Matroids on Partially Ordered Sets

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1. INTRODUCTION

Several generalizations of the notion of matroid have been proposed (Brylawski, Edmonds, Faigle, Korte, Lovasz, Welsh, White).

This work proposes yet another one, one that is motivated by two considerations.

The first motivation comes from projective geometry. Matroids are the natural setting for the study of arrangements of hyperplanes (or, equivalently, sets of points) in projective space. It is natural to ask whether arrangements of linear varieties of different dimensions in projective space may be ensconced into a similar axiomatic setting, one in which matroid-theoretic arguments, with circuits, rank, bases, etc., may be used.

The second motivation is the replacement of a Boolean algebra of sets by the distributive lattice of filters of a finite partially ordered set. This replacement has proved fruitful in other contexts, most successfully in the replacement of algebraic varieties by schemes in algebraic geometry.

Our definition of poset matroids allows the extension to this new setting of every notion of matroid theory. In fact, the extension of the notion of matroid to poset matroids sheds light on the mutual relation of the notions of matroid theory.

Every family of linear varieties in projective space defines a poset matroid, and a classification of the possible special positions of a set of linear varieties is reflected in the combinatorial structure of the poset matroid thereby obtained.

Two languages are available for poset matroids: the language of partially ordered sets and the language of distributive lattices. The translation of poset matroids into the language of distributive lattices leads to the definition of a combinatorial scheme. Again, the translation of matroid
notions into the language of combinatorial schemes uncovers hidden analogies, some of which are developed below. The theory of matroids is obtained by taking the underlying distributive lattice to be a Boolean algebra or, equivalently, by taking a partially ordered set that is trivially ordered. The present work is self-contained. It requires only a few elementary definitions from the theory of partially ordered sets. A concise presentation of Theorem 10.5 was published in [4].

2. SYNOPSIS

We begin by recalling the definitions (given in detail in the text) of the two fundamental notions of this work, namely, poset matroids and combinatorial schemes.

1. A poset matroid on the partially ordered set \( P \) is a nonempty family \( B \) of filters of \( P \) (called bases) satisfying the following two axioms:
   
   (a) For every \( B_1, B_2 \in B \): \( B_1 \not\subseteq B_2 \).

   (b) For every \( B_1, B_2 \in B \) and for every pair of filters \( X, Y \), such that \( X \subseteq B_1, B_2 \subseteq Y, X \subseteq Y \), there exists \( B \in B \) such that \( X \subseteq B \subseteq Y \).

   An independent set of a poset matroid \( B \) on the partially ordered set \( P \) is a filter \( I \) of \( P \) such that there exists a basis \( B \) such that \( I \subseteq B \).

2. Let \( L \) be a finite distributive lattice. A combinatorial scheme in \( L \) is a nonempty antichain \( A \) of \( L \) that satisfies the following axiom: for every \( a_1, a_2 \in A \) and for every \( x, y \in L \), with \( x \leq a_1, a_2 \leq y, x \leq y \), there exists \( a \in A \) such that \( x \leq a \leq y \).

By a fundamental theorem of G. Birkhoff, every finite distributive lattice is isomorphic to the lattice of all filters of a finite partially ordered set. Conversely, every finite partially ordered set is isomorphic to the partially ordered set of the meet-irreducible elements of a distributive lattice. By virtue of these isomorphisms, we may use the language of posets and their language of distributive lattice interchangeably. The notions of a poset matroid and a combinatorial scheme correspond to each other in this double language.

Our main result is the symmetric exchange axiom, to which Section 10 is devoted. The foremost example of a combinatorial scheme comes from projective geometry. Let \( PG \) be a projective space over any field, and let \( S \) be a finite family of finite-dimensional linear varieties in \( PG \), not necessarily of the same dimension. We associate with \( S \) the partially ordered set \( P \), which is
the disjoint union of chains

$$\mathcal{P} = \bigcup_{L \in S} C_L,$$

where the chain $C_L$ is of length $\dim(L) + 1$.

On the partially ordered set $\mathcal{P}$ we define a poset matroid by specifying its independent sets as follows. A filter $A$ of the partially ordered set $\mathcal{P}$ is uniquely determined by choosing a point $c_L$ in each $C_L$, and then setting

$$A = \bigcup_{L \in S} \{ x \in C_L; x \geq c_L \}.$$

Let $\text{height}(c_L)$ be the height of the element $c_L$ of $C_L$. Such a filter $A$ is declared to be an independent set if and only if there exists in each $L$ a linear variety $H_L$ such that $\dim(H_L) = \text{height}(c_L) - 1$, and

$$\dim \left( \bigvee_{L \in S} H_L \right) = \sum_{L \in S} (\dim(H_L) + 1).$$

Intuitively, the meaning of this example is the following: a filter in the poset $\mathcal{P}$ is independent whenever "generic" linear varieties $H_L$ of given dimension may be chosen within each of the subspaces $L$ in the set $S$ that are "in general position."

For example, in the real projective space of dimension 3, consider a plane $\pi$, a line $r$ not belonging to $\pi$, the point $P := r \cap \pi$, and two distinct points $Q, R$ both different from $P$, lying on the line $r$. The partially ordered set $\mathcal{P}$ associated with the family $\{\pi, r, P, Q, R\}$ is the disjoint union of five chains (see Fig. 1.1). The bases of the poset matroid associated with $\{\pi, r, P, Q, R\}$ are the following:

$$\{a, b, c, e\} \quad \{b, c, d, e\} \quad \{b, c, e, h\} \quad \{b, c, g, h\} \quad \{a, b, c, g\}$$
$$\{b, c, e, f\} \quad \{b, c, f, g\} \quad \{a, b, c, h\} \quad \{b, c, e, g\} \quad \{b, c, f, h\}.$$

3. PRELIMINARIES AND NOTATION

We summarize the known facts of poset and lattice theory that are needed in the present work.

If $\mathcal{P} := (\mathcal{P}, \leq)$ is a poset, its dual is the poset $\mathcal{P}^* := (\mathcal{P}, \geq)$.

A chain of a poset $\mathcal{P}$ is any subset $A$ of $\mathcal{P}$ such that for every $x, y \in A$: either $x \leq y$ or $y \leq x$.

The length of a finite chain $A$ is the natural number $\text{length}(A) := |A| - 1$. 
A \textit{non antichain (or incomparable set)} of \( P \) is any subset \( A \) of \( P \) such that for every \( x, y \in A \): \( x \not\preceq y \).

A \textit{decreasing set} of \( P \) is any subset \( A \) of \( P \) such that for every \( x, y \in P \), if \( x \leq y \) and \( y \in A \), then \( x \in A \).

Dually, a \textit{filter} of \( P \) is any subset \( A \) of \( P \) such that for every \( x, y \in P \), if \( x \geq y \) and \( y \in A \), then \( x \in A \).

For any given subset \( A \) of \( P \) we define Max \( (A) := \{ x \in A, x \text{ is maximal in } A \} \), Min \( (A) := \{ x \in A, x \text{ is minimal in } A \} \), Upp \( (A) := \{ x \in P, \text{there exists } y \in A \text{ such that } x \geq y \} \), Low \( (A) := \{ x \in P, \text{there exists } y \in A \text{ such that } x \leq y \} \), Comp \( (A) := P - A \).

We remark that, if \( A \) is a decreasing set of \( P \) and \( x \) is a maximal element of \( A \), then \( A - x \) is again a decreasing set. Dually, if \( A \) is a filter of \( P \) and \( x \) is a minimal element of \( A \), then \( A - x \) is again a filter.

For every \( x \) and \( y \) in a poset \( P \), \( x \leq y \), the \textit{interval} \( [x, y] \) is defined to be the set \( [x, y] := \{ z \in P; x \leq z \leq y \} \).

If \( \text{card} [x, y] = 2 \), then \( x \) is \textit{covered} by \( y \), in symbols, \( x < y \).

The least and greatest element of a finite lattice \( L \) will be denoted by \( 0 \) and \( 1 \), respectively.

It is well known that all maximal chains in an interval \( [x, y] \) of a finite distributive lattice \( L \) have the same length: we will refer to such a length as the \textit{length of the interval} \( [x, y] \), denoted by the symbol \text{length}[x, y]. The \textit{height} of an element \( x \) of a finite distributive lattice is defined to be the natural number \text{height}(x) := \text{length}(0, x)$. 
An element \( x \) of any lattice \( \mathbb{L} \) will be said to be meet-irreducible if there exists exactly one element in \( \mathbb{L} \) covering \( x \). Note that \( 1 \) is not meet-irreducible.

4. POSET MATROIDS

A poset matroid on the partially ordered set \( \mathbb{P} \) is a family \( \mathcal{B} \) of filters of \( \mathbb{P} \), called bases, satisfying the following axioms:

(b.0) \( \mathcal{B} \neq \emptyset \).

(b.1) For every \( B_1, B_2 \in \mathcal{B} \): \( B_1 \subsetneq B_2 \).

(b.2) For every \( B_1, B_2 \in \mathcal{B} \) and for every pair of filters \( X, Y \) of \( \mathbb{P} \), such that \( X \subseteq B_1 \), \( B_2 \subseteq Y \), \( X \subseteq Y \), there exists \( B \in \mathcal{B} \) such that \( X \subseteq B \subseteq Y \) (middle axiom).

If \( \mathbb{P} \) is a trivially ordered set, the preceding definition yields the classical notion of matroid (see [73]).

Most properties of bases in matroids remain valid for poset matroids. As an example, we extend to poset matroids the theorem stating the invariance of the number of elements of a basis.

**Theorem 4.1.** Let \( \mathcal{B} \) be a poset matroid on the partially ordered set \( \mathbb{P} \); then, for every \( B_1, B_2 \in \mathcal{B} \), we have \( |B_1| = |B_2| \).

**Proof.** We proceed by induction on \( k := |B_2 - B_1| + |B_1 - B_2| \).

The assertion is trivially true for \( k = 0, 1, \) and 2.

Suppose the assertion true for every \( k \leq n \), \( n \geq 2 \), and let \( B_1, B_2 \in \mathcal{B} \) such that \( |B_2 - B_1| + |B_1 - B_2| = n + 1 \). Without loss of generality, suppose \( |B_2 - B_1| \geq 2 \). Let \( x \in \text{Min}(B_2) - B_1 \), and set \( X := B_2 - x \), \( Y := B_1 \cup B_2 - x \). Since \( X \subset Y \), \( X \subset B_2 \), and \( B_1 \subset Y \), by the middle axiom, there exists \( B \in \mathcal{B} \) such that \( X \subset B \subset Y \). We have

\[
B - B_1 \subset B_1 \cup B_2 - x - B_1 \subset B_2 - B_1, \quad B_2 - B \subseteq B_1 - B_2,
\]

which gives \( |B - B_1| + |B_1 - B| < |B_2 - B_1| + |B_1 - B_2| \).

\[
B - B_2 \subset B_1 \cup B_2 - x - B_2 \subseteq B_1 - B_2, \quad B_2 - B = \{ x \} \subset B_2 - B_1,
\]

which gives \( |B - B_2| + |B_2 - B| < |B_2 - B_1| + |B_1 - B_2| \).

Hence, by the induction hypothesis, \( |B_1| = |B| = |B_2| \).}

As is the case for matroids, the notion of poset matroid is self-dual; indeed, axiom (b.2) is self-dual; hence we have
Theorem 4.2. Let $\mathcal{B}$ be a matroid on the partially ordered set $\mathcal{P}$. Then the family

$$\mathcal{B}^\perp := \{ \mathcal{P} - B; B \in \mathcal{B} \}$$

is a poset matroid on the dual poset $\mathcal{P}^\ast$.

The poset matroid $\mathcal{B}^\perp$ is the orthogonal matroid of $\mathcal{B}$.

Other characterizations of the bases of a poset matroid may be given, for example:

Proposition 4.3. Let $\mathcal{B}$ be a nonempty family of filters of the partially ordered set $\mathcal{P}$ such that any two elements of $\mathcal{B}$ are incomparable. The following properties are equivalent:

(b.2) For every $B_1, B_2 \in \mathcal{B}$ and for every pair of filters $X, Y$ of $\mathcal{P}$, with $X \subseteq B_1, B_2 \subseteq Y$, $X \subseteq Y$, there exists $B \in \mathcal{B}$ such that $X \subseteq B \subseteq Y$ (middle property).

(b.3) For every $B_1, B_2 \in \mathcal{B}$ and for every $x \in \text{Min}(B_1 - B_2)$, there exists $y \in \text{Max}(B_2 - B_1)$ such that $(B_1 - x) \cup y \in \mathcal{B}$ (exchange property).

(b.3') For every $B_1, B_2 \in \mathcal{B}$ and for every $x \in \text{Max}(B_2 - B_1)$, there exists $y \in \text{Min}(B_1 - B_2)$ such that $(B_1 \cup x) - y \in \mathcal{B}$ (dual exchange property).

Proof. (b.2) implies (b.3): Let $B_1, B_2 \in \mathcal{B}$ and take $x \in \text{Min}(B_1 - B_2)$. Then, $B_1 - x \subseteq B_1$ and $B_2 \subseteq (B_1 - x) \cup B_2$; hence, by the middle axiom, there exists a basis $B$ such that $B_1 - x \subseteq B \subseteq (B_1 - x) \cup B_2$. Since, by Proposition 4.1, $|B| = |B_1|$, there exists $y \in \text{Max}(B_2 - B_1)$ such that $B = (B_1 - x) \cup y$.

(b.3) implies (b.2): Let $B_1, B_2 \in \mathcal{B}$ and consider two filters $X, Y$ such that $X \subseteq B_1, B_2 \subseteq Y$, and $X \subseteq Y$. We proceed by induction on $k := |Y| - |X|$. The assertion is trivially true for $k = 0$. Suppose the assertion true for some $n \geq 0$, and suppose $|Y| - |X| = n + 1$. Since $X \subseteq B_1$, there exists an element $x \in \text{Min}(B_1)$ such that $x \notin X$, and, by the exchange property, there exists $y \in \text{Max}(B_2 - B_3)$ such that $B' := (B_1 - x) \cup y \in \mathcal{B}$. Set $X' := X \cup y$. The subset $X'$ is a filter such that $X' \subseteq B'$ and $X' \subseteq Y$; moreover, $|Y| - |X'| = n$; hence, by the induction hypothesis, there exists a basis $B_3$ with $X \subseteq B_3 \subseteq Y$. Since $X \subseteq X'$, we get the assertion.

Finally, the equivalence between (b.3) and (b.3') is immediate by duality.

In Section 10, a deeper property of bases (the symmetric exchange property) will be established.

Example 4.1. A single filter $B$ of $\mathcal{P}$ is a poset matroid.
EXAMPLE 4.2. For every nonnegative integer \( i \), the family \( \mathcal{B} \) of all filters of cardinality \( i \) of a given partially ordered set \( P \) is a poset matroid, called the \( i \)-uniform poset matroid.

EXAMPLE 4.3. Let \( P \) be the poset in Figure 4.1; then, the family of filters \( \mathcal{B} := \{(a, c, e), (b, c, e), (c, d, e)\} \) is a matroid on \( P \).

EXAMPLE 4.4. The following sets are not poset matroids:

1. In the trivially ordered set \( P = \{a, b, c\} \), the family of filters \( \{(a, b), (c)\} \) is not a matroid. This family will be denoted by \( \mathcal{P}_3 \).

2. In the three-element poset such that \( a \prec b \) and \( c \) is unrelated to \( a \) or \( b \), the family of filters \( \{(a, b), (c)\} \) is not a matroid. This family will be denoted by \( \mathcal{P}_{1,2} \) (see Fig. 4.2).

We will see in the following section that these two families play a role in other axiomatizations of poset matroids.

5. INDEPENDENT SETS AND SPANNING SETS

It is known that matroids can be cryptomorphically defined in several other ways, using families of independent sets, spanning sets, circuits, and hyperplanes. Each of these definitions of a matroid may be extended to poset matroids.

An independent set of a poset matroid \( \mathcal{B} \) on the partially ordered set \( P \) is a filter \( I \) of \( P \) such that there exists a basis \( B \in \mathcal{B} \) such that \( I \subseteq B \).

THEOREM 5.1. The family \( \mathcal{I} \) of all independent sets of a poset matroid \( \mathcal{B} \) on the partially ordered set \( P \) satisfies the following properties:

\( \mathcal{I} \neq \emptyset \).

(i.1) If \( X, Y \) are filters of \( P \) such that \( Y \in \mathcal{I} \) and \( X \subseteq Y \), then \( X \in \mathcal{I} \).

(i.2) For every \( X, Y \in \mathcal{I} \) with \( |X| < |Y| \), there exists \( y \in \text{Max}(Y - X) \) such that \( X \cup y \in \mathcal{I} \) (augmentation property).

Proof. Properties (i.0) and (i.1) are trivially satisfied. To prove (i.2), let \( X, Y \in \mathcal{I} \), with \( |X| < |Y| \). By definition, there exist two bases \( B_1, B_2 \), with \( X \subseteq B_1 \), \( Y \subseteq B_2 \). We proceed by induction on \( n := |B_2| - |Y| \).

\[ \text{FIGURE 4.1.} \]
If \( n = 0 \), the thesis follows immediately by the exchange property and by Theorem 4.1.

Suppose the assertion true for \( n = 1, 2, \ldots, k \), and let \( n = k + 1 \). Take \( z \in \text{Max}(B_2 - Y) \), and set \( Y' := Y \cup z \). \( Y' \) is an independent set contained in \( B_2 \) such that \( |X| < |Y'| \) and \( |B_2| - |Y'| = k \). By the induction hypothesis applied to the independent sets \( X \) and \( Y' \), there exists \( y' \in \text{Max}(Y' - X) \) such that the filter \( X' := X \cup y' \) is independent.

If \( y' \neq z \), we have \( y' \in \text{Max}(Y - X) \), and the assertion is true.

Suppose now \( y' = z \). Then \( Y' - X' = Y - X \). We have \( |X'| < |Y'| \) and \( |B_2| - |Y'| = k \). Hence, by the induction hypothesis, there exists \( y \in \text{Max}(Y' - X') \) such that \( X' \cup y \) is independent. The filter \( X' \cup y \) is independent by property (i.1); this completes the proof.

Obviously, a filter of the partially ordered set \( P \) is a basis of the poset matroid \( \mathcal{B} \) if and only if it is a maximal independent set.

**Proposition 5.2.** Let \( \mathfrak{A} \) be a family of filters of \( P \) satisfying properties (i.0), (i.1), (i.2); then the family

\[
\mathfrak{B} := \{ I \in \mathfrak{A} ; I \text{ is a maximal element of } \mathfrak{A} \}
\]

is a poset matroid.

**Proof.** To show that \( \mathfrak{B} \) is poset matroid, we have only to prove that \( \mathfrak{B} \) satisfies the exchange property.

First of all, we remark that the augmentation property implies immediately that all maximal filters of \( \mathfrak{A} \) have the same cardinality.

Now let \( B_1, B_2 \) be two different elements of \( \mathfrak{B} \) and take \( x \in \text{Min}(B_1) \).

Then \( B_1 - x \) and \( B_2 \) belong to \( \mathfrak{A} \), with \( |B_1 - x| < |B_2| \). Hence, by the augmentation property, there exists \( y \in \text{Max}(B_2 - (B_1 - x)) \) such that \( (B_1 - x) \cup y \in \mathfrak{A} \). Since \( |B_1 - x| \cup y| = |B_1| \), the filter \( (B_1 - x) \cup y \) is maximal in the family \( \mathfrak{A} \), namely, it belongs to \( \mathfrak{B} \).
The augmentation property of independent sets may be replaced by an apparently weaker condition:

**Proposition 5.3.** Let \( \mathcal{F} \) be a family of filters of \( P \) satisfying properties (i.0), (i.1); the following statements are equivalent:

(i.2) For every \( X, Y \in \mathcal{F} \) with \( |X| < |Y| \), there exists \( y \in \text{Max}(Y - X) \) such that \( X \cup y \in \mathcal{F} \) (augmentation property).

(i.3) For every \( X, Y \in \mathcal{F} \), with \( |Y| = 1 + |X| \) and \( |X| = 1 + |X \cap Y| \), there exists \( y \in \text{Max}(Y - X) \) such that \( X \cup y \in \mathcal{F} \) (local augmentation property).

**Proof.** It is sufficient to show that local augmentation implies “global” augmentation. Let \( \mathcal{F} \) be a family of filters of \( P \) satisfying properties (i.0), (i.1), and (i.3). Let \( U, V \in \mathcal{F} \), with \( |U| < |V| \), and set \( k = |U| - |U \cap V| \).

We proceed by induction on \( k \).

Suppose \( k = 1 \). Since the set \( V - U \) has cardinality at least 2, there exist at least two different elements \( x, y \), say, belonging to \( V - U \), such that \( W \equiv (U \cap V) \cup x \cup y \) is a filter. Note that \( W \subseteq V \), and hence \( W \in \mathcal{F} \) by property (i.1). We now apply the local augmentation property to \( X \equiv U \) and \( Y \equiv V \), and we get the required result. Suppose now the assertion is true for \( k = n \), and let \( k = n + 1 \). Let \( z \in \text{Min}(U - V) \); then, setting \( U' \equiv U - z \), we have \( U' \in \mathcal{F} \) and \( |U'| = |U \cap V| = n \). By the induction hypothesis, there exists \( y \in \text{Max}(V - U') \) such that \( U' \cup y \in \mathcal{F} \). Now, two cases can occur:

(a) If \( U \cup y \in \mathcal{F} \), the assertion holds.

(b) Suppose \( U \cup y \not\in \mathcal{F} \). Since \( |U'| = |U \cap V| = n \), by the induction hypothesis, there exists \( w \in \text{Max}(V - U') \) such that \( W \equiv U' \cup w \in \mathcal{F} \). We have \( |U| = |U \cap W| = 1 \). Therefore, by the induction hypothesis, there exists \( x \in \text{Max}(W - U) \) such that \( U \cup x \in \mathcal{F} \).

Dually, we define a spanning set of a poset matroid \( \mathcal{B} \) on the partially ordered set \( P \) to be a filter \( \mathcal{S} \) of \( P \), such that there exists a basis \( B \) such that \( B \subseteq \mathcal{S} \). Spanning sets of \( \mathcal{B} \) are independent sets of the orthogonal matroid \( \mathcal{B}^\perp \), and conversely. Hence, by duality, we obtain the following theorem.

**Theorem 5.4.** The family \( \mathcal{S} \) of all spanning sets of a poset matroid \( \mathcal{B} \) on the partially ordered set \( P \) satisfies the following properties:

(5.0) \( \mathcal{S} \neq \emptyset \).

(5.1) If \( X, Y \) are filters of \( P \) such that \( Y \in \mathcal{S} \) and \( X \supseteq Y \), then \( X \in \mathcal{S} \).

(5.2) For every \( X, Y \in \mathcal{S} \) with \( |X| > |Y| \), there exists \( x \in \text{Min}(X - Y) \) such that \( X - x \in \mathcal{S} \) (reduction property).
Conversely, let $\mathcal{S}$ be a family of filters of $\mathcal{P}$ satisfying properties (s.0), (s.1), (s.2); then the family

$$
\mathcal{B} := \{ S \in \mathcal{S} ; S \text{ is a minimal element of } \mathcal{S} \}
$$

is a poset matroid.

From Proposition 5.3 we have

**Proposition 5.5.** Let $\mathcal{S}$ be a family of filters of $\mathcal{P}$ satisfying properties (s.0), (s.1). The following properties are equivalent:

(s.2) for every $X, Y \in \mathcal{S}$ with $|X| > |Y|$, there exists $x \in \text{Min}(X - Y)$ such that $X - x \in \mathcal{S}$ (reduction property).

(s.3) For every $X, Y \in \mathcal{S}$, with $|X| = 1 + |Y|$ and $|Y| = 1 + |X \cap Y|$, there exists $x \in \text{Min}(X - Y)$ such that $X - x \in \mathcal{S}$ (local reduction property).

By the preceding theorems, independent sets, as well as spanning sets, can be taken as the primitive notion (and properties (i.0), (i.1), (i.2) as axioms) in the definition of a poset matroid.

### 6. Combinatorial Schemes

We now give an equivalent definition of poset matroids that uses the language of distributive lattices. Let $\text{Inc}(\mathcal{P})$ be the distributive lattice of all filters of the partially ordered set $\mathcal{P}$, ordered by inclusion.

A poset matroid $\mathcal{B}$ on $\mathcal{P}$ can be seen to be a nonempty antichain $\mathcal{A}$ of the distributive lattice $\text{Inc}(\mathcal{P})$ satisfying the following property:

(a.1) For every $a_1, a_2 \in \mathcal{A}$ and for every $x, y \in \text{Inc}(\mathcal{P})$, $x \leq a_1, a_2 \leq y$, $x \leq y$, there exists $a \in \mathcal{A}$ such that $x \leq a \leq y$.

Conversely, any nonempty antichain $\mathcal{A}$ of a finite distributive lattice $\mathcal{L}$ that satisfies (a.1) is the lattice counterpart of a poset matroid.

The previous considerations lead to the following definition: a nonempty antichain $\mathcal{A}$ of a distributive lattice $\mathcal{L}$ that satisfies the property:

(a.1) For every $a_1, a_2 \in \mathcal{A}$ and for every $x, y \in \mathcal{L}$, $x \leq a_1, a_2 \leq y$, $x \leq y$, there exists $a \in \mathcal{A}$ such that $x \leq a \leq y$ (middle property) will be called a combinatorial scheme. By abuse of language, the elements of a combinatorial scheme will also be called bases.

Proposition 4.1 yields immediately:

**Proposition 6.1.** Let $\mathcal{A}$ be a combinatorial scheme in a distributive lattice $\mathcal{L}$. Then all elements of $\mathcal{A}$ have the same height in $\mathcal{L}$. 
From Proposition 4.3 we get

**Proposition 6.2.** Let \( A \) be a nonempty antichain of a distributive lattice \( \mathbb{L} \). Property (a.1) is equivalent to either of the following:

(a.2) For every \( a_1, a_2 \in A \) and for every \( x < a_1 \), there exists \( y \leq a_2 \) such that \( x \lor y \in A \) (exchange property).

(a.2') For every \( a_1, a_2 \in A \) and for every \( x > a_1 \), there exists \( y \geq a_2 \) such that \( x \land y \in A \) (dual exchange property).

If \( \mathbb{L} \) is a Boolean algebra, then a combinatorial scheme in \( \mathbb{L} \) is isomorphic to a matroid in the ordinary sense of the word.

**Example 6.1.** In Figure 6.1, the combinatorial scheme associated with the poset matroid \( \mathcal{B} \) of Example 4.3 is shown. The conciseness of the lattice-theoretical language in describing bases is evident.

**Example 6.2.** In Figure 6.2 we show the antichains \( A_3 \) and \( A_{1,2} \), which correspond to the families \( \mathcal{P}_3 \) and \( \mathcal{P}_{1,2} \) of Example 4.4. Hence neither \( A_3 \) nor \( A_{1,2} \) is a combinatorial scheme.

We point out that \( A_3 \) and \( A_{1,2} \) are the only antichains in a distributive lattice of height 3 that are not combinatorial schemes.

### 7. Independent and Spanning Elements

Independent and spanning sets may be defined in lattice-theoretical language as follows: given a combinatorial scheme \( \mathcal{A} \) on a distributive lattice \( \mathbb{L} \), an element \( x \in \mathbb{L} \) is an independent element if \( x \leq y \) for some \( y \in A \). Dually, an element \( x \) is a spanning element if \( y \leq x \) for some \( y \in A \).
Theorems 5.1, 5.2, 5.3 and their duals may now be translated into lattice-theoretic language as follows:

**Theorem 7.1.** Let $I$ be a nonempty decreasing subset of the distributive lattice $L$. Then the set of maximal elements of $I$ is a combinatorial scheme in $L$ if and only if $I$ satisfies one of the following equivalent properties:

1. For every $i, j \in I$, if $\text{height}(i) < \text{height}(j)$, then there exists $z \in L$, $z \neq i$, such that $z \in I \cap [i, i \lor j]$ (augmentation property).

2. For every $i, j \in I$, if $\text{height}(i) < \text{height}(j)$ and $h[i \land j, i \lor j] = 3$, then there exists $z \in L$, $z \neq i$, such that $z \in I \cap [i, i \lor j]$ (local augmentation property).

Dually, we have:

**Theorem 7.2.** Let $F$ be a nonempty filter of the distributive lattice $L$. Then the set of minimal elements of $F$ is a combinatorial scheme in $L$ if and only if $F$ satisfies one of the following equivalent properties:

1. For every $x, y \in F$, if $\text{height}(x) > \text{height}(y)$, then there exists $z \in L$, $z \neq x$, such that $z \in F \cap [x \land y, x]$ (reduction property).

2. For every $x, y \in F$, if $\text{height}(x) > \text{height}(y)$ and $h[x \land y, x \lor y] = 3$, then there exists $z \in L$, $z \neq x$, such that $z \in F \cap [x \land y, x]$ (local reduction property).

**Example 7.1.** The decreasing set and the filter in Figure 7.1 are the set of independent elements and of spanning elements of the combinatorial scheme of Example 6.1.

**Example 7.2.** Let $B_3$ be the Boolean algebra of subsets of a three-element set. The subset $N_3$ in Figure 7.2 is not the set of independent elements of a combinatorial scheme in $B_3$, since the set of maximal elements of $N_3$ is the antichain $A_3$, which has been shown not to be a combinatorial scheme (see Example 6.2).
Similarly, let \( \mathbb{L}_{1,2} \) be the distributive lattice that is the product of a chain of length 1 and a chain of length 2. The subset \( N_{1,2} \) of in Figure 7.2 is not the set of independent elements of a combinatorial scheme in \( \mathbb{L}_{1,2} \), since its set of maximal elements is the antichain \( A_{1,2} \) (see Example 6.2).

Given a decreasing set \( D \) of a distributive lattice \( \mathbb{L} \), and given an interval \([x, y]\) of \( \mathbb{L} \), where \( x \leq y \), the set \([x, y] \cap D\) is a decreasing set of the lattice \([x, y]\).

**Theorem 7.3.** A nonempty decreasing set \( I \) of a distributive lattice \( \mathbb{L} \) is the set of independent elements of a combinatorial scheme if and only if for no interval \([x, y]\) of \( \mathbb{L} \) is the partially ordered set \( I \cap [x, y] \) isomorphic to either \( N_3 \) or \( N_{1,2} \).

**Proof.** It suffices to remark that the local augmentation property is equivalent to the condition that, for every interval \([x, y]\) of height 3 of \( \mathbb{L} \), the decreasing set \( I \cap [x, y] \) must satisfy the augmentation property in the lattice \([x, y]\). □

The following is an alternative characterization of the set of independent elements of a combinatorial scheme.
Theorem 7.4. A nonempty decreasing subset $I$ of the distributive lattice $\mathbb{L}$ is the set of independent elements of a combinatorial scheme if and only if it satisfies the following property:

\[(j,3) \quad \text{For every } x, y \in \mathbb{L} \text{ with } x < y, \text{ and for every element } i \in I \cap [0, x] \text{ of maximum height, there exists an element } j \in I \cap [0, y] \text{ of maximum height, such that } j \land x = i \text{ (greedy property).}\]

Proof. Suppose that $I$ is the set of independent elements of a combinatorial scheme in $\mathbb{L}$. Let $x, y$ be elements of $\mathbb{L}$ such that $x < y$, and let $i \in I \cap [0, x]$, of maximum height. Let $z$ be an element of maximum height in the set $I \cap [0, y]$. Set

$$S := \{ j \in I \cap [0, y]; j \land x = i \}.$$

By the augmentation property applied to $i$ and $z$, there exists an independent element $k$ such that $i < k \leq i \lor z$. Since $i$ has maximum height in $I \cap [0, x]$, we have $k \land x = i$. Hence, the set $S$ is nonempty. Set now

$$h := \max\{\text{height}(j); \ j \in S\}.$$

and suppose that $h < \text{height}(z)$. Then there exists an element $j \in S$ such that $\text{height}(j) = h < \text{height}(z)$. By the augmentation property applied to $j$ and $z$, there exists an independent element $j'$ such that $j < j' \leq j \lor z \leq y$. Since $i < j'$ and $i$ has maximum height in $I \cap [0, x]$, we have $j' \land x = i$. Hence, $j' \in S$, which gives a contradiction. Therefore, $h = \text{height}(z)$, and the greedy property is satisfied.

Conversely, let $I$ be a nonempty decreasing subset of $\mathbb{L}$ such that the partially ordered set $I \cap [x, y]$ is isomorphic to either $N_3$ or $N_{1,2}$ for some interval $[x, y]$ of $\mathbb{L}$. Then, taking elements $x, y$, and $i$ as in Figure 7.3, property $(j,3)$ is not valid.

\[\text{FIGURE 7.3.}\]
8. DEPENDENT ELEMENTS

Let \( I \) be the set of independent elements of a combinatorial scheme \( A \) in a distributive lattice \( L \). The elements of the set \( \text{Comp}(I) := L \setminus I \) are called dependent elements. Dually, the elements of the set \( \text{Comp}(S) := L \setminus S \), where \( S \) is the set of spanning elements of \( A \), are called nonspanning elements.

First of all, we remark that neither the set \( D_3 := \text{Comp}(N_3) \) nor the set \( D_{1,2} := \text{Comp}(N_{1,2}) \) in Figure 8.1 is the set of dependent elements of a combinatorial scheme (see Example 7.2).

From Theorem 7.3 we obtain the following characterization of dependent elements:

**Theorem 8.1.** A filter \( D \) properly contained in the distributive lattice \( L \) is the set of dependent elements of a combinatorial scheme if and only if for no interval \( [x, y] \) of \( L \) is the partially ordered set \( D \cap [x, y] \) isomorphic to either \( D_3 \) or \( D_{1,2} \).

The preceding result may be restated as follows (see Fig. 8.2):

**Theorem 8.2.** Let \( D \) be a proper filter in the distributive lattice \( L \). Then \( D \) is the set of dependent elements of a combinatorial scheme if and only if it satisfies the following properties:

(d.1) For every \( d_1, d_2 \in D \) such that \( d_1 \prec d_1 \lor d_2 \) and \( d_2 \prec d_1 \lor d_2 \), and for every \( z \neq d_i, i = 1, 2 \), \( z \prec d_1 \lor d_2 \), either \( z \in D \) or \( d_1 \land d_2 \in D \) (elimination property).

(d.2) For every \( a, b, c \in L \) such that \( b \lor a \lor b \), and \( [a, a \lor b] = \{a, c, a \lor b\} \), if \( c \in D \), then either \( a \in D \) or \( b \in D \) (replacement property).

We now translate the preceding results into the language of poset matroids. By abuse of reasoning, we assume that dependent sets of a poset matroid have been defined. Dependent sets in a poset matroid may be
The family $D$ of all dependent sets of $B$ is characterized by the following properties:

1. $\emptyset \notin D$.
2. If $X$ and $Y$ are filters in $P$, and if $Y \subseteq X$, then $X \in D$.
3. For every $X,Y \in D$ such that $X \cap Y \notin D$, for every minimal element $z$ of $X \cup Y$ we have $(X \cup Y) - z \in D$ (elimination property).
4. For every $X \in D$, for every $x,y \in \text{Min}(X)$, $x \neq y$, for every maximal element $z$ of $P - X$ such that $z < x$ and $z \not\in y$, we have either $X - x \in D$ or $(X - y) \cup z \in D$ (replacement property).

Proof. The family $D$ can be seen as a subset $D$ of the distributive lattice $L := \text{Inc}(P)$ of all filters of $P$, ordered by inclusion. We shall show that $D$ is the set of dependent elements of a combinatorial scheme in $L$ if and only if the family $D$ satisfies the four conditions above. More precisely, we have only to show that the elimination and replacement properties are equivalent to (d.1) and (d.2), respectively.

(a) It is immediately checked that the elimination property implies property (d.1) (see Fig. 8.3). To prove the converse, we proceed by induction on $n := |X \cup Y| - |X \cap Y|$.

If $n = 2$ the statement is trivially true (see Fig. 8.3).

Suppose the statement is true for every $n \leq k$, and take $X,Y \in D$ with $X \cap Y \notin D$ and $|X \cup Y| - |X \cap Y| = n + 1$. Without loss of generality, we can assume that $|X \cup Y| - |Y| \geq 2$. Since $X \cap Y$ is not dependent, $X = Y \neq \emptyset$; let $x \in \text{Max}(X - Y)$, and consider $Y \cup x$, which is dependent. Set $Z := (X \cap Y) \cup x$.

If $Z$ is not dependent, then, by the induction hypothesis applied to $X$ and $Y \cup x$, we get $(X \cup Y) - z \in D$ for every $z \in \text{Min}(X \cup Y)$. 

FIGURE 8.2.
Now let $Z$ be dependent, and take an element $z \in \text{Min}(X \cup Y)$; if $z \not\in X \cap Y$, the filter $(X \cup Y) - z$ contains $Z$, and hence it is dependent. If $z \in X \cap Y$, we apply the induction hypothesis to the filters $Z$ and $Y$, and we get that the filter $Y \cup Z = (Y \cup x) - z$ is dependent. Therefore, $(X \cup Y) - z$ is dependent.

(b) The equivalence between the replacement property and (d.2) follows immediately, as shown in Figure 8.3.

9. CIRCUITS

A minimal dependent set of a poset matroid is called a circuit. Similarly, a minimal dependent element in a combinatorial scheme is called a circuit, by abuse of language.

**Theorem 9.1.** Let $C$ be an antichain of a distributive lattice $\mathbb{L}$, not containing the zero element of $\mathbb{L}$. The antichain $C$ is the set of all circuits of a combinatorial scheme if and only if it satisfies the following properties:

(c.1) For every $c_1, c_2 \in C$ such that $c_1 \neq c_2$, and for every $x < c_1 \lor c_2$, $x \nleq c_1 \land c_2$, there exists $c \in C$ such that $c < x$, $c \nleq c_1 \land c_2$ (elimination property).

(c.2) For every $c \in C$, for every $x, y, z \in \mathbb{L}$ such that $x < y < z$, $[x, z] = \{x, y, z\}$ and $c \leq y$, for every $t < z$ there exists $c' \in C$ such that either $c' \leq x$ or $c' \leq t$ (replacement property).

**Proof.** Set $D := \text{Up}(C)$. The antichain $C$ is the set of all circuits of a combinatorial scheme if and only if the filter $D$ is the set of all dependent elements of a combinatorial scheme. Hence we shall prove the assertion by showing that $C$ satisfies (c.1) if and only if $D$ satisfies (d.1), and $C$ satisfies (c.2) if and only if $D$ satisfies (d.2).
(d.1) implies (c.1): Let \( c_1, c_2 \in C \) such that \( c_1 \neq c_2 \), and choose \( x < c_1 \lor c_2, x \neq c_1 \land c_2 \); without loss of generality, suppose that \( h[c_1, c_1 \lor c_2] \leq h[c_2, c_1 \lor c_2] \), and set \( h := h[c_1, c_1 \lor c_2] \). We proceed by induction on \( h \).

Suppose \( h = 1 \), that is, \( c_1 < c_1 \lor c_2 \); set \( d_1 := c_1 \) and choose \( d_2 \) such that \( c_2 \leq d_2 < c_1 \lor c_2 \). Since \( c_1 \lor c_2 = d_1 \lor d_2 \), we have \( d_1 \land d_2 < d_1 \), which is a minimal dependent element; hence, \( d_1 \land d_2 \notin D \). By property (d.1), \( x \) is forced to belong to \( D \); consequently, there exists \( c \in C \) such that \( c < x \), and, obviously, \( c \neq c_1 \land c_2 \).

Suppose now the assertion true for \( h \leq n \). Let \( h = n + 1 \), and choose \( d_1, d_2 < c_1 \lor c_2 \), with \( c_i \leq d_i, i = 1, 2 \). We have \( c_1 \lor c_2 = d_1 \lor d_2 \). If \( d_1 \land d_2 \notin D \), by (d.1), there exists \( c \in C \) such that \( c < c_2 \), and, obviously, \( c \neq c_1 \land c_2 \). If \( d_1 \land d_2 \in D \), there exists \( c_3 \in C \), \( c_3 \leq d_1 \land d_2 \). We have \( c_1 \lor c_2 \leq c_3 \lor (d_1 \land d_2) \leq c_1 \lor d_1 = d_1 \). Hence \( h[c_1, c_1 \lor c_3] \leq h[c_1, d_1] = n \). Setting \( x := x \land (c_1 \lor c_3) \), by the induction hypothesis there exists \( c \in C \) such that \( c < x \leq x' \).

(c.1) implies (d.1): Let \( d_1, d_2 \in D \) such that \( d_1 < d_1 \lor d_2, i = 1, 2 \), and choose \( x \neq d_i, i = 1, 2 \). Suppose that \( d_1 \land d_2 \notin D \); then there exist \( c_i, c_2 \in C \) with \( c_i \leq d_i, i = 1, 2 \), and \( c_i \neq d_i \) for \( i \neq j \).

If either \( c_1 \leq x \) or \( c_2 \leq x \), then \( x \in D \), and the assertion is true.

If this is not the case, recalling that \( x < d_1 \lor d_2 \), we have \( x \leq x \lor (c_1 \lor c_2) \leq d_1 \lor d_2 \) and \( x \neq x \lor (c_1 \lor c_2) \); hence \( x \lor (c_1 \lor c_2) = d_1 \lor d_2 \). By the modularity of the lattice \( L \), we have

\[
 x < d_1 \lor d_2 = x \lor (c_1 \lor c_2) \implies x \land (c_1 \lor c_2) < c_1 \lor c_2.
\]

Therefore, setting \( x' := x \land (c_1 \lor c_2) \) and applying property (c.1) to the elements \( c_1, c_2 \), and \( x' \), we get the assertion.

(d.2) is equivalent to (c.2): Immediate.

As for combinatorial schemes, the axioms for circuits of a poset matroid are easily deduced from those of dependent sets:

**Theorem 9.2.** The family \( \mathcal{C} \) of all circuits of a poset matroid \( \mathfrak{M} \) satisfies the following properties:

1. (e.0) \( \emptyset \notin \mathcal{C} \).
2. (e.1) For every \( C_1, C_2 \in \mathcal{C} \): \( C_1 \notin C_2 \).
3. (e.2) For every \( C_1, C_2 \in \mathcal{C} \) with \( C_1 \neq C_2 \), and for every \( z \in \text{Min}(C_1 \cup C_2) \), there exists \( C_3 \in \mathcal{C} \) such that \( C_3 \subseteq (C_1 \cup C_2) - z \) (elimination property).
4. (e.3) For every \( C \in \mathcal{C} \), for every \( x, y \in \text{Min}(C) \), \( x \neq y \), and for every \( z < x \) such that \( z < y \), there exists \( C' \in \mathcal{C} \) such that either \( C' \subseteq C \cup \text{Upp}(z) \) or \( (x \cup z) \) or \( C' \subseteq (C - y) \cup \text{Upp}(z) \) (replacement property).
Conversely, let $\mathfrak{C}$ be a family in $\text{Inc}(\mathfrak{P})$ satisfying the four properties above. Then the family

$$\mathfrak{M} := \text{Max}(\text{Comp}(\text{Upp} \mathfrak{C}))$$

is a matroid on $\mathfrak{P}$ and $\mathfrak{C}$ is the family of its circuits.

Proof. We consider the family $\mathfrak{C}$ as an antichain $C$ in the lattice $L := \text{Inc}(\mathfrak{P})$, and show that $C$ is the set of all circuits of a combinatorial scheme in $L$ if and only if the family $\mathfrak{C}$ satisfies the four conditions above. Moreover precisely, we have only to show that the elimination and replacement properties are equivalent to (c.1) and (c.2), respectively.

(e.2) is equivalent to (c.1): Immediate.

(c.2) implies (e.3): Let $C \in \mathfrak{C}$, and let $x, y \in \text{Min}(C)$, $x \neq y$, and $z < x$ with $z \notin y$. Set $D := C \cup \text{Upp}(z)$; then $x, y, z \in \text{Min}(D)$; now the assertion is easily proved, as shown in Figure 9.1.

(e.3) implies (c.2): Let $X, Y, Z, T$ be different sets in $\text{Inc}(\mathfrak{P})$ such that $Y = X \cup x$, $Z = X \cup x \cup z$, $T = Z - y$. This implies that $x$ and $y$ are minimal elements of $Y$, $z < x$, and $z \notin y$. Suppose there exists a circuit $C \subseteq Y$, and assume that there is no circuit contained in $X$. Then $x$ is a minimal element of $C$ (otherwise, $C \subseteq X$). If $y$ does not belong to $C$, then $C \subseteq Z - y = T$, and the statement is true. On the contrary, if $y$ is minimal in $C$, by property (e.3) there exists a circuit $C'$ contained in $C \cup \text{Upp}(z) - y$; since $C \cup \text{Upp}(z) - y \subseteq T$, we get the assertion.

If $L$ is a Boolean algebra, the replacement axiom is trivially satisfied by any antichain. As a consequence, there is always a matroid over a set $S$ having a given subset as its only circuit. For combinatorial schemes in a distributive lattice $L$, it is not true that any element $c$ of $L$ may be taken as the only circuit of a combinatorial scheme. For instance, Figure 9.2 shows two examples of lattices with a distinct element $c$ such that the singleton $\{c\}$ is not the set of circuits of any combinatorial scheme in $L$. In fact, in both cases, the antichain $\{c\}$ does not satisfy property (c.2).
Dually, we define a *hyperplane* of a poset matroid \( \mathcal{B} \) on the partially ordered set \( \mathcal{P} \) to be a maximal nonspanning set. Hyperplanes of \( \mathcal{B} \) are circuits of the orthogonal matroid \( \mathcal{B}^\perp \), and conversely. A characterization of hyperplanes of a poset matroid can be obtained from Theorem 9.2 by duality.

**Example 9.1.** Let \( B \) be a filter of the partially ordered set \( \mathcal{P} \). The family of circuits of the poset matroid \( (B) \) is

\[
\mathcal{C} := \{ \{p\} ; p \notin B \text{ and } p \text{ is maximal in } \mathcal{P} \},
\]

and the family of hyperplanes of \( (B) \) is

\[
\mathcal{H} := \{ \mathcal{P} - p ; p \in B \text{ and } p \text{ is minimal in } \mathcal{P} \}.
\]

**Example 9.2.** Let \( \mathcal{B}_i \) be the \( i \)-uniform poset matroid on the partially ordered set \( \mathcal{P} \), as defined in Example 4.2. A filter \( C \) of \( \mathcal{P} \) is a circuit of the poset matroid \( \mathcal{B}_i \) if and only if \( |C| = i + 1 \). Dually, a filter \( H \) of \( \mathcal{P} \) is a hyperplane of \( \mathcal{B}_i \) if and only if \( |H| = i - 1 \).

**10. THE SYMMETRIC EXCHANGE PROPERTY**

**Theorem 10.1.** Let \( \mathcal{B} \) be a poset matroid on the poset \( \mathcal{P} \), and let \( B \subset \mathcal{B} \). For every \( c \in \text{Max}(\mathcal{P} - B) \), there exists a unique circuit \( C \) such that \( C \subseteq B \cup c \); in addition, \( c \in \text{Min}(C) \).

**Proof.** The filter \( B \cup c \) is dependent; hence it must contain a circuit \( C \). Since \( B \) is independent, we have \( c \in C \), and \( c \in \text{Max}(\mathcal{P} - B) \) implies \( c \in \text{Min}(C) \). Suppose now that the filter \( B \cup c \) contains another circuit \( C' \), different from \( C \); by the same argument as before, the element \( c \) is minimal in \( C' \). And \( C \) and \( C' \) are different circuits and \( c \in \text{Min}(C \cap C') \); hence, by the elimination property, there exists a circuit \( C'' \) contained in

\[\ldots\]

\[\ldots\]
(C ∩ C') − c. Since the filter (C ∩ C') − c is contained in the basis B, we get a contradiction.

If B is a basis and c is a maximal element in P − B, the unique circuit contained in B − c is called the fundamental circuit of c with respect to B.

**Proposition 10.2.** Let B be a poset matroid on the partially ordered set P, and let B ∈ B. Choose a maximal element c of the poset P − B, and let C be the fundamental circuit of c with respect to B. Then, setting

\[ B^0 := \{ b ∈ \text{Min}(B); (B ∪ c) − b ∈ \mathcal{B} \}, \]

we have

\[ \text{Min}(C) = B^0 ∪ c. \]

*Proof.* Take b ∈ B^0; since (B ∪ c) − b is independent, and C ⊆ B ∪ c, the element b is minimal in C. Hence B^0 ∪ c ⊆ Min(C). Choose now x ∈ Min(C), x ≠ c; we have that C − x is an independent set contained in B ∪ c, which is a spanning set; hence, by the middle property, there exists a basis B' such that C − x ⊆ B' ⊆ B ∪ c. Since c ∈ B', we have B' ≠ B; recalling that |B'| − |B|, this implies that there exists b ∈ Min(B) such that B' = (B ∪ c) − b; hence, b ∈ B^0. We claim that x = b. In fact, x ∉ (B ∪ c) − b, since C − x ⊆ (B ∪ c) − b, while C ⊇ (B ∪ c) − b. On the other hand, we have x ∈ C ⊆ B ∪ c; hence x = b. This implies that Min(C) ⊆ B^0 ⊆ B^0 ∪ c. 

Dually, we have:

**Proposition 10.3.** Let B be a poset matroid on the partially ordered set P, and let B ∈ B. For every minimal element b in B, there exists a unique hyperplane H such that H ≥ B − b; in addition, b ∈ Max(P − H).

If B is a basis and b is a minimal element in B, the unique hyperplane containing B − b is called the fundamental hyperplane of b with respect to B.

**Proposition 10.4.** Let B be a poset matroid on the partially ordered set P, and let B ∈ B. For every minimal element b in B, let H be the fundamental hyperplane of b with respect to B. Then, setting

\[ B_a := \{ a ∈ \text{Max}(P − B); (B − b) ∪ a ∈ \mathcal{B} \}, \]

we have \( \text{Max}(P − H) = B_a ∪ b. \)
Let $U, V$ be two filters in $P$. A pair $(u, v)$ of elements of $P$ will be called a staple relative to the pair $(U, V)$ whenever it satisfies the following conditions:

$$u \in \text{Min}(U), \quad v \in \text{Max}(P - V), \quad \text{and } u \leq v.$$ 

We are now in position to state our main result:

**Theorem 10.5.** Let $P$ be a finite poset. A nonempty, incomparable family $\mathcal{B}$ of filters of $P$ is a poset matroid on $P$ if and only if satisfies the following property:

(b.4) For every $B_1, B_2 \in \mathcal{B}$ and for every staple $(x_1, y_1)$ relative to $(B_1, B_2)$, there exists a staple $(y_2, x_2)$ relative to $(B_2, B_1)$ such that $(B_1 - x_1) \cup x_2 \in \mathcal{B}$ and $(B_2 - y_2) \cup y_1 \in \mathcal{B}$ (symmetric exchange property).

**Proof.** If $\mathcal{B}$ is a nonempty, incomparable family of filters of $P$ satisfying (b.4), it is obviously a poset matroid on $P$. Conversely, let $\mathcal{B}$ be a poset matroid on $P$; choose $B_1, B_2 \in \mathcal{B}$ and let $(x_1, y_1)$ be a staple relative to $(B_1, B_2)$. Let $H$ be the fundamental hyperplane of $x_1$ with respect to $B_1$, and let $C$ be the fundamental circuit of $y_1$ with respect to $B_2$. Set

$$K := \text{Up}(\text{Max}(P - H) - x_1), \quad D := \text{Low}(\text{Min}(C) - y_1).$$

We recall that $\text{Max}(P - H)$ consists of all those elements $x$ such that $(B_1 - x_1) \cup x$ is a basis, while $\text{Min}(C)$ consists of all those elements $y$ such that $(B_2 - y) \cup y_1$ is a basis; hence all we have to prove is that $D \cap K \neq \emptyset$.

Suppose $D \cap K = \emptyset$. Since, by definition, $P - (H \cup x_1)$ is contained in $K$, while $C - y_1$ is contained in $D$, $D \cap K = \emptyset$ implies

$$C - y_1 \subseteq H \cup x_1.$$ 

$C - y_1$ is an independent set and $H \cup x_1$ is a spanning set; hence, by the middle property, there exists a basis $B_3$ such that

$$C - y_1 \subseteq B_3 \subseteq H \cup x_1.$$ 

Obviously, $C$ cannot be contained in $B_3$, and $H$ cannot contain $B_3$; this implies that $y_1 \notin B_3$, while $x_1 \in B_3$. This gives a contradiction, since $x_1 \leq y_1$ and $B_3$ is a filter. $\blacksquare$
11. RANK

Let $A$ be a nonempty antichain in a distributive lattice $L$. The rank $\rho_A$ associated with $A$ is the nonnegative integer valued function on $L$ defined as follows: for every $x \in L$,

$$\rho_A(x) := \max\{\text{height}(y) ; y \leq x \text{ and } y \leq a \text{ for some } a \in A\}.$$ 

**Proposition 11.1.** Let $A$ be a nonempty antichain in a distributive lattice $L$. The rank $\rho_A$ satisfies the following properties:

1. For every $x \in L$ there exists $y \in L$ such that $y \leq x$ and $\text{height}(y) = \rho_A(y) = \rho_A(x)$.
2. For every $x, y \in L$ such that $x < y$, we have $\rho_A(x) \leq \rho_A(y) \leq \rho_A(x) + 1$.

Conversely, let $\rho$ be a nonnegative integer valued function defined on the distributive lattice $L$ satisfying conditions (r.1) and (r.2); then, setting

$$A = \text{Max}\{x \in L ; \text{height}(x) = \rho(x)\},$$

one obtains an antichain $A$ and $\rho = \rho_A$.

**Proof.** Property (r.1) is trivially satisfied.

Now let $x, y \in L$ with $x < y$, and let $y' \in \text{Low}(A)$ such that $y' \leq y$ and $\text{height}(y') = \rho_A(y)$. $x < y$ implies that either $y' \land x < y'$ or $y' \land x = y'$; in both cases,

$$\rho_A(x) \geq \text{height}(y' \land x) \geq \text{height}(y') - 1 = \rho_A(y) - 1,$$

and we get (r.2). Moreover,

$$\text{Max}\{x \in L ; \text{height}(x) = \rho_A(x)\} = A.$$

Conversely, suppose that $\rho : L \to \mathbb{N}$ satisfies conditions (r.1) and (r.2). Set

$$A := \text{Max}\{x \in L ; \text{height}(x) = \rho(x)\}$$

and let $\rho_A$ be the rank associated with the antichain $A$.

We remark that the decreasing set $\text{Low}(A)$ contains the set

$$D := \{x \in L ; \text{height}(x) = \rho(x)\}.$$ 

Furthermore, if $y \in D$ and $x < y$, by property (r.2) $\rho(x) = \text{height}(x)$. Hence $D$ is a decreasing set and $\text{Low}(A) = D$. Let now $x \in L$ and choose $y \in D$, such that $y \leq x$ and $\rho(y) = \rho(x)$. Such an element $y$ exists by
property (r.1). We have $\rho_A(x) \geq \rho(y) = \rho(x)$. On the other hand, (r.2) implies that, for every $z \leq x$, $\rho(z) \leq \rho(x) = \text{height}(y)$; hence, $\rho_A(x) \leq \rho(x)$. This implies $\rho_A(x) = \rho(x)$. 

**Proposition 11.2.** Property (r.2) may be equivalently replaced by the following:

(r.3) for every $x, y \in \mathbb{L}$ such that $x \leq y$, we have $0 \leq \rho_A(y) - \rho_A(x) \leq \text{height}(y) - \text{height}(x)$.

We next characterize ranks associated with combinatorial schemes:

**Theorem 11.3.** Let $A$ be a nonempty antichain in a distributive lattice $\mathbb{L}$, and $\rho_A$ be the associated rank. Then $A$ is a combinatorial scheme if and only if $\rho_A$ satisfies one of the following equivalent conditions:

(r.4) For every $x, y \in \mathbb{L}$, $x \leq y$, there exists $z \in \mathbb{L}$ such that $x \leq z \leq y$, $\rho_A(z) = \rho_A(y)$ and $\text{height}(z) - \text{height}(x) = \rho_A(y) - \rho_A(x)$ (greedy property).

(r.4') For every $x, y \in \mathbb{L}$, $x \leq y$ there exists a complete chain, $x = x_0 < x_1 < \cdots < x_i = y$, such that $\rho_A(x_{j+1}) = 1 + \rho_A(x_j)$ for $j = 0, 1, \ldots, i - 1$, and $\rho_A(x_{j+1}) = \rho_A(x_j)$ for $j \geq i$ (greedy chain property).

**Proof.** We prove the assertion for the greedy property. The equivalence between (r.4) and (r.4') is straightforward.

Suppose $A$ is a combinatorial scheme. Take $x, y \in \mathbb{L}$, $x \leq y$, and let $i$ be an independent element in $[0, x]$, of maximum height. By property (j.3) there exists an independent element $j$ in $[0, y]$ of maximum height, such that $i = j \wedge x$. Set $z := j \vee x$. It is immediately seen that $\rho_A(z) = \rho_A(y)$ and, by the modularity of the lattice $\mathbb{L}$, we have $\text{height}(z) - \text{height}(x) = \text{height}(j) - \text{height}(i) = \rho_A(y) - \rho_A(x)$.

Conversely, suppose that the rank $\rho_A$ satisfies the greedy property; we shall show that the decreasing set $I := \text{Low}(A)$ is the set of independent elements of a combinatorial scheme in $\mathbb{L}$, by proving that $I$ satisfies property (j.3).

Take $x, y \in \mathbb{L}$, $x \leq y$; let $i \in \text{Low}(A) \cap [0, x]$, of maximum height. By the hypothesis, there exists $z \in \mathbb{L}$ such that $i \leq z \leq y$, $\rho_A(z) = \rho_A(y)$ and $\text{height}(z) - \text{height}(i) = \rho_A(y) - \rho_A(i)$; hence $\rho_A(z) = \text{height}(z)$. This implies that $z \in \text{Low}(A) \cap [0, y]$, and its height is maximum. Moreover, $z \wedge x \in \text{Low}(A) \cap [0, x]$, and $i \leq z \wedge x$; since $i$ was supposed to have maximum height in $\text{Low}(A) \cap [0, x]$, we conclude that $i = z \wedge x$, and we get the assertion.

Either of these two greedy properties may be replaced by a “local” version, first discovered by Henry Crapo, as follows:

**Proposition 11.4.** Let $\rho_A$ be the rank associated with an antichain $A$. The greedy properties (r.4) and (r.4') are respectively equivalent to the
following properties:

(r.5) For every \( x, y \in \mathbb{L} \) such that \( x \leq y \) and \( \text{height}(\{x, y\}) = 2 \), there exists \( z \in \mathbb{L} \) such that \( x \leq z \leq y \), \( \rho_\mathbb{A}(z) = \rho_\mathbb{A}(y) \) and \( \text{height}(z) - \text{height}(x) = \rho_\mathbb{A}(y) - \rho_\mathbb{A}(x) \) (Crapo’s property).

(r.5’) For every \( x, y \in \mathbb{L} \) with \( x < y \), \( \rho_\mathbb{A}(x) < \rho_\mathbb{A}(y) \) and \( \text{height}(y) - \text{height}(x) \leq 2 \), there exists \( z \in \mathbb{L} \) such that \( x < z < y \) and \( \rho_\mathbb{A}(z) = \rho_\mathbb{A}(x) + 1 \) (local chain property).

Proof. The equivalence between (r.4) and (r.5’) is trivial; we have only to show that (r.5) implies (r.4). In fact, let \( \rho_3, \rho_{1,2} \) be the rank functions associated with the two antichains \( A_3 \) and \( A_{1,2} \) of Example (6.2), respectively. Neither \( \rho_3 \) nor \( \rho_{1,2} \) satisfies Crapo’s property. To check this, take \( x \) and \( y \) as in Figure 11.1.

**Proposition 11.5.** Let \( \rho_\mathbb{A} \) be the rank associated with a combinatorial scheme \( \mathbb{A} \). An element \( x \) of \( \mathbb{L} \) is independent in \( \mathbb{A} \) if and only if \( \rho_\mathbb{A}(x) = \text{height}(x) \).

Proof. As we remarked above, for every element \( x \in \mathbb{L} \) such that \( \text{height}(x) = \rho_\mathbb{A}(x) \), there exists a basis \( a \in \mathbb{A} \) such that \( x \leq a \), and conversely. This gives the assertion.

The following characterization of ranks associated with combinatorial schemes will be used in the sequel:

**Proposition 11.6.** A map \( \rho: \mathbb{L} \to \mathbb{N} \) is the rank associated with a combinatorial scheme if and only if it satisfies the following conditions:

(r.0) \( \rho(0) = 0 \).

(r.2) For every \( x, y \in \mathbb{L} \) such that \( x < y \), we have \( \rho(x) \leq \rho(y) \leq \rho(x) + 1 \).

(r.5’) For every \( x, y \in \mathbb{L} \), with \( x < y \), \( \rho(x) < \rho(y) \) and \( \text{height}(y) - \text{height}(x) \leq 2 \), there exists \( z \in \mathbb{L} \) such that \( x < z < y \) and \( \rho(z) = \rho(x) + 1 \) (local chain property).

**Figure 11.1.**
Recall that an integer valued function $\rho$ defined on a distributive lattice $L$ is said to be semimodular if and only if it satisfies the condition

$$\rho(x \vee y) + \rho(x \wedge y) \leq \rho(x) + \rho(y), \quad x, y \in L.$$ 

When $L$ is a Boolean algebra, the rank of a matroid is characterized by the semimodular property above, together with $\rho(0) = 0$ and $\rho(x) \leq \rho(y) \leq \rho(x) + 1$ if $x < y$.

Rank functions of combinatorial schemes are also semimodular:

**Proposition 11.7.** The rank function $\rho_A$ of a combinatorial scheme $A$ in a distributive lattice $L$ is semimodular.

*Proof.* Let $\rho_A$ be a rank function over the lattice $L$. Let $x, y \in L$, and let $i$ be an independent element of maximum height under $x \wedge y$; then there exists an independent element $j$ of maximum height under $x \vee y$ such that $i = j \wedge (x \wedge y)$. We have $j = j \wedge (x \vee y) = (j \wedge x) \vee (j \wedge y)$, whence

$$\rho_A(x \vee y) + \rho_A(x \wedge y)
= \text{height}(j) + \text{height}(j \wedge x \wedge y)
= \text{height}(j \wedge x) + \text{height}(j \wedge y)
- \text{height}(j \wedge x \wedge y) + \text{height}(j \wedge x \wedge y)
\leq \rho_A(x) + \rho_A(y).$$

Semimodularity alone is not sufficient to characterize rank functions of combinatorial schemes in a distributive lattice. Semimodularity is equivalent to the greedy property only in the case of combinatorial schemes over lattices that are not “too far” from being a Boolean algebra. The next result describes the situation.

We denote by $C_{1,2}$ the distributive lattice that is the product of a chain of height one and a chain of height 2.

**Proposition 11.8.** If the distributive lattice $L$ does not contain an interval isomorphic to $C_{1,2}$, then a semimodular nonnegative integer valued function $\rho$ on $L$ is the rank function of a combinatorial scheme if and only if it satisfies conditions (r.1) and (r.2).

*Proof.* Suppose that $L$ has no interval isomorphic to $C_{1,2}$, and let $\rho_A$ be the rank associated with an antichain $A$ in $L$. Consider the decreasing set $D$ of $L$ defined as

$$D = \{x \in L; x \leq y \text{ for some } y \in A\}.$$
If \( \mathcal{A} \) is not a combinatorial scheme, by Theorem 7.3 there exists an interval \([a, b]\) of \( \mathbb{L} \) such that the partially ordered set \( D \cap [a, b] \) is isomorphic to \( \mathcal{D}_3 \), and this implies that \( \rho_A \) is not semimodular, as can be seen in Figure 11.2.

We remark that the converse of the preceding proposition is false; the rank associated with the antichain \( A_{1,2} \) defined in Example 6.2 is semimodular (see Fig. 11.3), but the corresponding antichain is not a combinatorial scheme. This shows that if the lattice \( \mathbb{L} \) has some interval isomorphic to \( \mathcal{C}_{1,2} \), then the class of semimodular ranks associated with an antichain of \( \mathbb{L} \) properly contains the class of rank functions of combinatorial schemes in \( \mathbb{L} \).

In closing, we remark that, as in classical matroid theory, the nullity \( \nu_A \) associated with the antichain \( \mathcal{A} \) of a distributive lattice \( \mathbb{L} \) is defined as follows:

\[
\nu_A(x) := \text{height}(x) - \rho_A(x).
\]

Characterizations of combinatorial schemes in terms of nullity may be given.

### 12. The Fundamental Example

We show that a poset matroid may be associated with every subset of linear varieties in a projective space. Such a poset matroid codes information pertaining to linear dependence of the varieties.

Let \( K \) be a field, and \( \mathbb{P}_d(K) \) be the projective space of finite dimension \( d \) over \( K \). Let \( \mathbb{L}(\mathbb{P}_d) \) be the lattice of linear subspaces of \( \mathbb{P}_d(K) \), ordered by inclusion.

A **linear configuration** in \( \mathbb{P}_d(K) \) is an \( n \)-tuple \( (s_1, s_2, \ldots, s_n) \) of elements of \( \mathbb{L}(\mathbb{P}_d) \).

If \( \rho_A(x_0) = h \), then

\[
\rho_A(x_1) = h+1 = \rho_A(x_2),
\]

\[
\rho_A(x_3) = h+1 = \rho_A(x_4) = \rho_A(x_6),
\]

\[
\rho_A(x_5) = h+2 = \rho_A(x_7),
\]

and the function \( \rho_A \) is not semimodular.

**Figure 11.2.**
With a linear configuration \((s_1, s_2, \ldots, s_n)\) in \(\mathbb{P}_d(K)\) we associate a poset \(P\) that is the disjoint union of \(n\) chains \(C_1, C_2, \ldots, C_n\), with \(\text{Card}(C_i) = \text{height}(s_i)\), where \(\text{height}(s_i)\) is the height of \(s_i\) in the lattice \(\mathbb{L}(\mathbb{P}_d)\), that is, \(\text{height}(s_j) = \text{dim}(s_j) + 1\).

Let \(L\) be a distributive lattice that is isomorphic to the distributive lattice of filters of the partially ordered set \(P\), and let \(\phi\) be such an isomorphism. Thus, for \(x\) in \(L\), \(\phi(x) = A = A_1 \cup A_2 \cup \cdots \cup A_n\), where \(A\) is a filter of \(P\), and \(A_i\) is a filter of \(C_i\).

Given an element \(x\) of \(L\), \(\phi(x) = A_1 \cup A_2 \cup \cdots \cup A_n\), a realization of \(x\) will be any \(n\)-tuple \((t_1, t_2, \ldots, t_n)\) of elements of the lattice \(\mathbb{L}(\mathbb{P}_d)\), where \(t_i \leq s_i\) and \(\text{dim}(t_i) = |A_i| - 1\) for \(i = 1, 2, \ldots, n\).

We define a nonnegative integer valued function \(\rho\) on the distributive lattice \(L\) by setting \(\rho(x)\) to be \(1 + \text{the maximum dimension of } t_1 \vee t_2 \vee \cdots \vee t_n\) as \((t_1, t_2, \ldots, t_n)\) range over all realizations of \(x\). In symbols,

\[
\rho(x) := \max\{\text{dimension}(t_1 \vee t_2 \vee \cdots \vee t_n) = 1; (t_1, t_2, \ldots, t_n) \text{ is a realization of } x\}.
\]

**Theorem 12.1.** The function \(\rho\) thus defined on the distributive lattice \(L\) is the rank function of a combinatorial scheme.

**Proof.** We will show that \(\rho\) satisfies conditions (r.0), (r.2), and (r.5'); hence \(\rho\) is the rank function of a combinatorial scheme by Proposition 11.6. First of all, the unique realization of the minimum of \(L\) is the minimum of the lattice \(\mathbb{L}(\mathbb{P}_d)\), whence \(\rho(0) = 0\). Furthermore, let \(x, y\) be elements of \(L\) such that \(x < y\), and let \((t_1, t_2, \ldots, t_n)\) be a realization of \(y\) such that \(\text{dim}(t_1 \vee t_2 \vee \cdots \vee t_n)\) is maximum. Since \(x < y\), it is easily seen that there exists a realization of \(x\), \((t'_1, t'_2, \ldots, t'_n)\), and a positive integer
Since \( r \) dimension and \( t \) such that \( A \) \( \rightarrow \) \( r \) hence \( F \) assertion.

Finally, let \( x, y \) be elements of \( L \) such that \( x < y \), height\( (y) \) − height\( (x) \) = 2 and \( \rho (x) < \rho (y) \). Note that setting

\[
\phi (x) = A_1 \cup A_2 \cup \cdots \cup A_n, \quad \phi (y) = B_1 \cup B_2 \cup \cdots \cup B_n,
\]

\( x < y \) implies that \( A_i \subseteq B_i \subseteq C_j \), and there exists at least an index \( j \leq n \) such that \( A_j \subset B_j \). Now let \( (t_1, t_2, \ldots, t_n) \) be a realization of \( x \) of maximum dimension and \( (t'_1, t'_2, \ldots, t'_n) \) be a realization of \( y \) of maximum dimension. Since \( \rho (x) < \rho (y) \), there exists an atom \( p \) of the lattice \( \mathbb{L}(\mathbb{P}_n) \) such that \( p \leq t'_1 \lor t'_2 \lor \cdots \lor t'_n \) and \( p \nleq t_1 \lor t_2 \lor \cdots \lor t_n \). In fact, if every atom of \( \mathbb{L}(\mathbb{P}_n) \) lying under \( t'_1 \lor t'_2 \lor \cdots \lor t'_n \) also lies under \( t_1 \lor t_2 \lor \cdots \lor t_n \), by the atomicity of \( \mathbb{L}(\mathbb{P}_n) \) we get \( t_1 \lor t_2 \lor \cdots \lor t_n \leq t'_1 \lor t'_2 \lor \cdots \lor t'_n \), contradicting \( \rho (x) < \rho (y) \).

Now the \( n \)-tuple \( (t_1, t_2, \ldots, t_{j-1}, t_j \lor p, t_{j+1}, \ldots, t_n) \) is a realization of an element \( z \) of \( L \) such that \( x < z < y \) and \( \rho (z) = \rho (x) + 1 \), and we get the assertion.

**Proposition 12.2.** In the notation of the preceding theorem, an element \( x \) of \( \mathbb{L} \) is independent if and only if \( \rho (x) = |\phi (x)| = |A| \), that is, whenever there exist \( t_i \leq s \) such that \( \dim (t_i) = |A_i| - 1 \) and such that

\[
\text{height}(t_1 \lor t_2 \lor \cdots \lor t_n) = \text{height}(t_1) + \text{height}(t_2) + \cdots + \text{height}(t_n),
\]

or, equivalently,

\[
\dim (t_1 \lor t_2 \lor \cdots \lor t_n) = \dim (t_1) + \dim (t_2) + \cdots + \dim (t_n) + n - 1.
\]

In projective geometry, a set of \( n \) subspaces of a projective space satisfying the condition above is said to be independent (see [5]).

**Example 12.1.** In the real projective space of dimension 3, consider a plane \( \pi \), two incident lines \( r, s \) not belonging to \( \pi \), an the two distinct points \( P := r \cap \pi, Q := r \cap s \). The poset \( \mathbb{P} \) associated with the linear configuration \( (\pi, r, s, P, Q) \) is the disjoint union of five chains (see Fig. 12.1). We list here the bases of the poset matroid that corresponds to
EXAMPLE 12.2. In the real projective space of dimension 3, consider two planes $\pi, \sigma$, a line $r$ belonging neither to $\pi$ nor to $\sigma$, and the lines $s := \pi \cap \sigma$. The poset $P$ associated with the linear configuration $(\pi, \sigma, r, s)$ is the disjoint union of four chains (see Fig. 12.2). The bases of the poset matroid associated with $(\pi, \sigma, r, s)$ are the following:

\begin{align*}
\{a, b, c, e\} & \quad \{b, c, d, e\} & \quad \{b, c, e, i\} & \quad \{b, c, g, i\} & \quad \{c, d, e, i\} \\
\{a, d, f, i\} & \quad \{a, f, h, i\} & \quad \{a, b, c, g\} & \quad \{b, c, e, g\} & \quad \{b, c, f, g\} \\
\{b, c, h, i\} & \quad \{c, e, f, g\} & \quad \{c, e, h, i\} & \quad \{e, g, h, i\} & \quad \{a, b, c, i\} \\
\{b, c, e, h\} & \quad \{b, c, g, h\} & \quad \{c, d, e, g\} & \quad \{c, e, g, h\} & \quad \{c, f, g, h\}.
\end{align*}

EXAMPLE 12.3. In the real projective space of dimension 3, consider two planes plane $\pi, \sigma$, a line $r$ belonging neither to $\pi$ nor to $\sigma$ and
intersecting $\pi$ and $\sigma$ in two different points, and a line $s$ lying on $\pi$ and intersecting $r$. The poset $\mathcal{P}$ associated with the linear configuration $(\pi, \sigma, r, s)$ is the same as that of Example 12.2 (see Fig. 12.3). The bases of the poset matroid associated with $(\pi, \sigma, r, s)$ are the following:

\[
\begin{align*}
\{a, b, c, f\} & \quad \{b, c, f, l\} & \quad \{c, d, e, f\} & \quad \{c, f, g, h\} & \quad \{c, g, h, l\} \\
\{d, e, f, l\} & \quad \{e, f, i, l\} & \quad \{a, b, c, h\} & \quad \{b, c, g, h\} & \quad \{c, e, f, h\} \\
\{c, f, h, l\} & \quad \{c, h, i, l\} & \quad \{e, f, g, h\} & \quad \{f, g, h, l\} & \quad \{b, c, f, h\} \\
\{b, c, h, l\} & \quad \{c, e, f, l\} & \quad \{c, f, i, l\} & \quad \{d, e, f, h\} & \quad \{e, f, h, l\} \\
\{f, h, i, l\}.
\end{align*}
\]

We conclude with a list of a few open problems:

1. The symmetric exchange property that is the main result of this paper generalizes only one of several exchange properties known for matroids. Which of these exchange properties can be extended to combinatorial schemes, and how?

2. Just as one visualizes matroids as arrangements of hyperplanes or as sets of points in projective space, one expects to visualize combinatorial schemes by sets of linear varieties in projective space. However, the only combinatorial schemes that we currently obtain from linear varieties in projective space are those associated with distributive lattices in which the
meet-irreducibles are disjoint unions of chains. Is there a way of associating projective configurations with more general combinatorial schemes?

3. Neil White has developed a theory of representation of matroids by bracket rings. Using techniques of supersymmetric algebra, one may conjecture that certain combinatorial schemes may also be representable by brackets over a positive alphabet, in which divided powers stand for varieties.

4. The Tutte-Grothendieck ring, as developed by T. Brylawski, may be generalized to combinatorial schemes.

5. The greedy algorithm for matroids may also be generalized to combinatorial schemes.

REFERENCES

43. J. Lawrence, Oriented matroids and multiply ordered sets, Linear Algebra Appl. 48 (1982), 1–12.
45. S. MacLane, Some interpretations of abstract linear dependence in terms of projective geometry, Amer. J. Math. 58 (1936), 236–240.