed to prove:

THEOREM 2.6. *Let ρ be a σ-subadditive function norm, and let L_ρ satisfy the equivalent conditions* (a)–(c) *in Theorem* 2.5. *Then*

(i) *L_ρ is a subalgebra of L^∞.*

(ii) *ρ has M- hence Q-factors on L_ρ.*

(iii) *The sets of M- and Q-factors for ρ coincide.*

(iv) *If* $L_\rho \neq \{0\}$ *then*

$$\mu_\rho \equiv \sup\{\|f\|_\infty : f \in L_\rho, \rho(f) \leqslant 1\} \tag{2.6}$$

is an M-factor for ρ.

(v) *If* $L_\rho = \{0\}$ *then every* $\mu > 0$ *is an M-factor for* ρ.

Proof. Part (i) is merely (b) and (c) of Theorem 2.5; and since (v) is trivial let us prove (iv).

As $L_\rho \neq \{0\}$ and closed under multiplication, we have

$$\mu_{\text{inf}} \equiv \sup\{\rho(fg): f, g \in L_\rho, \rho(f) \leqslant 1, \rho(g) \leqslant 1\} > 0. \tag{2.7}$$

Further,

$$\rho(fg) \leqslant \|f\|_\infty \rho(g), \qquad \forall f, g \in L_\rho,$$

and so

$$\mu_{\text{inf}} \leqslant \sup\{\|f\|_\infty : f \in L_\rho, \rho(f) \leqslant 1\} = \mu_\rho. \tag{2.8}$$

Next, since $L_\rho \subseteq L^\infty$, Theorem 2.4 provides a constant $C > 0$ such that

$$\|f\|_\infty \leqslant C\rho(f), \qquad \forall f \in L_\rho;$$

thus

$$\mu_\rho \leqslant C. \tag{2.9}$$

Combining (2.7)–(2.9) we get

$$0 < \mu_{\text{inf}} \leqslant \mu_\rho < \infty; \tag{2.10}$$

so μ_ρ is an M-factor by Theorem 1.1, and (iv) follows.

Now (ii) holds by (iv) and (v); and as L_ρ is closed under absolute values and ρ is absolute, Corollary 1.1 implies (iii). ∎

Note that μ_ρ in (2.6) is the operator norm $\|E\|$ of the embedding $E: L_\rho \to L^\infty$ defined by

$$Ef = f, \qquad f \in L_\rho.$$

Further note that if $L_\rho = \{0\}$, then μ_ρ in (2.6) vanishes, hence it is not an M-factor by definition.

In Theorem 2.9 we give conditions under which the best M- and Q-factors for ρ are both given by μ_ρ.

Following [LZ4] we call a function norm ρ *saturated* if for every set $\mathbf{A} \in \Omega$ of finite positive measure, there exists a measurable subset $\mathbf{B} \subseteq \mathbf{A}$ of positive measure, such that the characteristic function $\chi_{\mathbf{B}}$ is in L_ρ, i.e., $\rho(\chi_{\mathbf{B}}) < \infty$.

Given a function norm ρ we now consider its *associate* ρ', defined on $\mathcal{M}$ by

$$\rho'(f) = \sup\left\{\int_{\mathbf{T}} |f(t)\, g(t)|\, dm : g \in L_\rho, \rho(g) \leqslant 1\right\}, \qquad f \in \mathcal{M}. \quad (2.11)$$

Thus we obtain a function

$$\rho' : \mathcal{M} \to [0, \infty]$$

with the following property:

THEOREM 2.7 [LZ4, Theorems 9.2 and 9.7]. *Let ρ be a function norm. Then the associate function ρ' is a function norm if and only if ρ is saturated.*

We define now the space $L_{\rho'}$ analogously to L_ρ. Again we follow standard notation letting $L^1 = L^1(\mathbf{T}, \Omega, m)$ be the space of all measurable functions on $\mathbf{T}$ with

$$\|f\|_1 \equiv \int_{\mathbf{T}} |f(t)|\, dm < \infty.$$

We note that our function norm ρ and its associate ρ' satisfy the Hölder Inequality

$$\|fg\|_1 \leqslant \rho(f)\, \rho'(g) \qquad \forall f \in L_\rho, \quad g \in L_{\rho'}. \quad (2.12)$$

For by (2.11),

$$\rho(f)\, \rho'(g) = \rho(f) \sup\left\{\int_{\mathbf{T}} |g(t)\, h(t)|\, dm : h \in L_\rho, \rho(h) \leqslant 1\right\}$$

$$\geqslant \rho(f) \int_{\mathbf{T}} \left| g(t) \frac{f(t)}{\rho(f)} \right| dm = \int_{\mathbf{T}} |f(t)\, g(t)|\, dm = \|fg\|_1.$$

Since as mentioned before, σ-subadditivity and the Riesz–Fischer property are equivalent, we rephrase now a factorization theorem by T. A. Gillespie:

THEOREM 2.8 [G, Theorem 2(ii)]. *Let $(\mathbf{X}, \Lambda, \bar{m})$ be a finite measure space with $\bar{m}(\mathbf{X}) = 1$. Let η be a σ-subadditive, saturated function norm, and* *let η' be the associate of η. Then for every $\varepsilon > 0$ there exist functions $u \in L_\eta$, $v \in L_{\eta'}$ such that $uv = 1$ and*

$$\eta(u)\,\eta'(v) \leqslant 1 + \varepsilon.$$

For sets $\mathbf{X}$ of arbitrary finite measure we need the following obvious modification of Gillespie's theorem:

COROLLARY 2.2. *Let $(\mathbf{X}, \Lambda, m)$ be a finite measure space, and let η and η' be as in Theorem 2.8. Then for every $\varepsilon > 0$ there exist $u \in L_\eta$, $v \in L_{\eta'}$ with $uv = 1$ and*

$$\eta(u)\,\eta'(v) \leqslant (1 + \varepsilon) m(\mathbf{X}).$$

Proof. Consider the measure space $(\mathbf{X}, \Lambda, \bar{m})$ where $\bar{m} = m/m(\mathbf{X})$. Then $\bar{m}(\mathbf{X}) = 1$, and by observing that the associate function norm of η with respect to the measure $\bar{m}$ is $\eta'/m(\mathbf{X})$, the required result follows from Theorem 2.8. ∎

Returning to our measure space $(\mathbf{T}, \Omega, m)$, we recall that a set $\mathbf{A} \in \Omega$ is an *infinite atom* if $m(\mathbf{A}) = \infty$, and for every $\mathbf{B} \in \Omega$ satisfying $\mathbf{B} \subseteq \mathbf{A}$, either $m(\mathbf{B}) = 0$ or $m(\mathbf{B}) = \infty$.

We are finally ready to prove:

THEOREM 2.9 (Second Main Theorem). *Let $(\mathbf{T}, \Omega, m)$ be free of infinite atoms, and let ρ be a σ-subadditive, saturated function norm. If L_ρ satisfies the equivalent conditions* (a)–(c) *in Theorem* 2.5, *then*:

(i) $L_\rho \neq \{0\}$.

(ii) *L_ρ is a subalgebra of L^∞.*

(iii) *ρ has M- hence Q-factors on L_ρ.*

(iv) *The sets of M- and Q-factors for ρ coincide.*

(v) *The best (least) M- and Q-factors for ρ on L_ρ are both given by μ_ρ in* (2.6).

Proof. As $(\mathbf{T}, \Omega, m)$ is free of infinite atoms and ρ is saturated, (i) follows. So the heart of the theorem is part (v) since (ii)–(iv) are contained in Theorem 2.6.

We begin the proof of (v) by recalling (2.10) and Theorem 1.1, by which μ_{inf} in (2.7) is the best M-factor for ρ and μ_ρ in (2.6) satisfies

$$\mu_\rho \geqslant \mu_{\mathrm{inf}}.$$

So it remains to show that

$$\mu_\rho \leqslant \mu_{\mathrm{inf}}.$$

To this end, let ρ' be the associate of ρ defined in (2.11), and let $f, g \in L_\rho$, $h \in L_{\rho'}$. By hypothesis, L_ρ is an algebra; hence $fg \in L_\rho$ and so by (2.12),

$$\|fgh\|_1 \leqslant \rho(fg)\, \rho'(h).$$

Consequently,

$$\|fgh\|_1 \leqslant \rho(fg) \qquad \forall f, g \in L_\rho,\ h \in L_{\rho'},\ \rho'(h) \leqslant 1;$$

thus

$$\begin{aligned}\mu_{\text{inf}} &\equiv \sup\{\rho(fg): f, g \in L_\rho, \rho(f) \leqslant 1, \rho(g) \leqslant 1\}\\ &\geqslant \sup\{\|fgh\|_1: f, g \in L_\rho, h \in L_{\rho'}, \rho(f) \leqslant 1, \rho(g) \leqslant 1, \rho'(h) \leqslant 1\}. \qquad (2.13)\end{aligned}$$

Next, select $f \in L_\rho$. Then $f \in L^\infty$ since by hypothesis $L_\rho \subseteq L^\infty$. Further, given $\delta > 0$, the definition of $\|\cdot\|_\infty$ implies that

$$\mathbf{A} = \mathbf{A}(f, \delta) = \{t \in \mathbf{T}: |f(t)| \geqslant \|f\|_\infty - \delta\} \qquad (2.14\text{a})$$

is a set of positive measure. As $\mathbf{A}$ is not an infinite atom, there exists a subset $\mathbf{X} = \mathbf{X}(f, \delta) \subseteq \mathbf{A}$, such that $0 < m(\mathbf{X}) < \infty$, so by (2.14a),

$$|f(t)| \geqslant \|f\|_\infty - \delta \qquad \text{on } \mathbf{X}. \qquad (2.14\text{b})$$

Consider now the finite measure space $(\mathbf{X}, \Lambda, m)$ where

$$\Lambda \equiv \{\mathbf{B} \cap \mathbf{X}: \mathbf{B} \in \Omega\}.$$

For each $k \in \mathcal{M}(\mathbf{X}, \Lambda, m)$ define a function $\hat{k} \in \mathcal{M}(\mathbf{T}, \Omega, m)$ by

$$\hat{k}(t) = \begin{cases} k(t), & t \in \mathbf{X} \\ 0 & t \in \mathbf{T} \backslash \mathbf{X}, \end{cases} \qquad (2.15\text{a})$$

and set

$$\eta(k) = \rho(\hat{k}) \qquad \forall k \in \mathcal{M}(\mathbf{X}, \Lambda, m). \qquad (2.15\text{b})$$

Clearly, η is a function norm on $\mathcal{M}(\mathbf{X}, \Lambda, m)$; and as ρ is σ-subadditive and saturated, so is η. Moreover, by (2.15) and (2.11), the associate of η satisfies

$$\begin{aligned}\eta'(k) &= \sup\left\{\int_{\mathbf{X}} |k(t)\, h(t)|\, dm: h \in L_\eta, \eta(h) \leqslant 1\right\}\\ &= \sup\left\{\int_{\mathbf{T}} |\hat{k}(t)\, h(t)|\, dm: h \in L_\rho, \rho(h) \leqslant 1\right\} = \rho'(\hat{k}), \qquad k \in \mathcal{M}(\mathbf{X}, \Lambda, m).\end{aligned} \qquad (2.16)$$

Now given $\varepsilon > 0$, Corollary 2.2 yields functions $u \in L_\eta$, $v \in L_{\eta'}$, such that

$$uv = 1 \qquad \text{on } \mathbf{X} \tag{2.17}$$

and

$$\eta(u)\,\eta'(v) \leqslant (1+\varepsilon)\, m(\mathbf{X}). \tag{2.18}$$

Defining $\hat{u}$, $\hat{v}$ as in (2.15a) and putting

$$u_0 = \frac{\hat{u}\eta'(v)}{(1+\varepsilon)\, m(\mathbf{X})}, \qquad v_0 = \frac{\hat{v}}{\eta'(v)}, \tag{2.19}$$

we have

$$u_0 \in L_\rho, \qquad v_0 \in L_{\rho'}, \tag{2.20a}$$

where by (2.19), (2.15b), (2.18), and (2.16),

$$\rho(u_0) = \frac{\rho(\hat{u})\,\eta'(v)}{(1+\varepsilon)\, m(\mathbf{X})} = \frac{\eta(u)\,\eta'(v)}{(1+\varepsilon)\, m(\mathbf{X})} \leqslant 1 \tag{2.20b}$$

and

$$\rho'(v_0) = \frac{\rho'(\hat{v})}{\eta'(v)} = 1. \tag{2.20c}$$

By (2.20), (2.19), (2.17), and (2.14b) therefore,

$$\begin{aligned}
&\sup\{\|fgh\|_1 : g \in L_\rho,\ h \in L_{\rho'},\ \rho(g) \leqslant 1,\ \rho'(h) \leqslant 1\} \\
&\quad \geqslant \|fu_0v_0\|_1 = \frac{1}{(1+\varepsilon)\, m(\mathbf{X})}\, \|f\hat{u}\hat{v}\|_1 = \frac{1}{(1+\varepsilon)\, m(\mathbf{X})}\, \|f\chi_{\mathbf{X}}\|_1 \\
&\quad \geqslant \frac{\|f\|_\infty - \delta}{(1+\varepsilon)\, m(\mathbf{X})}\, \|\chi_{\mathbf{X}}\|_1 = \frac{\|f\|_\infty - \delta}{1+\varepsilon}.
\end{aligned}$$

And as ε and δ are arbitrarily small, we get

$$\|f\|_\infty \leqslant \sup\{\|fgh\|_1 : g \in L_\rho,\ h \in L_{\rho'},\ \rho(g) \leqslant 1,\ \rho'(h) \leqslant 1\} \qquad \forall f \in L_\rho. \tag{2.21}$$

Finally, by (2.21) and (2.13),

$$\begin{aligned}
\mu_\rho &\equiv \sup\{\|f\|_\infty : f \in L_\rho,\ \rho(f) \leqslant 1\} \\
&\leqslant \sup\{\|fgh\|_1 : f, g \in L_\rho,\ h \in L_{\rho'},\ \rho(f) \leqslant 1,\ \rho(g) \leqslant 1,\ \rho'(h) \leqslant 1\} \leqslant \mu_{\inf},
\end{aligned}$$

and the proof is complete. ∎

For certain function norms we are able to compute the best M-factors by elementary means rather than via μ_ρ in (2.6). For example, let $c \in \mathcal{M}(\mathbf{T}, \Omega, m)$, $|c| > 0$ a.e., be a fixed function, and in analogy with (1.3) define the σ-subadditive function norm

$$\rho_c(f) = \|cf\|_\infty, \qquad f \in \mathcal{M}. \tag{2.22}$$

We can prove:

THEOREM 2.10. *Let ρ_c be the function norm in (2.22) where $c \in \mathcal{M}(\mathbf{T}, \Omega, m)$, $|c| > 0$ a.e., is fixed. Then*:

(a) $L_{\rho_c} \neq \{0\}$.

(b) $L_{\rho_c} \subseteq L^\infty$ *if and only if* $1/c \in L^\infty$, *where*

$$\frac{1}{c}(t) \equiv \begin{cases} \dfrac{1}{c(t)} & \textit{when} \quad c(t) \neq 0 \\ 0 & \textit{otherwise}. \end{cases}$$

(c) *If* $1/c \in L^\infty$ *then*:

(i) L_{ρ_c} *is closed under multiplication, hence an algebra.*

(ii) ρ_c *has M- hence Q-factors.*

(iii) *The M- and Q-factors for* ρ_c *coincide.*

(iv) *The best (least) M- and Q-factors for* ρ_c *are both given by* $\|1/c\|_\infty$.

Proof. (a) $1/c \in L_{\rho_c}$ since

$$\rho_c\left(\frac{1}{c}\right) = \left\|\frac{1}{c}\,c\right\|_\infty = 1. \tag{2.23}$$

(b) If $L_{\rho_c} \subseteq L^\infty$ then $1/c \in L^\infty$ by (a). Conversely, suppose $1/c \in L^\infty$. Then

$$\|f\|_\infty = \left\|\frac{1}{c}\,cf\right\|_\infty \leqslant \left\|\frac{1}{c}\right\|_\infty \|cf\|_\infty = \left\|\frac{1}{c}\right\|_\infty \rho_c(f) < \infty \qquad \forall f \in L_{\rho_c},$$

so $L_{\rho_c} \subseteq L^\infty$.

(c) Let $1/c \in L^{\infty}$. Since $0 < \|1/c\|_{\infty} < \infty$ it follows that for all $f, g \in L_{\rho_c}$,

$$\rho_c(fg) = \|cfg\|_{\infty} \leqslant \left\|\frac{1}{c}\right\|_{\infty} \|cf\|_{\infty} \|cg\|_{\infty}$$

$$= \left\|\frac{1}{c}\right\|_{\infty} \rho_c(f)\, \rho_c(g), \qquad \forall f, g \in L_{\rho_c}; \tag{2.24}$$

hence (i) and (ii) follow.

Denote now the infima of all M- and Q-factors for ρ_c by μ_0 and λ_0, respectively. Then by (2.24),

$$\left\|\frac{1}{c}\right\|_{\infty} \geqslant \mu_0 \geqslant \lambda_0.$$

By Theorem 1.2, however,

$$\lambda_0 = \sup\{\rho_c(f^2) : f \in L_{\rho_c},\ \rho_c(f) \leqslant 1\};$$

so (2.23) yields

$$\lambda_0 \geqslant \rho_c\left(\frac{1}{c^2}\right) = \left\|\frac{1}{c}\right\|_{\infty}.$$

Thus $\mu_0 = \lambda_0 = \|1/c\|_{\infty}$ and the proof is complete. ∎

Note that $c \in L^{\infty}$ with $|c| > 0$ a.e. implies

$$\left\|\frac{1}{c}\right\|_{\infty} = \text{ess.sup}\left\{\frac{1}{|c(t)|} : t \in \mathbf{T},\ c(t) \neq 0\right\}$$

$$= [\text{ess.inf}\{|c(t)| : t \in \mathbf{T},\ c(t) \neq 0\}]^{-1}.$$

Hence Theorem 2.10 is seen to be a special case of Theorem 1.3 when it is recognized that L^{∞}, as a normed algebra, is equivalent to a space of continuous functions on a compact Hausdorff space [GN].

In concluding the paper we mention that

$$\rho : \mathcal{M} \to [0, \infty]$$

is a *function seminorm* on $\mathcal{M} = \mathcal{M}(\mathbf{T}, \Omega, m)$ if it satisfies (2.1) with (2.1a) replaced by

$$\rho(f) = 0 \qquad \text{for} \quad f = 0 \text{ a.e.}$$

The reason we avoid such seminorms in the present paper is that in this case it is no longer possible to embed L_ρ into L^∞ in the manner discussed above.

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