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Stability of the weighted splitting finite-difference scheme for a two-dimensional parabolic equation with two nonlocal integral conditions

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ABSTRACT

Nonlocal conditions arise in mathematical models of various physical, chemical or biological processes. Therefore, interest in developing computational techniques for the numerical solution of partial differential equations (PDEs) with various types of nonlocal conditions has been growing fast. We construct and analyse a weighted splitting finite-difference scheme for a two-dimensional parabolic equation with nonlocal integral conditions. The main attention is paid to the stability of the method. We apply the stability analysis technique which is based on the investigation of the spectral structure of the transition matrix of a finite-difference scheme. We demonstrate that depending on the parameters of the finite-difference scheme and nonlocal conditions the proposed method can be stable or unstable. The results of numerical experiments with several test problems are also presented and they validate theoretical results.

1. Introduction

We consider the two-dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t), \quad 0 < x < L_x,\ 0 < y < L_y, 0 < t \leqslant T, \tag{1}$$

subject to nonlocal integral conditions

$$u(0, y, t) = \gamma_1 \int_0^{L_x} \alpha(x) u(x, y, t) \mathrm{d}x + \mu_1(y, t), \tag{2}$$

$$u(L_x, y, t) = \gamma_2 \int_0^{L_x} \beta(x) u(x, y, t) \mathrm{d}x + \mu_2(y, t), \quad 0 < y < L_y, 0 < t \leqslant T, \tag{3}$$

boundary conditions

$$u(x, 0, t) = \mu_3(x, t), \qquad u(x, L_y, t) = \mu_4(x, t), \quad 0 < x < L_x, 0 < t \leqslant T, \tag{4}$$

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and initial condition

$$u(x, y, 0) = \varphi(x, y), \quad 0 \leqslant x \leqslant L_x, 0 \leqslant y \leqslant L_y, \tag{5}$$

where $f(x, y, t)$, $\mu_1(y, t)$, $\mu_2(y, t)$, $\mu_3(x, t)$, $\mu_4(x, t)$, $\alpha(x)$, $\beta(x)$, $\varphi(x, y)$ are given functions, γ_1, γ_2 are given parameters, and function $u(x, y, t)$ is unknown. We assume that for all t, $0 < t \leqslant T$, nonlocal integral conditions (2), (3) and boundary conditions (4) are compatible, i.e., the following compatibility conditions are satisfied:

$$\gamma_1 \int_0^{L_x} \alpha(x)\mu_3(x, t)\mathrm{d}x + \mu_1(0, t) = \mu_3(0, t),$$

$$\gamma_1 \int_0^{L_x} \alpha(x)\mu_4(x, t)\mathrm{d}x + \mu_1(L_y, t) = \mu_4(0, t),$$

$$\gamma_2 \int_0^{L_x} \beta(x)\mu_3(x, t)\mathrm{d}x + \mu_2(0, t) = \mu_3(L_x, t),$$

$$\gamma_2 \int_0^{L_x} \beta(x)\mu_4(x, t)\mathrm{d}x + \mu_2(L_y, t) = \mu_4(L_x, t).$$

Nonlocal integral conditions of type (2), (3) often arise in mathematical models of various physical, chemical or biological processes. For example, we can mention the mathematical model of the quasi-static flexure of a thermoelastic rod [1–4]. It is proved that the entropy is the solution of a certain parabolic equation subject to initial condition and nonlocal integral conditions with special types of weight functions $\alpha(x)$ and $\beta(x)$.

Various differential problems with nonlocal integral conditions are investigated both in theoretical (see, e.g., [1–7]) and numerical (see, e.g., [8–12] and some of below mentioned references) aspects. The review of results related with the numerical solution of one-dimensional parabolic equation subject to various type of nonlocal integral specifications as well as the examples of the applications of such problems are presented in paper [13].

The present paper is devoted to the finite-difference scheme for the two-dimensional differential problem (1)–(5). We construct the weighted finite-difference scheme and analyse its stability. The proposed method is based on the splitting of the two-dimensional differential problem into two finite-difference subproblems. With particular values of the weights of the scheme we have so-called locally one-dimensional (LOD), alternating direction implicit (ADI) or fully-explicit splitting finite-difference schemes.

The stability of implicit and explicit finite-difference schemes for the corresponding one-dimensional parabolic problems with nonlocal integral conditions similar to conditions (2), (3) has been investigated by many authors (see, e.g., [14–17]). In paper [17], the differential problem (1)–(5) is formulated as an example of a problem for the possible extension of the proposed stability analysis technique. Paper [18] is devoted to the stability of implicit, explicit and Crank–Nicolson (symmetric) finite-difference schemes for one- and two-dimensional parabolic equations with a special case of Bitsadze–Samarskii type nonlocal conditions. Various LOD and ADI methods for two-dimensional parabolic problems with nonlocal integral condition (the specification of mass/energy) have been investigated by M. Dehghan (see, e.g., [19–21]).

Paper [22] deals with the ADI method for the two-dimensional parabolic Eq. (1) with Bitsadze–Samarskii type nonlocal boundary condition. We use a similar technique and argument in order to construct the weighted splitting finite-difference scheme for the two-dimensional differential problem (1)–(5) and to investigate the stability of that method.

The paper is organised as follows. In Section 2, the notation is introduced and the details of the finite-difference scheme are described. In the same section, the stability analysis technique based on the spectral structure of the transition matrix of a finite-difference scheme is applied in order to analyse the stability of the proposed method. The results of numerical experiments with several test problems are presented in Section 3. Some remarks in Section 4 conclude the paper.

2. Finite-difference scheme and its stability

2.1. Notation

To solve the two-dimensional differential problem (1)–(5) numerically, we apply the finite-difference technique [23]. Let us define discrete grids with uniform steps,

$$\omega_{h_1} = \{x_i = ih_1, i = 1, 2, \ldots, N_1 - 1, N_1h_1 = L_x\}, \qquad \overline{\omega}_{h_1} = \omega_{h_1} \cup \{x_0 = 0, x_{N_1} = L_x\},$$
$$\omega_{h_2} = \{y_j = jh_2, j = 1, 2, \ldots, N_2 - 1, N_2h_2 = L_y\}, \qquad \overline{\omega}_{h_2} = \omega_{h_2} \cup \{y_0 = 0, y_{N_2} = L_y\},$$
$$\omega = \omega_{h_1} \times \omega_{h_2}, \qquad \overline{\omega} = \overline{\omega}_{h_1} \times \overline{\omega}_{h_2},$$
$$\omega^\tau = \{t^k = k\tau, k = 1, 2, \ldots, M, M\tau = T\}, \qquad \overline{\omega}^\tau = \omega^\tau \cup \{t^0 = 0\}.$$

We use the notation $U_{ij}^k = U(x_i, y_j, t^k)$ for functions defined on the grid $\overline{\omega} \times \overline{\omega}^\tau$ or its parts, and the notation $U_{ij}^{k+1/2} = U(x_i, y_j, t^k + 0.5\tau)$ (some of the indices can be omitted). We define one-dimensional discrete operators

$$\Lambda_1 U_{ij} = \frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h_1^2}, \qquad \Lambda_2 U_{ij} = \frac{U_{i,j-1} - 2U_{ij} + U_{i,j+1}}{h_2^2}.$$

In order to approximate nonlocal integral conditions (2), (3), we will use the trapezoidal rule. For functions U and V defined on the grid $\overline{\omega}_{h_1}$ we introduce the notation

$$(U, V) = h_1\left(\frac{U_0V_0 + U_{N_1}V_{N_1}}{2} + \sum_{i=1}^{N_1-1} U_iV_i\right).$$

Let E_N be the identity matrix of order N and $A \otimes B$ denotes the Kronecker (tensor) product of matrices A and B. We denote the eigenvalues of matrix A by $\lambda(A)$. The spectral radius of matrix A is denoted by $\rho(A)$, i.e.,

$$\rho(A) = \max_{\lambda(A)} |\lambda(A)|.$$

2.2. Development of the finite-difference scheme

We explain the main steps of the method for the numerical solution of problem (1)–(5).

First of all, we replace the initial condition (5) by equations

$$U_{ij}^0 = \varphi_{ij}, \quad (x_i, y_j) \in \overline{\omega}. \tag{6}$$

Then, for any k, $0 \leqslant k < M - 1$, the transition from the kth layer of time to the $(k+1)$th layer can be carried out by splitting it into two stages and solving one-dimensional finite-difference subproblems in each of them. By evaluating the derivative with respect to x explicitly and the derivative with respect to y implicitly, we get the first one-dimensional subproblem, i.e., the set of linear algebraic equations systems for $i = 1, 2, \ldots, N_1 - 1$:

$$\frac{U_{ij}^{k+1/2} - U_{ij}^k}{\tau} = (1-\sigma_1)\Lambda_1 U_{ij}^k + \sigma_2\Lambda_2 U_{ij}^{k+1/2} + \sigma_2 f_{ij}^{k+1/2}, \quad y_j \in \omega_{h_2}, \tag{7}$$

$$U_{i0}^{k+1/2} = (\widetilde{\mu}_3)_i, \tag{8}$$

$$U_{iN_2}^{k+1/2} = (\widetilde{\mu}_4)_i, \tag{9}$$

where σ_1 and σ_2 are the weights of the finite-difference scheme,

$$\widetilde{\mu}_3 = \sigma_2(\mu_3)^{k+1} + (1-\sigma_2)(\mu_3)^k - \tau\sigma_1\sigma_2\Lambda_1(\mu_3)^{k+1} + \tau(1-\sigma_1)(1-\sigma_2)\Lambda_1(\mu_3)^k,$$
$$\widetilde{\mu}_4 = \sigma_2(\mu_4)^{k+1} + (1-\sigma_2)(\mu_4)^k - \tau\sigma_1\sigma_2\Lambda_1(\mu_4)^{k+1} + \tau(1-\sigma_1)(1-\sigma_2)\Lambda_1(\mu_4)^k.$$

The second subproblem (the set of linear algebraic equations systems for $j = 1, 2, \ldots, N_2 - 1$) is implicit with respect to x and explicit with respect to y:

$$\frac{U_{ij}^{k+1} - U_{ij}^{k+1/2}}{\tau} = \sigma_1\Lambda_1 U_{ij}^{k+1} + (1-\sigma_2)\Lambda_2 U_{ij}^{k+1/2} + (1-\sigma_2)f_{ij}^{k+1}, \quad x_i \in \omega_{h_1}, \tag{10}$$

$$U_{0j}^{k+1} = \gamma_1\big(\alpha, U\big)_j^{k+1} + (\mu_1)_j^{k+1}, \tag{11}$$

$$U_{N_1j}^{k+1} = \gamma_2\big(\beta, U\big)_j^{k+1} + (\mu_2)_j^{k+1}. \tag{12}$$

Every transition is finished by computing

$$U_{i0}^{k+1} = (\mu_3)_i^{k+1}, \qquad U_{iN_2}^{k+1} = (\mu_4)_i^{k+1}, \quad x_i \in \overline{\omega}_{h_1}. \tag{13}$$

Thus, the procedure of numerical solution can be stated as follows:

procedure `The Weighted Splitting Finite-Difference Scheme`
begin
  Compute U_{ij}^0 ($i = 0, 1, \ldots, N_1, j = 0, 1, \ldots, N_2$) from Eqs. (6) ;
  for $k = 0, 1, \ldots, M - 1$
    for $i = 1, 2, \ldots, N_1 - 1$
      Solve system (7)–(9) and compute $U_{ij}^{k+1/2}$ ($j = 0, 1, \ldots, N_2$);
    end for
    for $j = 1, 2, \ldots, N_2 - 1$
      Solve system (10)–(12) and compute U_{ij}^{k+1} ($i = 0, 1, \ldots, N_1$);
    end for
    Compute U_{i0}^{k+1} and $U_{iN_2}^{k+1}$ ($i = 0, 1, \ldots, N_1$) from Eqs. (13) ;
  end for
end

If $\sigma_1 = \sigma_2 = 1$ or $\sigma_1 = \sigma_2 = 1/2$, we have LOD or ADI methods, respectively. The splitting finite-difference scheme is fully-explicit for $\sigma_1 = \sigma_2 = 0$. The finite-difference subproblems which appear when executing the transition from the kth layer of time to the $(k+1)$th layer in case of LOD method are fully-implicit. In case of ADI method these subproblems are semi-implicit. The LOD method approximates the differential problem (1)–(5) with error $O(\tau + h_1^2 + h_2^2)$ while the approximation errors of ADI and fully-explicit methods are $O(\tau^2 + h_1^2 + h_2^2)$ and $O(\tau + h_1 + h_2)$, respectively [23].

If one or the both of the finite-difference subproblems (7)–(9), (10)–(12) are fully-explicit (i.e., $\sigma_1 = 0$ and/or $\sigma_2 = 0$), then the corresponding subproblem(-s) can be solved explicitly. However, if the finite-difference subproblem (7)–(9) is not fully-explicit ($\sigma_2 \neq 0$), then it is noteworthy that we can use the well-known Thomas algorithm and efficiently solve systems (7)–(9) because of the tridiagonality of their matrices. In order to solve the implicit finite-difference subproblem (10)–(12) ($\sigma_1 \neq 0$), the modification of the general algorithm for solving linear equations systems with quasi-tridiagonal matrices [24] can be used.

Now let us transform the finite-difference scheme (7)–(12) to the matrix form. From Eqs. (11) and (12) we obtain

$$U_{0j}^{k+1} = \gamma_1 h_1 \sum_{i=1}^{N_1-1} a_i U_{ij}^{k+1} + (\overline{\mu}_1)_j^{k+1},$$

$$U_{N_1 j}^{k+1} = \gamma_2 h_1 \sum_{i=1}^{N_1-1} b_i U_{ij}^{k+1} + (\overline{\mu}_2)_j^{k+1},$$

where

$$a_i = \frac{1}{D}\Big(\alpha_i - \frac{\gamma_2 h_1 \alpha_i \beta_{N_1}}{2} + \frac{\gamma_2 h_1 \alpha_{N_1} \beta_i}{2}\Big),$$
$$b_i = \frac{1}{D}\Big(\beta_i + \frac{\gamma_1 h_1 \alpha_i \beta_0}{2} - \frac{\gamma_1 h_1 \alpha_0 \beta_i}{2}\Big),$$
$$(\overline{\mu}_1)_j^{k+1} = \frac{1}{D}\Big((\mu_1)_j^{k+1} - \frac{\gamma_2 h_1 \beta_{N_1}}{2}(\mu_1)_j^{k+1} + \frac{\gamma_1 h_1 \alpha_{N_1}}{2}(\mu_2)_j^{k+1}\Big),$$
$$(\overline{\mu}_2)_j^{k+1} = \frac{1}{D}\Big((\mu_2)_j^{k+1} + \frac{\gamma_2 h_1 \beta_0}{2}(\mu_1)_j^{k+1} - \frac{\gamma_1 h_1 \alpha_0}{2}(\mu_2)_j^{k+1}\Big),$$
$$D = \Big(1 - \frac{\gamma_1 h_1 \alpha_0}{2}\Big)\Big(1 - \frac{\gamma_2 h_1 \beta_{N_1}}{2}\Big) - \frac{\gamma_1 h_1 \alpha_{N_1}}{2} \cdot \frac{\gamma_2 h_1 \beta_0}{2}.$$

We assume that the grid step h_1 is chosen so that $D > 0$.

Let us introduce $(N_1 - 1) \times (N_1 - 1)$ and $(N_2 - 1) \times (N_2 - 1)$ matrices

$$\widetilde{\Lambda}_1 = h_1^{-2} \begin{pmatrix} -2+\delta_1^{(1)} & 1+\delta_1^{(2)} & \delta_1^{(3)} & \cdots & \delta_1^{(N_1-3)} & \delta_1^{(N_1-2)} & \delta_1^{(N_1-1)} \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ \delta_2^{(1)} & \delta_2^{(2)} & \delta_2^{(3)} & \cdots & \delta_2^{(N_1-3)} & 1+\delta_2^{(N_1-2)} & -2+\delta_2^{(N_1-1)} \end{pmatrix}$$

and

$$\widetilde{\Lambda}_2 = h_2^{-2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix},$$

where

$$\delta_1^{(i)} = \gamma_1 h_1 a_i, \qquad \delta_2^{(i)} = \gamma_2 h_1 b_i, \quad x_i \in \omega_{h_1}.$$

Now we define matrices of order $(N_1 - 1) \cdot (N_2 - 1)$,

$$A_1 = -E_{N_2-1} \otimes \widetilde{\Lambda}_1, \qquad A_2 = -\widetilde{\Lambda}_2 \otimes E_{N_1-1}.$$

We can directly verify that A_1 and A_2 are commutative matrices, i.e.,

$$A_1A_2 = A_2A_1 = \widetilde{\Lambda}_2 \otimes \widetilde{\Lambda}_1.$$

Introducing the matrices A_1 and A_2 allow us to rewrite the finite-difference scheme (7)–(12) in the following form:

$$\big(E + \sigma_2\tau A_2\big)U^{k+1/2} = \big(E - (1-\sigma_1)\tau A_1\big)U^k + \sigma_2\tau F^{k+1/2}, \tag{14}$$

$$\big(E + \sigma_1\tau A_1\big)U^{k+1} = \big(E - (1-\sigma_2)\tau A_2\big)U^{k+1/2} + (1-\sigma_2)\tau F^{k+1}, \tag{15}$$

where E is the identity matrix of order $(N_1 - 1)\cdot(N_2 - 1)$,

$$U = (\widetilde{U}_1, \widetilde{U}_2, \ldots, \widetilde{U}_j, \ldots, \widetilde{U}_{N_2-1})^T, \quad \widetilde{U}_j = (U_{1j}, U_{2j}, \ldots, U_{ij}, \ldots, U_{N_1-1,j})^T,$$

and

$$F^{k+1/2} = \big(F_1^{k+1/2}, F_2^{k+1/2}, \ldots, F_j^{k+1/2}, \ldots, F_{N_2-1}^{k+1/2}\big)^T,$$

$$F_1^{k+1/2} = \left(\frac{(\widetilde{\mu}_3)_1}{h_2^2} + f_{11}^{k+1/2}, \frac{(\widetilde{\mu}_3)_2}{h_2^2} + f_{21}^{k+1/2}, \ldots, \frac{(\widetilde{\mu}_3)_{N_1-1}}{h_2^2} + f_{N_1-1,1}^{k+1/2}\right)^T,$$

$$F_j^{k+1/2} = \big(f_{1j}^{k+1/2}, f_{2j}^{k+1/2}, \ldots, f_{ij}^{k+1/2}, \ldots, f_{N_1-1,j}^{k+1/2}\big)^T, \quad j = 2, 3, \ldots, N_2 - 2,$$

$$F_{N_2-1}^{k+1/2} = \left(\frac{(\widetilde{\mu}_4)_1}{h_2^2} + f_{1,N_2-1}^{k+1/2}, \frac{(\widetilde{\mu}_4)_2}{h_2^2} + f_{2,N_2-1}^{k+1/2}, \ldots, \frac{(\widetilde{\mu}_4)_{N_1-1}}{h_2^2} + f_{N_1-1,N_2-1}^{k+1/2}\right)^T,$$

$$F^{k+1} = \big(F_1^{k+1}, F_2^{k+1}, \ldots, F_j^{k+1}, \ldots, F_{N_2-1}^{k+1}\big)^T,$$

$$F_j^{k+1} = \left(\frac{(\overline{\mu}_1)_j^{k+1}}{h_1^2} + f_{1j}^{k+1}, f_{2j}^{k+1}, \ldots, f_{N_1-2,j}^{k+1}, \frac{(\overline{\mu}_2)_j^{k+1}}{h_1^2} + f_{N_1-1,j}^{k+1}\right)^T, \quad j = 1, 2, \ldots, N_2 - 1.$$

From Eqs. (14) and (15) it follows that

$$U^{k+1} = SU^k + \overline{F}^k, \tag{16}$$

where

$$S = \big(E + \sigma_1\tau A_1\big)^{-1}\big(E - (1-\sigma_2)\tau A_2\big)\big(E + \sigma_2\tau A_2\big)^{-1}\big(E - (1-\sigma_1)\tau A_1\big),$$

$$\overline{F}^k = \tau\big(E + \sigma_1\tau A_1\big)^{-1}\Big[\sigma_2\big(E - (1-\sigma_2)\tau A_2\big)\big(E + \sigma_2\tau A_2\big)^{-1}F^{k+1/2} + (1-\sigma_2)F^{k+1}\Big].$$

We assume that the existence of the matrices $\big(E + \sigma_1\tau A_1\big)^{-1}$ and $\big(E + \sigma_2\tau A_2\big)^{-1}$ is ensured by the formulation of the considered two-dimensional differential problem and the proposed finite-difference scheme.

2.3. Spectral structure of the matrix S

The spectral structure of finite-difference and differential operators with nonlocal conditions are investigated by many authors (see, e.g., [25–29] and references therein). Papers [17,30,31] deal with the eigenvalue problem for the matrix $(-\widetilde{\Lambda}_1)$ in general or special cases of functions $\alpha(x)$ and $\beta(x)$. Depending on the values of parameters γ_1, γ_2 and the expressions of functions $\alpha(x)$, $\beta(x)$, the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ can be both positive or non-positive real numbers and complex numbers with positive or non-positive real parts (see [17,30,31]).

It is well-known (see, e.g., [23]) that all the eigenvalues of the matrix $(-\widetilde{\Lambda}_2)$ are real, positive and algebraically simple:

$$\lambda_j(-\widetilde{\Lambda}_2) = \frac{4}{h_2^2}\sin^2\frac{j\pi h_2}{2}, \quad j = 1, 2, \ldots, N_2 - 1. \tag{17}$$

Hence, the matrix A_2 is a simple-structured matrix (i.e., all the eigenvalues of the matrix are distinct) as a Kronecker product of two simple-structured matrices, and its eigenvalues $\lambda(A_2)$ are real and positive numbers. Let us denote

$$\Delta_2 = \max_{\lambda(A_2)}\lambda(A_2) = \max_{1\leqslant j\leqslant N_2-1}\lambda_j(-\widetilde{\Lambda}_2) = \lambda_{N_2-1}(-\widetilde{\Lambda}_2) = \frac{4}{h_2^2}\sin^2\frac{(L_x - h_2)\pi}{2}.$$

If A_1 is a simple-structured matrix, then S is a simple-structured matrix, too. The eigenvalues of the matrix S can be expressed by the formula

$$\lambda(S) = \frac{\big(1 - (1-\sigma_1)\tau\lambda(A_1)\big)\big(1 - (1-\sigma_2)\tau\lambda(A_2)\big)}{\big(1 + \sigma_1\tau\lambda(A_1)\big)\big(1 + \sigma_2\tau\lambda(A_2)\big)}. \tag{18}$$

2.4. Analysis of the stability

Let us recall some facts related with the stability of the finite-difference schemes [17,23,32].

We know (see [23]) that a sufficient stability condition for the finite-difference scheme (16) can be written in the form

$$\|S\| \leqslant 1 + c_0\tau,$$

where a non-negative constant c_0 is independent on τ and h_1, h_2. Since in our case the matrix S is nonsymmetric (this property is typical for problems with nonlocal conditions), the norm $\|S\|$ can not be defined as spectral radius $\rho(S)$.

Let us assume that S is a simple-structured matrix, i.e., all the eigenvectors of the matrix S are linearly independent. Then it is possible to define the transformed matrix norm [17]

$$\|B\|_* = \|P^{-1}BP\|_\infty,$$

which is compatible with the vector norm

$$\|V\|_* = \|P^{-1}V\|_\infty,$$

where the columns of the matrix P are linearly independent eigenvectors of S,

$$\|B\|_\infty = \max_{1\leqslant i\leqslant m}\sum_{j=1}^{m}|b_{ij}|, \qquad \|V\|_\infty = \max_{1\leqslant i\leqslant m}|v_i|,$$

m is the order of the matrix $B = (b_{ij})_{i,j=1}^{m}$ and vector $V = (v_1, v_2, \dots, v_m)^T$. The matrix $P^{-1}SP$ is diagonal matrix and its elements are eigenvalues of S. As a result, the norm $\|S\|_*$ is equal to the spectral radius of S:

$$\|S\|_* = \|P^{-1}SP\|_\infty = \rho(S).$$

Such definition of transformed matrix norm $\|\cdot\|_*$ was formerly used to analyse the stability of the finite-difference schemes (see, e.g., [17,18,31]) and to investigate the convergence of iterative methods for solution of finite-difference schemes with nonlocal conditions (see, e.g., [25]).

Therefore, we will use the stability condition $\rho(S) < 1$ in the analysis of the stability of the finite-difference scheme (16). This condition ensures the stepwise stability of the scheme [32]. We recall that the finite-difference scheme (16) is called stepwise stable if for all fixed τ and h_1, h_2 there exists a constant $C = C(\tau, h_1, h_2)$ such that $|U_{ij}^k| \leqslant C$, $i = 0, 1, \dots, N_1$, $j = 0, 1, \dots, N_2$, $k = 0, 1, \dots$.

Let us assume that A_1 is a simple-structured matrix. Under this assumption we prove several statements related with the stability of the finite-difference scheme (16).

Theorem 1. *If all the eigenvalues of the matrix A_1 are real and non-negative numbers, then the finite-difference scheme (16) is stable under the constrains*

$$\sigma_1 > \sigma_1^* = \frac{1}{2} - \frac{1}{\tau\rho(A_1)}, \qquad \sigma_2 > \sigma_2^* = \frac{1}{2} - \frac{1}{\tau\Delta_2}. \tag{19}$$

Proof. From Eq. (18) it follows that

$$|\lambda(S)| = \left|\frac{1-(1-\sigma_1)\tau\lambda(A_1)}{1+\sigma_1\tau\lambda(A_1)}\right| \cdot \left|\frac{1-(1-\sigma_2)\tau\lambda(A_2)}{1+\sigma_2\tau\lambda(A_2)}\right|.$$

Thus, we conclude that $\rho(S) < 1$, if conditions (19) are fulfilled. □

Now let us assume that some of the eigenvalues of the matrix A_1 are conjugate complex numbers and denote them by

$$\lambda(A_1) = \mathrm{Re}\lambda(A_1) \pm i\mathrm{Im}\lambda(A_1).$$

Theorem 2. *If $\mathrm{Re}\lambda(A_1) \geqslant 0$ for all the eigenvalues of the matrix A_1, then the finite-difference scheme (16) is stable when*

$$\sigma_1 > \sigma_1^* = \frac{1}{2} - \frac{1}{\tau}\min_{\lambda(A_1)}\frac{\mathrm{Re}\lambda(A_1)}{|\lambda(A_1)|^2}, \quad \sigma_2 > \sigma_2^* = \frac{1}{2} - \frac{1}{\tau\Delta_2}. \tag{20}$$

Proof. From Eq. (18) we have

$$\begin{aligned}|\lambda(S)|^2 &= \left|\frac{1-(1-\sigma_1)\tau\lambda(A_1)}{1+\sigma_1\tau\lambda(A_1)}\right|^2 \cdot \left|\frac{1-(1-\sigma_2)\tau\lambda(A_2)}{1+\sigma_2\tau\lambda(A_2)}\right|^2 \\ &= \left|\frac{1-(1-\sigma_1)\tau\big(\mathrm{Re}\lambda(A_1)\pm i\mathrm{Im}\lambda(A_1)\big)}{1+\sigma_1\tau\big(\mathrm{Re}\lambda(A_1)\pm i\mathrm{Im}\lambda(A_1)\big)}\right|^2 \cdot \left|\frac{1-(1-\sigma_2)\tau\lambda(A_2)}{1+\sigma_2\tau\lambda(A_2)}\right|^2 \\ &= \frac{\big(1-(1-\sigma_1)\tau\mathrm{Re}\lambda(A_1)\big)^2+\big((1-\sigma_1)\tau\mathrm{Im}\lambda(A_1)\big)^2}{\big(1+\sigma_1\tau\mathrm{Re}\lambda(A_1)\big)^2+\big(\sigma_1\tau\mathrm{Im}\lambda(A_1)\big)^2} \cdot \left|\frac{1-(1-\sigma_2)\tau\lambda(A_2)}{1+\sigma_2\tau\lambda(A_2)}\right|^2.\end{aligned}$$

Now we can conclude that $\rho(S) < 1$ under conditions (20). □

Corollary 1. *If $\sigma_1 = \sigma_2 = \sigma$ and all the eigenvalues of the matrix A_1 are real and non-negative numbers, then the finite-difference scheme (16) is stable when $\sigma > \sigma^* = \max\{\sigma_1^*, \sigma_2^*\}$, i.e. when*

$$\sigma > \sigma^* = \frac{1}{2} - \frac{1}{\tau}\min\left\{\frac{1}{\rho(A_1)}, \frac{1}{\Delta_2}\right\}. \tag{21}$$

Corollary 2. *If $\sigma_1 = \sigma_2 = \sigma$ and $\mathrm{Re}\lambda(A_1) \geqslant 0$ for all the eigenvalues of the matrix A_1, then the finite-difference scheme (16) is stable when $\sigma > \sigma^* = \max\{\sigma_1^*, \sigma_2^*\}$, i.e. when*

$$\sigma > \sigma^* = \frac{1}{2} - \frac{1}{\tau}\min\left\{\min_{\lambda(A_1)}\frac{\mathrm{Re}\lambda(A_1)}{|\lambda(A_1)|^2}, \frac{1}{\Delta_2}\right\}. \tag{22}$$

The following corollaries state the sufficient conditions for the stability of LOD ($\sigma_1 = \sigma_2 = 1$), ADI ($\sigma_1 = \sigma_2 = 1/2$) and fully-explicit splitting ($\sigma_1 = \sigma_2 = 0$) finite-difference schemes.

Corollary 3. *If all the eigenvalues of the matrix A_1 are real and non-negative numbers, then for $\sigma_1 = \sigma_2 = 1$ or $\sigma_1 = \sigma_2 = 1/2$ the finite-difference scheme (16) is unconditionally stable and for $\sigma_1 = \sigma_2 = 0$ it is stable under condition*

$$\tau < \tau^* = 2\min\left\{\frac{1}{\rho(A_1)}, \frac{1}{\Delta_2}\right\}. \tag{23}$$

Corollary 4. *If $\mathrm{Re}\lambda(A_1) \geqslant 0$ for all the eigenvalues of the matrix A_1, then for $\sigma_1 = \sigma_2 = 1$ or $\sigma_1 = \sigma_2 = 1/2$ the finite-difference scheme (16) is unconditionally stable and for $\sigma_1 = \sigma_2 = 0$ it is stable under condition*

$$\tau < \tau^* = 2\min\left\{\min_{\lambda(A_1)}\frac{\mathrm{Re}\lambda(A_1)}{|\lambda(A_1)|^2}, \frac{1}{\Delta_2}\right\}. \tag{24}$$

We note that if all the eigenvalues of the matrix A_1 are real and non-negative numbers, then conditions (19), (21), (23) coincide with conditions (20), (22), (24), respectively.

Since the eigenvalues of the matrix A_1 coincide with the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ and they are multiple, the main point of the analysis of the stability of the finite-difference scheme (16) is to investigate the spectrum of the matrix $(-\widetilde{\Lambda}_1)$ and to verify whether $(-\widetilde{\Lambda}_1)$ is a simple-structured matrix and $\lambda_i(-\widetilde{\Lambda}_1) \geqslant 0$ or $\mathrm{Re}\lambda_i(-\widetilde{\Lambda}_1) \geqslant 0$, $i = 1, 2, \ldots, N_1 - 1$. Together with satisfaction of some of constraints (19)–(24), the non-negativity of the eigenvalues $\lambda_i(-\widetilde{\Lambda}_1)$ or their real parts $\mathrm{Re}\lambda_i(-\widetilde{\Lambda}_1)$ ensures the stability of the finite-difference scheme (16), but, as noted in [22], the scheme can be stable even if the matrix $(-\widetilde{\Lambda}_1)$ has a negative eigenvalue or a complex eigenvalue with a negative real part.

3. Numerical experiments

3.1. Technical details

In order to demonstrate the efficiency of the considered numerical method and practically justify the stability analysis technique, several test problems with different types of weight functions $\alpha(x)$ and $\beta(x)$ were solved. Functions $f(x, y, t)$, $\mu_1(y, t)$, $\mu_2(y, t)$, $\mu_3(x, t)$, $\mu_4(x, t)$ and $\varphi(x, y)$ were chosen so that particular functions $u(x, y, t)$ would be solutions to the differential problem (1)–(5). We also used results related with the structure of the spectrum of the matrix $(-\widetilde{\Lambda}_1)$ which were obtained in papers [17,30], where the corresponding one-dimensional problems have been investigated.

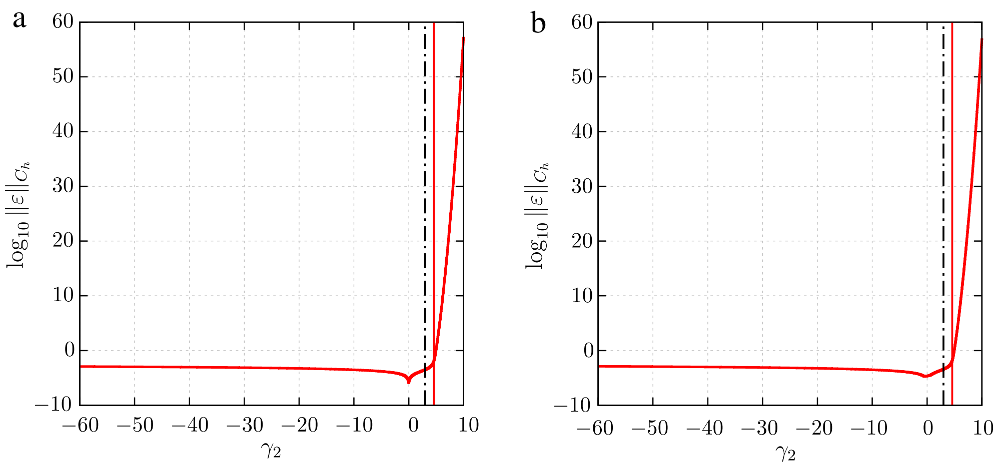

Fig. 1. The dependence of $\log_{10} \|\varepsilon\|_{C_h}$ on the values of parameter γ_2 in cases of (a) LOD method and (b) ADI method (Example 1). The dash-dot and solid vertical straight lines denote the lines $\gamma_2 = \widetilde{\gamma}_2$ and $\gamma_2 = \gamma_2^*$, respectively.

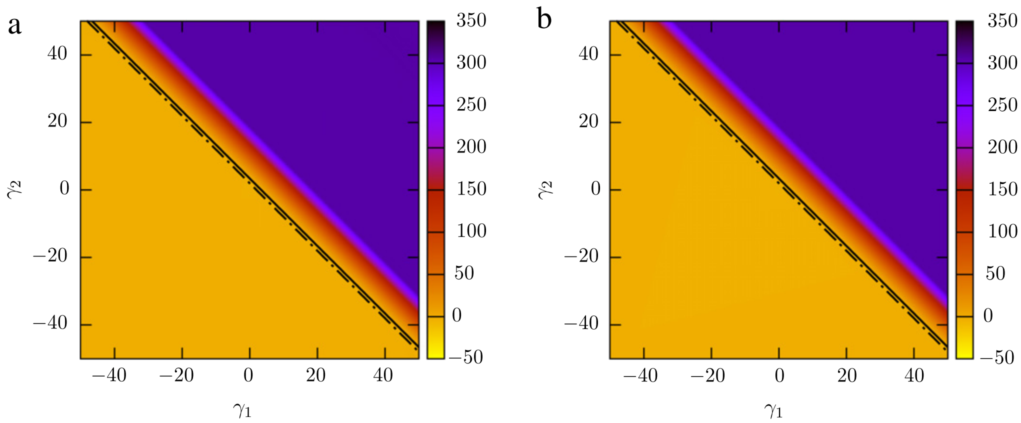

Fig. 2. The dependence of $\log_{10} \|\varepsilon\|_{C_h}$ on the values of parameters γ_1 and γ_2 in cases of (a) LOD method and (b) ADI method (Example 2). The dash-dot and solid straight lines denote the lines $\gamma_1 + \gamma_2 = 2$ and $\gamma_1 + \gamma_2 = \gamma^*$, respectively.

In this paper, we present the results of the numerical analysis of three test examples with different expressions of functions $\alpha(x)$ and $\beta(x)$. In all the examples, functions $f(x, y, t)$, $\mu_1(y, t)$, $\mu_2(y, t)$, $\mu_3(x, t)$, $\mu_4(x, t)$ and $\varphi(x, y)$ were chosen so that the function

$$u(x, y, t) = x^3 + y^3 + t^3$$

would be the solution of the differential problem (1)–(5) formulated in a unit square ($L_x = L_y = 1$), i.e.,

$$f(x, y, t) = -3(2x + 2y - t^2),$$
$$\mu_1(y, t) = y^3 + t^3 - \gamma_1 \int_0^1 \alpha(x)(x^3 + y^3 + t^3)\mathrm{d}x,$$
$$\mu_2(y, t) = 1 + y^3 + t^3 - \gamma_2 \int_0^1 \beta(x)(x^3 + y^3 + t^3)\mathrm{d}x,$$
$$\mu_3(x, t) = x^3 + t^3, \qquad \mu_4(x, t) = x^3 + 1 + t^3,$$
$$\varphi(x, y) = x^3 + y^3.$$

All numerical experiments were performed with $\tau = 10^{-4}$, $h_1 = h_2 = 10^{-2}$, $T = 2.0$ and with different values of parameters γ_1, γ_2, if it is not mentioned otherwise. To estimate the accuracy of the numerical solution, we calculated the maximum norm of computational error,

$$\|\varepsilon\|_{C_h} = \max_{0 \leqslant k \leqslant M} \max_{\substack{0 \leqslant i \leqslant N_1 \\ 0 \leqslant j \leqslant N_2}} |U_{ij}^k - u(x_i, y_j, t^k)|.$$

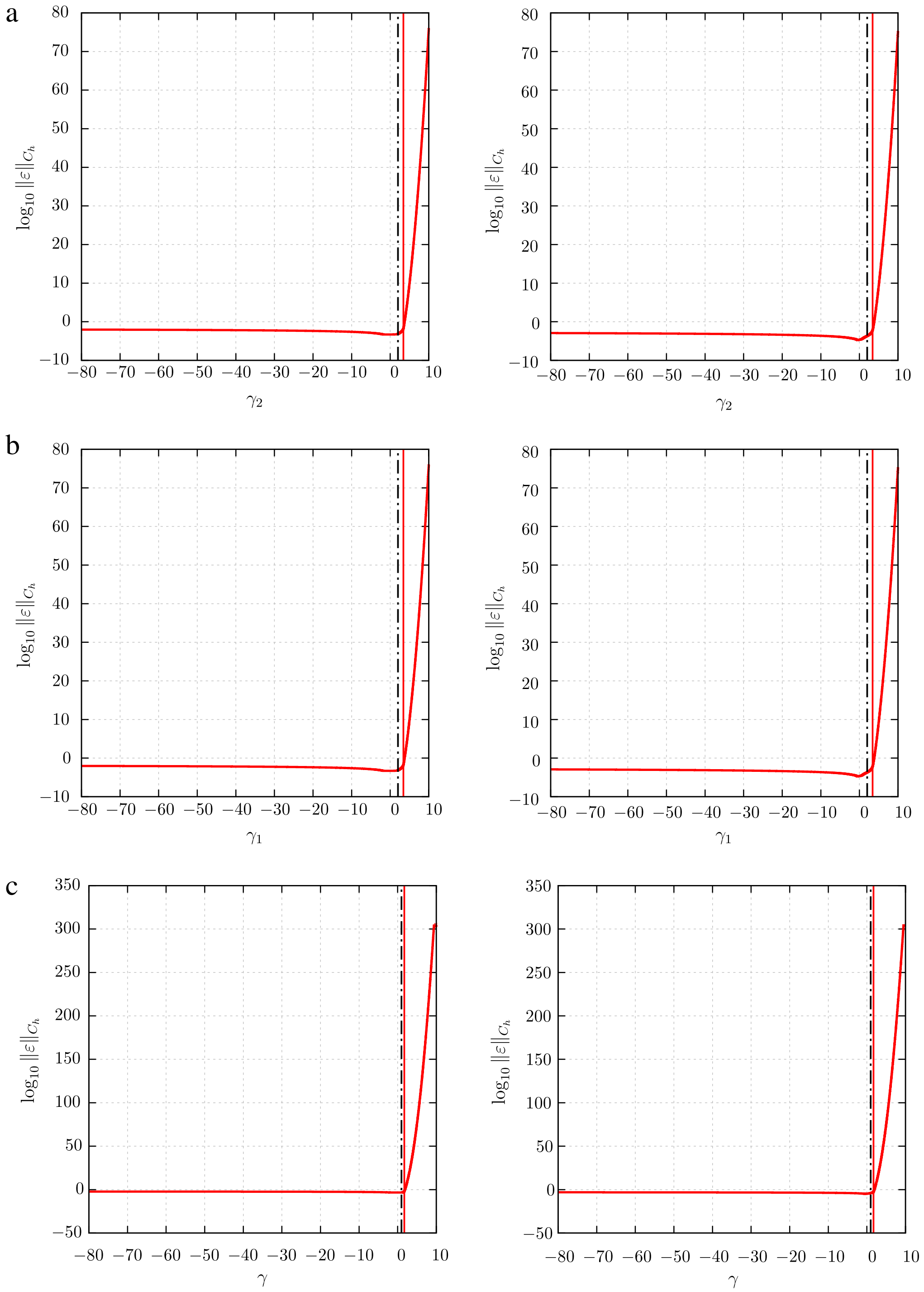

Fig. 3. The dependence of $\log_{10} \|\varepsilon\|_{C_h}$ on the values of parameters γ_1 and γ_2 in cases of LOD method (left) and ADI method (right) (Example 2): (a) $\gamma_1 = 0$, (b) $\gamma_2 = 0$, (c) $\gamma_1 = \gamma_2 = \gamma$. The dash-dot and solid vertical straight lines denote the lines (a) $\gamma_2 = 2$ and $\gamma_2 = \gamma^*$, (b) $\gamma_1 = 2$ and $\gamma_1 = \gamma^*$ or (c) $\gamma = 1$ and $\gamma = \gamma^*/2$, respectively.

Note that

$$\min_{0\leqslant t\leqslant T} \min_{\substack{0\leqslant x\leqslant L_x\\ 0\leqslant y\leqslant L_y}} u(x, y, t) = u(0, 0, 0) = 0, \qquad \max_{0\leqslant t\leqslant T} \max_{\substack{0\leqslant x\leqslant L_x\\ 0\leqslant y\leqslant L_y}} u(x, y, t) = u(L_x, L_y, T) = 10.$$

The finite-difference scheme was implemented in a stand-alone C application [33]. Numerical experiments were performed using the technologies of grid computing [34].

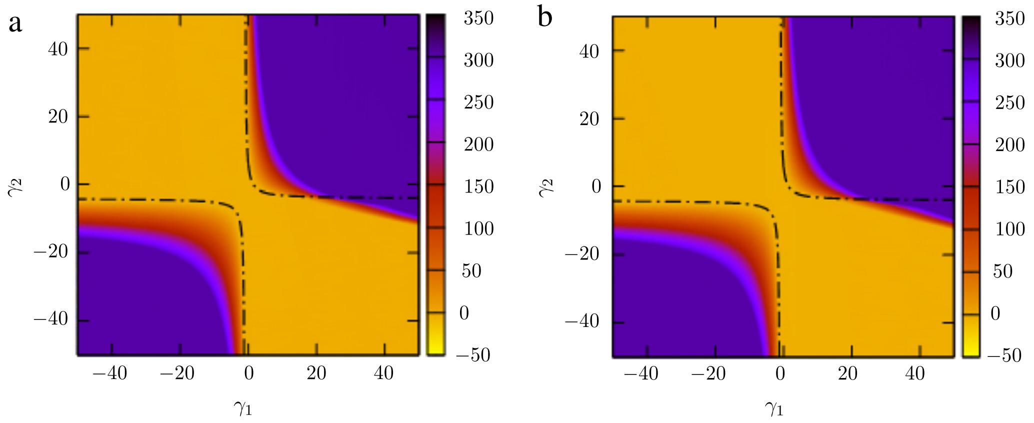

Fig. 4. The dependence of $\log_{10}\|\varepsilon\|_{C_h}$ on the values of parameters γ_1 and γ_2 in cases of (a) LOD method and (b) ADI method (Example 3). The dash-dot curves denote the branches of the hyperbola (25).

Similar as in paper [26], for the numerical analysis of the spectrum of the matrix S, MATLAB (The MathWorks, Inc.) software package [35] was used. The eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ were calculated numerically. Then all different eigenvalues of the matrix S were calculated using expressions (17) and formula (18).

In next subsections we will consider three test examples with different functions $\alpha(x)$, $\beta(x)$ and investigate the stability of LOD and ADI methods for the corresponding two-dimensional differential problems. The influence of conditions (19)–(24) will be considered in a separate subsection.

3.2. *Example 1:* $\alpha(x) = 0$, $\beta(x) = x$

This example corresponds to the differential problem with classical boundary conditions (2), (4) and one nonlocal integral condition (3). In this case, all the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ are real, non-negative and algebraically simple when $\gamma_2 \leqslant \widetilde{\gamma}_2 = 3 - 3h_1^2/(2 + h_1^2)$ ($\widetilde{\gamma}_2 \approx 2.99985$, when $h_1 = 10^{-2}$), and there exists only one negative eigenvalue when $\gamma_2 > \widetilde{\gamma}_2$ [17]. The numerical analysis of the spectrum of the matrix S shown that all the eigenvalues of the matrix S hold property $|\lambda(S)| < 1$ when $\gamma_2 \leqslant \gamma_2^*$; where $\gamma_2^* \approx 4.58114$ in the case of the LOD method and $\gamma_2^* \approx 4.58243$ in the case of the ADI method.

Fig. 1 presents the dependence of $\log_{10}\|\varepsilon\|_{C_h}$ on the values of parameter γ_2. In cases of both the LOD and ADI methods, the values of $\|\varepsilon\|_{C_h}$ grow slowly when $\widetilde{\gamma}_2 < \gamma_2 \leqslant \gamma_2^*$ and the growing becomes extremely fast when $\gamma_2 > \gamma_2^*$. The case of $\gamma_2 = 0$ corresponds to the differential problem with classical boundary conditions and it is known [23] that the both methods are stable in this case.

3.3. *Example 2:* $\alpha(x) = \mathrm{const}$, $\beta(x) = \mathrm{const}$

Let us assume that $\alpha(x) \equiv 1$ and $\beta(x) \equiv 1$. Preliminary results on the stability of ADI, LOD and fully-explicit splitting finite-difference schemes for the differential problem (1)–(5) with $\alpha(x) \equiv 1$ and $\beta(x) \equiv 1$ have been presented in papers [36–38]. The eigenvalue problem for the corresponding matrix $(-\widetilde{\Lambda}_1)$ is investigated in paper [30]. When $\gamma_1+\gamma_2 \leqslant 2$, then all the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ are real and non-negative numbers. If $\gamma_1 + \gamma_2 > 2$, then there exists one and only one negative eigenvalue of the matrix $(-\widetilde{\Lambda}_1)$. From the results of the numerical analysis of the spectrum of the matrix S it follows that absolute values of all the eigenvalues of the matrix S are less than 1 when $\gamma_1 + \gamma_2 \leqslant \gamma^*$, where $\gamma^* \approx 3.42366$ in the case of the LOD method and $\gamma^* \approx 3.42489$ in the case of the ADI method.

The dependence of $\log_{10}\|\varepsilon\|_{C_h}$ on the values of parameters γ_1 and γ_2 are depicted in Fig. 2. We see how the norm $\|\varepsilon\|_{C_h}$ grows when $\gamma_1 + \gamma_2$ becomes greater than γ^*.

If $\gamma_1 = 0$, $\gamma_2 \neq 0$ or $\gamma_1 \neq 0$, $\gamma_2 = 0$, then conditions (2) or (3) become classical boundary conditions. From Fig. 3((a) and (b)) we see that in these cases the norm $\|\varepsilon\|_{C_h}$ starts to grow when $2 < \gamma_2 \leqslant \gamma^*$ or $2 < \gamma_1 \leqslant \gamma^*$, and the growing becomes extremely fast when $\gamma_2 > \gamma^*$ or $\gamma_1 > \gamma^*$. The similar situations appear with $\gamma_1 = \gamma_2 = \gamma$, when $1 < \gamma \leqslant \gamma^*/2$ and $\gamma > \gamma^*/2$ (see Fig. 3(c)).

3.4. *Example 3:* $\alpha(x) = 1 + x$, $\beta(x) = 1 - x$

In this case, the spectrum of the matrix $(-\widetilde{\Lambda}_1)$ has a more complicated structure than in previous examples. Indeed, depending on the values of γ_1 and γ_2, both real and complex numbers can be the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ [17]. In

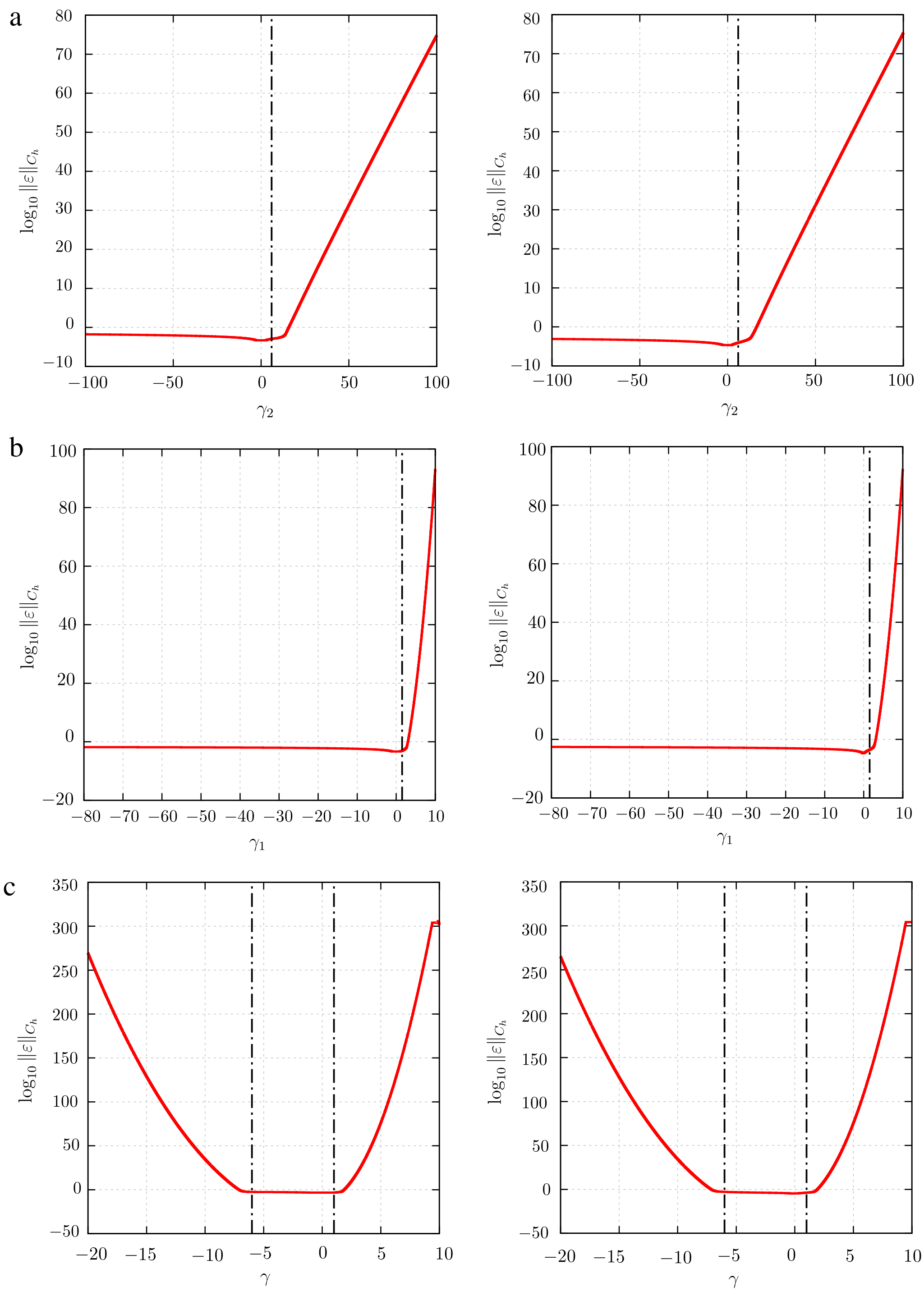

Fig. 5. The dependence of $\log_{10}\|\varepsilon\|_{C_h}$ on the values of parameters γ_1 and γ_2 in cases of LOD method (left) and ADI method (right) (Example 3): (a) $\gamma_1 = 0$, (b) $\gamma_2 = 0$, (c) $\gamma_1 = \gamma_2 = \gamma$. The dash-dot vertical straight lines denote the lines (a) $\gamma_2 = \widetilde{\gamma}_2$, (b) $\gamma_1 = \widetilde{\gamma}_1$ and (c) $\gamma = \widetilde{\gamma}$ and $\gamma = 1$.

paper [17] also it is noted that all real eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ are non-negative when points (γ_1, γ_2) are located anywhere between two branches of the hyperbola

$$\gamma_1\gamma_2(1 + 2h_1^2) + \gamma_1(4 - h_1^2) + \gamma_2(1 - h_1^2) - 6 = 0 \tag{25}$$

or belong to it. However, the existence of complex eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ in this particular ($\alpha(x) = 1 + x$, $\beta(x) = 1 - x$) or in the general case still remains an open problem.

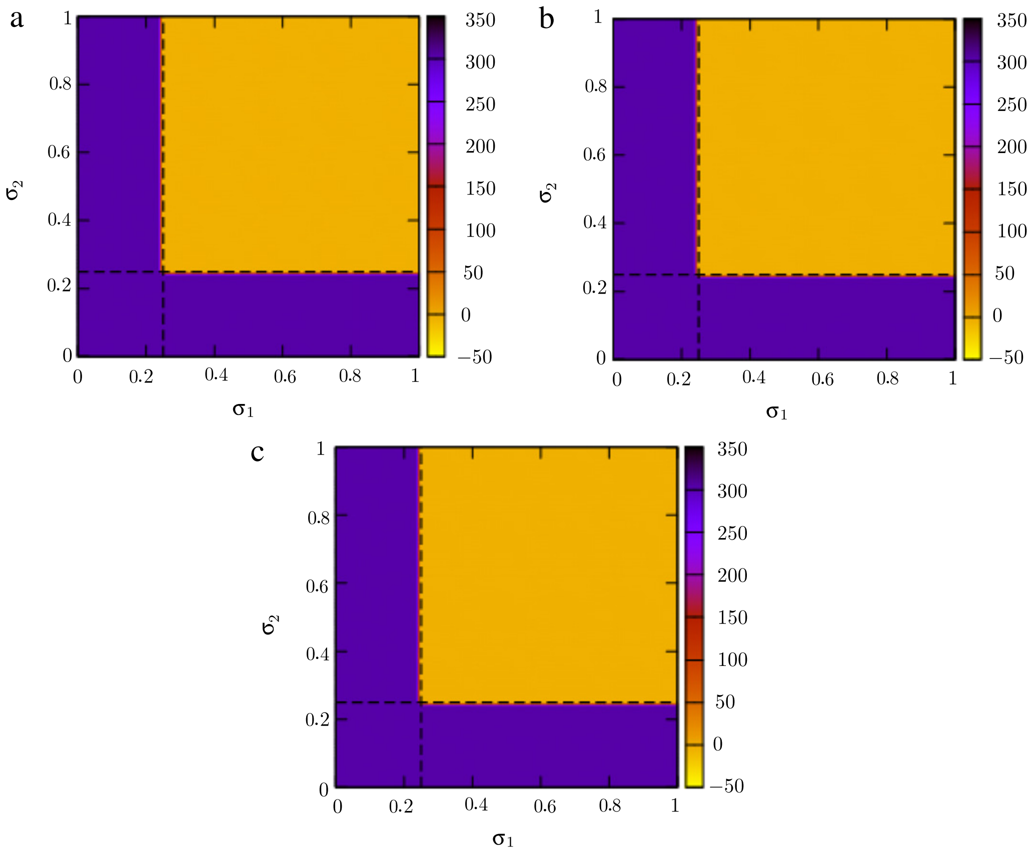

Fig. 6. The dependence of $\log_{10}\|\varepsilon\|_{C_h}$ on the values of parameters σ_1 and σ_2 ($\gamma_1 = -\gamma_2 = 1$) in cases of (a) Example 1, (b) Example 2 and (c) Example 3. The dashed straight lines denote the lines $\sigma_1 = \sigma_1^*$ and $\sigma_2 = \sigma_2^*$.

Without a comprehensive description of properties of the spectrum of the matrix $(-\widetilde{\Lambda}_1)$ (see [17] for more details), in Fig. 4 we present the values of $\log_{10}\|\varepsilon\|_{C_h}$ for different values of the parameters γ_1 and γ_2. We see that the finite-difference scheme is stable when the parameters γ_1 and γ_2 belong to almost all the region between two branches of the hyperbola (25) except the part near one of the half-branches (see quadrant in the southeast direction from the origin of coordinates in Fig. 4) where the instability of the finite-difference scheme is observable. The finite-difference scheme also becomes unstable when the parameters cross the hyperbola and get into one of two other regions outside the branches of the hyperbola.

The results of numerical analysis of the spectrum of the matrix $(-\widetilde{\Lambda}_1)$ show that when the parameters γ_1 and γ_2 belong to the above mentioned region near one of the half-branches of the hyperbola (25), there exist conjugate complex eigenvalues with negative real parts which transform into two negative or into one negative and one positive real eigenvalues as parameters vary. The similar properties of the spectrum of a particular quasi-tridiagonal matrix were observed in paper [26].

Similarly as in the previous example, from Fig. 5((a) and (b)) we see that in case when only one of conditions (2) and (3) are nonlocal ($\gamma_1 = 0$ or $\gamma_2 = 0$), the norm $\|\varepsilon\|_{C_h}$ grows fast if $\gamma_2 > \widetilde{\gamma}_2 = 6/(1-h_1^2)$ or $\gamma_1 > \widetilde{\gamma}_1 = 6/(4-h_1^2)$ ($\widetilde{\gamma}_1 \approx 1.50038$ and $\widetilde{\gamma}_2 \approx 6.00060$, when $h_1 = 10^{-2}$). If $\gamma_1 = \gamma_2 = \gamma$, then the norm $\|\varepsilon\|_{C_h}$ starts to grow when γ decreases in the region $\gamma < \widetilde{\gamma} = -6/(1+2h_1^2)$ ($\widetilde{\gamma} \approx -5.99880$, if $h_1 = 10^{-2}$) or γ increases in the region $\gamma > 1$ (see Fig. 5(c)).

3.5. Additional remarks

The influence of conditions (19)–(24) for the stability of the finite-difference scheme (16) was also investigated. We present the numerical results obtained with $\gamma_1 = -\gamma_2 = 1$ and with various values of σ_1, σ_2 or τ (the values of other parameters are the same as mentioned previously). When $\gamma_1 = -\gamma_2 = 1$, then all the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ are real, non-negative and algebraically simple numbers in all three considered cases.

The values of $\rho(A_1)$, σ_1^*, σ_2^*, σ^*, τ^* in all three examples are presented in Table 1. The values of σ^* and τ^* coincide in cases of Example 2 and 3, since in these cases

$$\min\left\{\frac{1}{\rho(A_1)}, \frac{1}{\Delta_2}\right\} = \min\left\{\min_{\lambda(A_1)}\frac{\mathrm{Re}\lambda(A_1)}{|\lambda(A_1)|^2}, \frac{1}{\Delta_2}\right\} = \frac{1}{\Delta_2}.$$

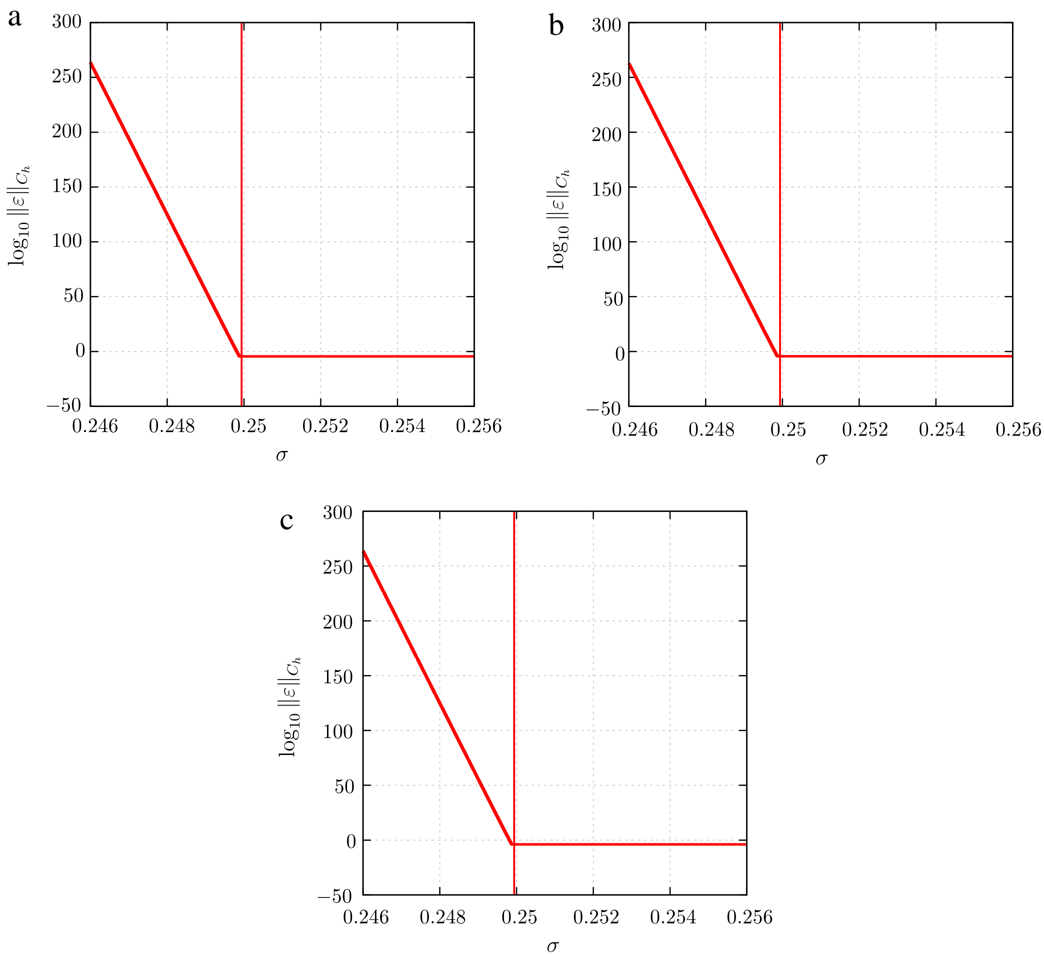

Fig. 7. The dependence of $\log_{10}\|\varepsilon\|_{C_h}$ font the values of σ ($\sigma_1 = \sigma_2 = \sigma$, $\gamma_1 = -\gamma_2 = 1$) in cases of (a) Example 1, (b) Example 2 and (c) Example 3. The vertical straight lines denote the lines $\sigma = \sigma^*$.

Table 1
The values of $\rho(A_1)$, σ_1^*, σ_2^*, σ^*, τ^* in cases of Examples 1–3 ($\gamma_1 = -\gamma_2 = 1$); $\Delta_2 = 3.999013120731463 \cdot 10^4$.

	Example 1	Example 2	Example 3
$\rho(A_1)$	$3.999013170072345 \cdot 10^4$	$3.999013120731456 \cdot 10^4$	$3.999013022046018 \cdot 10^4$
σ_1^*	$2.499383079096213 \cdot 10^{-1}$	$2.499383048242935 \cdot 10^{-1}$	$2.499382986534089 \cdot 10^{-1}$
σ_2^*	$2.499383048242940 \cdot 10^{-1}$	$2.499383048242940 \cdot 10^{-1}$	$2.499383048242940 \cdot 10^{-1}$
σ^*	$2.499383079096213 \cdot 10^{-1}$	$2.499383048242940 \cdot 10^{-1}$	$2.499383048242940 \cdot 10^{-1}$
τ^*	$5.001233841807574 \cdot 10^{-5}$	$5.001233903514120 \cdot 10^{-5}$	$5.001233903514120 \cdot 10^{-5}$

From Figs. 6–8 we can see that in all three cases the norm $\|\varepsilon\|_{C_h}$ is quite small when $\sigma_1 > \sigma_1^*$, $\sigma_2 > \sigma_2^*$, $\sigma > \sigma^*$ or $\tau < \tau^*$. We note that the constraints (19)–(24) are quite precise.

4. Conclusions

We developed a weighted splitting finite-difference scheme for the two-dimensional parabolic equation with nonlocal integral conditions. Applying quite a simple technique allows us to investigate the stability of the method. The technique is based on the analysis of the spectrum of the transition matrix of a finite-difference scheme. We demonstrate that depending on the parameters of the finite-difference scheme and nonlocal conditions the proposed method can be stable or unstable. The results of numerical experiments with several test problems justify theoretical results.

The proposed weighted splitting finite-difference scheme can be generalised and the same stability analysis technique can be applied in case of two-dimensional parabolic equation with more general integral or other type of nonlocal conditions.

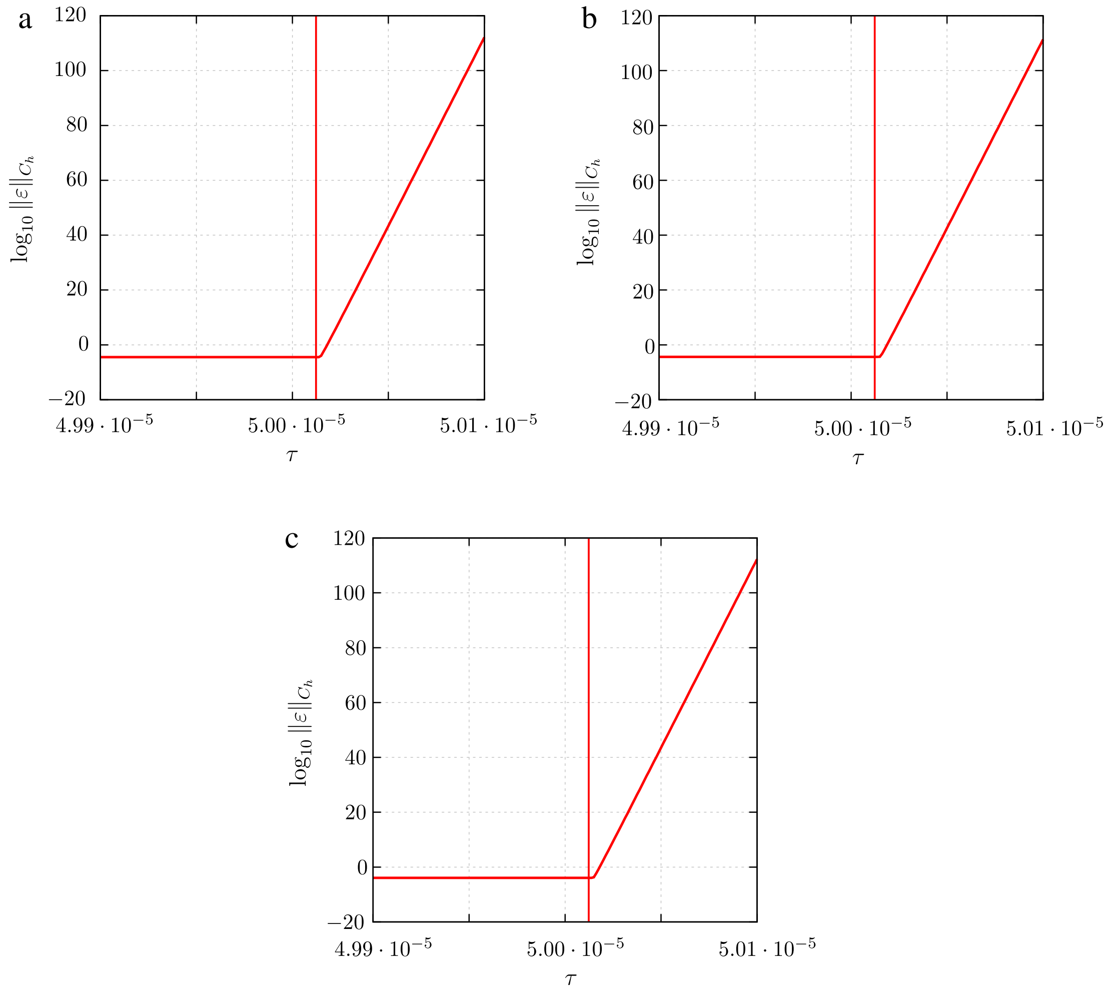

Fig. 8. The dependence of $\log_{10} \|\varepsilon\|_{C_h}$ on the values of τ ($\sigma_1 = \sigma_2 = 0$, $\gamma_1 = -\gamma_2 = 1$) in cases of (a) Example 1, (b) Example 2 and (c) Example 3. The vertical straight lines denote the lines $\tau = \tau^*$.

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