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A noncommutative Brooks–Jewett Theorem

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ABSTRACT

In classical measure theory the Brooks–Jewett Theorem provides a finitely-additive-analogue to the Vitali–Hahn–Saks Theorem. In this paper, it is studied whether the Brooks–Jewett Theorem allows for a noncommutative extension. It will be seen that, in general, a bona-fide extension is not valid. Indeed, it will be shown that a C^*-algebra A satisfies the Brooks–Jewett property if, and only if, it is Grothendieck, and every irreducible representation of A is finite-dimensional; and a von Neumann algebra satisfies the Brooks–Jewett property if, and only if, it is topologically equivalent to an abelian algebra.

1. Introduction and preliminaries

This paper continues to study possible noncommutative extensions of classical convergence theorems of measure theory. The results obtained here build on our previous work [7,8] and extend that in [4–6]. Besides clarifying the status of the Brooks–Jewett Theorem in the noncommutative setting, this paper exhibits an interesting interplay between Banach space properties, measure-theoretic properties and structural properties of operator algebras. The paper is organized as follows. After recalling basic facts in the introduction, we then continue to collect needed results on Grothendieck C^*-algebras and weakly compact operators in Section 2. The main results concerning the noncommutative Brooks–Jewett property are given in Section 3. This is followed by a small Appendix A, in which a new convergence theorem for weakly compact operators is proved.

Let us recall needed concepts and fix the notation. For a normed space X we use the symbol X_1 to denote its closed unit ball and X^* its dual. For normed spaces X and Y the symbol $B(X,Y)$ denotes the space of all bounded linear mappings of X into Y equipped with the operator norm. We write $B(X)$ instead of $B(X,X)$. The strong operator topology on $B(X,Y)$ is induced by the seminorms $T \in B(X,Y) \mapsto \|Tx\|$, $x \in X$. For $T \in B(X,Y)$ the symbol T^* denotes the adjoint of T.

Throughout the paper, A is a unital C^*-algebra and A_+ (respectively A^*_+) denotes the positive part of A (respectively of A^*). Our standard reference for operator algebras is [12,18,22]. For every $\psi \in A^*_+$ the mapping $a \mapsto \sqrt{\psi(a^*a) + \psi(aa^*)}$ defines a seminorm on A which we denote by η_ψ.

We write M to denote a generic von Neumann algebra and M_* its predual. We identify the elements of M_* with the normal functionals in M^*. We write M_{*+} to denote the positive part of M_*. We recall that the weak*-topology on M (denoted by $\sigma(M, M_*)$) is the weakest topology compatible with the duality $\langle M_*, M\rangle$. The strongest topology on M compatible with this duality is the Mackey topology and is denoted by $\tau(M, M_*)$. The $\tau(M, M_*)$ topology on M coincides with the topology of uniform convergence on weakly relatively compact subsets of M_*. Lying between these topologies we have the σ-strong topology $s(M, M_*)$ determined by the family of seminorms $\{\varrho_\psi \mid \psi \in M_{*+}\}$ where $\varrho_\psi(x) = \sqrt{\psi(x^*x)}$; and the σ-strong* topology $s^*(M, M_*)$ determined by the family of seminorms $\{\eta_\psi \mid \psi \in M_{*+}\}$. Recall that on bounded parts

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of M the σ-strong* topology coincides with the Mackey topology. We denote by $P(M)$ the projection lattice of M, i.e. $P(M) = \{p \in M \mid p = p^2 = p^*\}$. An operator $T \in B(M, X)$ is said to be completely additive if $T(\sum_{\alpha \in I} p_\alpha) = \sum_{\alpha \in I} T(p_\alpha)$ whenever $\{p_\alpha \mid \alpha \in I\}$ is a set of pairwise orthogonal projections in M. (The sum on the left-hand side is equal to $\bigvee_{\alpha \in I} p_\alpha$ and is the $s^*(M, M_*)$ limit of the net $\{\sum_{\alpha \in J} p_\alpha \mid J$ is a finite subset of $I\}$. The sum on the right-hand side is considered in the norm topology.) It is well known that any one-dimensional operator on M is completely additive if, and only if, it is normal (see [22, Corollary 3.11, p. 136]).

A subset $\mathcal{K}$ of $B(A, X)$ is said to be *pointwise absolutely continuous* with respect to $\psi \in A_+^*$ (in symbols $\mathcal{K} \ll_p \psi$) if the restriction of every $T \in \mathcal{K}$ to A_1 is continuous with respect to the seminorm η_ψ. If $\mathcal{K} = \{T\}$ we write $T \ll \psi$ instead of $\{T\} \ll_p \psi$. $\mathcal{K}$ is said to be *uniformly absolutely continuous* with respect to ψ (in symbols $\mathcal{K} \ll_u \psi$) if the set $\{T|_{A_1} \mid T \in \mathcal{K}\}$ is uniformly continuous on A_1 with respect to η_ψ.

In particular, for $K \subset M_*$ we define $K_p = \{\psi \in M_{*+} \mid K \ll_p \psi\}$ and $K_u = \{\psi \in M_{*+} \mid K \ll_u \psi\}$. A von Neumann algebra M is said to have the *Vitali–Hahn–Saks property* if $K_p = K_u$ is satisfied for every weakly relatively compact subset $K \subset M_*$. In view of [22, Theorem 5.4, p. 149], a von Neumann algebra has the Vitali–Hahn–Saks property if, and only if, the weakly relatively compact subsets of M_* are precisely the bounded subsets for which $\emptyset \neq K_p = K_u$. The following theorem was established in [8]. Let us recall that the von Neumann algebra M is finite if, and only if, the $*$-operation is σ-strongly continuous. Hence M is finite if, and only if, the Mackey topology coincides with the σ-strong topology on bounded parts of M.

Theorem 1. *(See [8, Theorem 2.3].) A von Neumann algebra has the Vitali–Hahn–Saks property if, and only if, it is finite.*

We recall that for a C^*-algebra A, we can identify A^{**} with the von Neumann envelope of A in its universal representation. Let $\varphi \in A^*$. The double adjoint φ^{**} of φ is a linear functional acting on A^{**}. It is well known that φ^{**} is the unique extension of φ from A (embedded canonically into the second dual A^{**}) to a normal functional φ^{**} on the von Neumann algebra A^{**}. We usually identify φ with φ^{**} but sometimes we insist on the notational distinction to aid clarity. We recall that the image of A_1 under the canonical embedding of A into A^{**} is dense in the unit ball of A^{**} with respect to the $s^*(A^{**}, A^*)$-topology (Kaplansky's Density Theorem); and therefore with respect to the seminorm $\eta_{\psi^{**}}$ for any $\psi \in A_+^*$. So if $K \subset A^*$ and $\psi \in A_+^*$ such that $K \ll_u \psi$, then $K^{**} = \{\varphi^{**} \mid \varphi \in K\} \ll_u \psi^{**}$ (see also [5, Lemma 4.2]). We recall that every $\varphi \in A^*$ can be uniquely decomposed into a linear combination of four positive linear functionals $\varphi = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$ where φ_1 and φ_2 (respectively φ_3 and φ_4) have orthogonal support projections in A^{**}. We denote by $[\varphi]$ the positive linear functional defined by $[\varphi] = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$.

In classical measure theory, the Brooks–Jewett Theorem can be considered as the finitely-additive analogue of the Vitali–Hahn–Saks Theorem and reads as follows:

Theorem 2. *Let Σ be a σ-field and (μ_n) a sequence of s-bounded vector-valued (finitely-additive) measures on Σ. Suppose further that (μ_n) is pointwise convergent on the elements of Σ and that each μ_n is absolutely continuous with respect to some positive (finitely-additive) measure μ on Σ. Then the set $\{\mu_n \mid n \in \mathbb{N}\}$ is uniformly absolutely continuous with respect to μ.*

It must be remarked here that this theorem was first proved for finitely-additive scalar measures by Andô [2] using sliding hump arguments. The vector case can be easily reduced to the scalar case. Without knowing this, Brooks and Jewett proved it directly again in [3] (see discussion on p. 35 in [11]).

To elucidate the generalization made in this paper, we recall that any commutative von Neumann algebra M is $*$-isomorphic to the algebra $L^\infty(\Sigma, \nu)$ of essentially bounded Σ-measurable functions, where ν is a (possibly infinite-valued) positive σ-additive measure on Σ. On the other hand, except for some pathological cases, every algebra $L^\infty(\Sigma, \nu)$ is a commutative von Neumann algebra. Any element of Σ can be identified with its corresponding characteristic function in the von Neumann algebra $L^\infty(\Sigma, \nu)$. We recall the well-known fact from integration theory that every finitely-additive vector measure μ on Σ that is absolutely continuous with respect to ν extends uniquely to a bounded operator on $L^\infty(\Sigma, \nu)$. Thus, one can obtain an algebraic version of Theorem 2 by replacing the structure of sets Σ with the algebra $L^\infty(\Sigma, \nu)$; the control measure μ with a positive functional on $L^\infty(\Sigma, \nu)$; and each μ_n with an operator on $L^\infty(\Sigma, \nu)$. We further recall that every commutative von Neumann algebra $L^\infty(\Sigma, \nu)$ can be identified with the algebra $C(X)$ of continuous functions on a suitable hyperstonean space X (see [13,22]). Thus the convergence theorem we study in this paper gives a two-fold generalization of the Vitali–Hahn–Saks Theorem. Basically, we ask whether the same conclusion as in Theorem 2 can be drawn if one were to replace (μ_n) by a sequence (T_n) of elements of $B(A, X)$ where A is a (not necessarily commutative) C^*-algebra and X a Banach space. Let us recall that noncommutative extensions of convergence theorems of classical measure theory have received a considerable attention and led to many deep results recently (see [4–8,15] and extensive references therein).

Definition 1. A C^*-algebra A has the *Brooks–Jewett property* if the following holds: If (φ_n) is a weakly* convergent sequence in A^* and $\{\varphi_n \mid n \in \mathbb{N}\} \ll_p \psi \in A_+^*$, then $\{\varphi_n \mid n \in \mathbb{N}\} \ll_u \psi$.

Let us comment on the difference between the definition of the Vitali–Hahn–Saks property and the Brooks–Jewett property. While the Vitali–Hahn–Saks property postulates uniform continuity of weakly compact sets of functionals, the

Brooks–Jewett property allows to conclude uniform continuity only for convergent sequences of functionals. The reason is that unlike the Vitali–Hahn–Saks property, the "set version" of the Brooks–Jewett property would imply the reflexivity of the algebra; and therefore this would imply that the algebra is finite-dimensional. Indeed, let A be a C^*-algebra satisfying the "set version" of the Brooks–Jewett property. Every countable subset $\{\varphi_n \mid n \in \mathbb{N}\}$ of A_1^* admits a functional such that each φ_n is absolutely continuous with respect to it, namely $\psi = \sum_n \frac{1}{2^n}[\varphi_n]$. Clearly, $\{\varphi_n \mid n \in \mathbb{N}\}$ is weak* relatively compact, and therefore $\{\varphi_n \mid n \in \mathbb{N}\} \ll_u \psi$. Thus $\{\varphi_n^{**} \mid n \in \mathbb{N}\} \ll_u \psi^{**}$ and, in view of Akemann's Theorem [1], this implies that $\{\varphi_n^{**} \mid n \in \mathbb{N}\}$ is weakly relatively compact. Hence A_1^* is weakly compact by the Eberlein–Šmulian Theorem, i.e. A is reflexive and so finite-dimensional.

2. Grothendieck property and its consequences for operator algebras

It turns out that the convergence theorems are closely connected with the Grothendieck property. For this reason we isolate in this section some facts about Grothendieck operator algebras. We recall that a Banach space has the Grothendieck property (or is a Grothendieck Banach space) if every weakly* null sequence in its dual is weakly null. It was proved by H. Pfitzner [19] that all von Neumann algebras have the Grothendieck property. Later on this result was generalized by K. Saitô and J.D.M. Wright to the effect that all monotone σ-complete C^*-algebras are Grothendieck spaces [20].

The Grothendieck property for Banach spaces is defined in terms of linear functionals, i.e. one-dimensional operators. However, this property carries over to linear operators, as shown in the following lemma. Although this follows implicitly from [17, Lemma 1], we here provide a direct proof for the sake of completeness.

Lemma 3. *Let X be a Grothendieck Banach space and let Y be a normed space. Let (T_n) be a sequence in $B(X,Y)$ convergent to T in the strong operator topology. Then (T_n^{**}) converges to T^{**} in the strong operator topology.*

Proof. Suppose on the contrary that there is $f \in X^{**}$ such that $(T_n^{**} f)_{n\in\mathbb{N}}$ does not converge to $T^{**}f$. Without loss of generality assume that there is an $\varepsilon > 0$ such that $\|T_n^{**} f - T^{**} f\| > \varepsilon$ for each n. Let $\varphi_n \in Y_1^*$ with

$$\left|\left(T_n^{**} f - T^{**} f\right)\varphi_n\right| > \varepsilon/2. \tag{1}$$

Put

$$\psi_n = \left(T_n^* - T^*\right)\varphi_n \in X^*.$$

By hypothesis, $\|(T_n - T)x\| \to 0$ for each $x \in X$ and so $\psi_n(x) \to 0$ for each $x \in X$. By the Grothendieck property $\psi_n^{**}(g) \to 0$ for each $g \in X^{**}$. But this contradicts (1). □

We recall that an operator T in $B(X,Y)$ is weakly compact if TX_1 is weakly relatively compact in Y. It is well known that T is weakly compact if, and only if, $T^{**}(X^{**}) \subset Y$ (see [13, Theorem 2, p. 482]). If T is weakly compact, then the same holds for its adjoint T^*. If Y is a Banach space, Gantmacher's Theorem states that T is weakly compact if, and only if, T^* is weakly compact (see [13, Theorem 8, p. 485]).

Proposition 4. *Let X be a Grothendieck Banach space and let Y be a Banach space. Let (T_n) be a sequence of weakly compact operators in $B(X,Y)$ convergent to T in the strong operator topology. Then T is weakly compact.*

Proof. The images of the T_n^{**} are in Y because they are weakly compact.

From Lemma 3 it follows that T_n^{**} converges to T^{**} in the strong operator topology. Thus the image of T^{**} is in Y, too, and T is weakly compact. □

We remark that the statement above does not hold if X does not have the Grothendieck property. Indeed, take $A = c_0$ and operators $T_n \in B(A)$ defined by $T_n(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. All T_n's are weakly compact and converge to the identity in the strong operator topology. But the identity is not weakly compact since A is not reflexive (see also [24]).

The absolute continuity of an operator with respect to some positive functional is a ubiquitous assumption made in the paper. In the following proposition, it is shown that we are every time working with weakly compact operators. For the theory of weakly compact operators on C^*-algebras we refer the reader to [16,21,23].

Proposition 5. *Let X be a Banach space. The following four conditions for an operator T in $B(A,X)$ are equivalent.*

(1) *T is weakly compact.*
(2) *Whenever (a_n) is an orthogonal sequence of self-adjoint elements in the unit ball of A, then $\lim_{n\to\infty} \|T(a_n)\| = 0$.*
(3) *There exists $\psi \in A_+^*$ such that $T \ll \psi$.*
(4) *T^{**} is completely additive.*

Proof. ((1) $\Leftrightarrow$ (2)) T is weakly compact if, and only if, $T^*X_1^*$ is weakly relatively compact. Result now follows from the deep Pfitzner's Theorem [19, Theorem 1].

((1) $\Rightarrow$ (3)) If T is weakly compact, then $T^*X_1^*$ is weakly relatively compact. Since $T^*X_1^*$ is bounded, it follows from Akemann's Theorem [1] that there exists ψ in A_+^* such that $T^*X_1^* \ll_u \psi$. Consequently $T^{**} \ll \psi^{**}$ and therefore $T \ll \psi$.

((3) $\Rightarrow$ (4)) If $T \ll \psi$, then $T^*X_1^* \ll_u \psi$. As explained in the Introduction (paragraph before Theorem 2) this implies that $(T^*X_1^*)^{**} \ll_u \psi^{**}$, i.e. $\{\varphi^{**} \circ T^{**} \mid \varphi \in X_1^*\} \ll_u \psi^{**}$. Consequently $T^{**} \ll \psi^{**}$. Let $\{p_\alpha \mid \alpha \in I\}$ be a set of pairwise orthogonal projections in A^{**}. For each finite subset J of I, put $e_J = \sum_{\alpha\in J} p_\alpha$. Then the net $\{e_J \mid J$ is a finite subset of $I\}$ converges in the $s^*(A^{**}, A^*)$-topology to $\bigvee_{\alpha\in I} p_\alpha = \sum_{\alpha\in I} p_\alpha$. In particular, $e_J \to \sum_{\alpha\in I} p_\alpha$ in the $\eta_{\psi^{**}}$ seminorm and since $T^{**} \ll \psi^{**}$, it follows that $T^{**}(e_J) \to T^{**}(\sum_{\alpha\in I} p_\alpha)$; i.e. T^{**} is completely-additive.

((4) $\Rightarrow$ (1)) Let (p_n) be a decreasing sequence of projections in A^{**} converging in the weak*-topology to zero. We have $\lim_{n\to\infty}\|T^{**}(p_n)\| = 0$; i.e. $\lim_{n\to\infty}\sup_{\varphi\in X_1^*}|\varphi^{**}\circ T^{**}(p_n)| \to 0$ as $n \to \infty$. This implies that $T^*X_1^*$ is weakly relatively compact (by Akemann's Theorem—see [22, Theorem 5.4, p. 149]), and therefore T^* is weakly compact. Consequently, T is weakly compact. □

The following proposition is a basis for various convergence theorems for weakly compact operators on C^*-algebras [5] and was proved in the setup of monotone σ-complete algebras in [6]. Here we show that what is essential is the Grothendieck property of the underlying algebra.

Proposition 6. *Let A be a C^*-algebra with the Grothendieck property and X a normed space. Let (T_n) be a strong operator convergent sequence of weakly compact operators in $B(A, X)$. Then the set*

$$K = \{\varphi \circ T_n \mid \varphi \in X_1^*,\ n \in \mathbb{N}\}$$

is a weakly relatively compact subset of A^*.

Proof. Since each T_n^{**} restricts to a completely-additive vector-valued measure on $P(A^{**})$, result follows from Lemma 3 and [7, Theorem 3.3]. □

Corollary 7 *(Vitali–Hahn–Saks–Nikodým Theorem). Let A be a C^*-algebra with the Grothendieck property and X a normed space. Let (T_n) be a strong operator convergent sequence of weakly compact operators in $B(A, X)$. Then (T_n^{**}) is uniformly completely additive on A^{**}.*

Proof. This follows from Proposition 6 and Akemann's Theorem. □

3. Brooks–Jewett property

We start with an auxiliary lemma.

Lemma 8. *Let X be a Banach space and (φ_n) a weakly* null sequence in X^*. If $\{\varphi_n \mid n \in \mathbb{N}\}$ is weakly relatively compact, then $(\varphi_n)_{n\in\mathbb{N}}$ is weakly null.*

Proof. For a contradiction, suppose that $(\varphi_n)_{n\in\mathbb{N}}$ is not weakly null. This implies that there is positive real number ε, $f \in X^{**}$ and a subsequence (φ_{n_k}) such that $|f(\varphi_{n_k})| > \varepsilon$ for all $k \in \mathbb{N}$. By the Eberlein–Šmulian Theorem, one can pass to a weakly convergent subsequence of (φ_{n_k}). Let $\varphi_0 \in X^*$ be the weak limit of this subsequence. Then $\varphi_0(x) = 0$ for all $x \in X$ and therefore $\varphi_0 = 0$. This contradicts the fact that $|f(\varphi_0)| \geqslant \varepsilon$. □

In the following theorem we characterize the Brooks–Jewett property of a C^*-algebra in terms of measure-theoretic properties of A^{**}.

Theorem 9. *A C^*-algebra A has the Brooks–Jewett property if, and only if, A is Grothendieck and A^{**} has the Vitali–Hahn–Saks property. Consequently, A has the Brooks–Jewett property if, and only if, A is Grothendieck and its bidual is a finite von Neumann algebra.*

Proof. Assume that A has the Brooks–Jewett property and let $K \subset A^*$ be weakly relatively compact. Seeking a contradiction, let $\psi \in K_p$, (φ_n) a sequence in K and (a_n) a sequence in A_1^{**} such that $\eta_\psi(a_n) \to 0$ and $\inf\{|\varphi_n(a_m)| \mid n \in \mathbb{N},\ m \in \mathbb{N}\} > 0$. By the Eberlein–Šmulian Theorem, $(\varphi_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence (φ_{n_k}). From the hypothesis it follows that $\{\varphi_{n_k} \mid k \in \mathbb{N}\}$ is uniformly continuous on A_1 when endowed with the seminorm η_ψ, and therefore, as was already remarked in the introduction, $\{\varphi_{n_k} \mid k \in \mathbb{N}\}$ is uniformly continuous on A_1^{**} with respect to η_ψ. This implies that

$$0 = \inf\{|\varphi_{n_k}(a_m)| \mid k \in \mathbb{N},\ m \in \mathbb{N}\} \geqslant \inf\{|\varphi_n(a_m)| \mid n \in \mathbb{N},\ m \in \mathbb{N}\} > 0,$$

which is a contradiction. This proves that if A has the Brooks–Jewett property then A^{**} has the Vitali–Hahn–Saks property. To show that the Brooks–Jewett property implies the Grothendieck property, suppose that (φ_n) is a weakly* null sequence in A^* and let $\psi = \sum_n \frac{1}{2^n}[\varphi_n]$. Observe that $\{\varphi_n \mid n \in \mathbb{N}\} \ll_p \psi$ and therefore by hypothesis $\{\varphi_n \mid n \in \mathbb{N}\} \ll_u \psi$. This implies that $\{\varphi_n \mid n \in \mathbb{N}\}$ is uniformly continuous on A_1^{**} with respect to η_ψ (i.e. $\{\varphi_n^{**} \mid n \in \mathbb{N}\} \ll_u \psi^{**}$). By Akemann Theorem [1] it implies that $\{\varphi_n \mid n \in \mathbb{N}\}$ is weakly relatively compact and therefore, by Lemma 8, $(\varphi_n)_{n\in\mathbb{N}}$ is weakly null.

For the converse, let (φ_n) be a weakly* convergent sequence in A^* and $\psi \in A_+^*$ such that $\{\varphi_n \mid n \in \mathbb{N}\} \ll_p \psi$. Then $(\varphi_n)_{n\in\mathbb{N}}$ is weakly convergent and $\{\varphi_n^{**} \mid n \in \mathbb{N}\} \ll_p \psi^{**}$. From the Vitali–Hahn–Saks property of A^{**} it follows that $\{\varphi_n^{**} \mid n \in \mathbb{N}\} \ll_u \psi^{**}$ and consequently A has the Brooks–Jewett property. The last statement follows from Theorem 1. □

Combining Theorem 9, Lemma 3, Proposition 6 and [8, Theorems 2.1] one deduces that a C^*-algebra with the Brooks–Jewett property automatically has the vector form of the Brooks–Jewett property.

Theorem 10. *Let A be a C^*-algebra having the Brooks–Jewett property and X a Banach space. Let (T_n) be a strong operator convergent sequence in $B(A, X)$ such that $\{T_n \mid n \in \mathbb{N}\} \ll_p \psi \in A_+^*$. Then $\{T_n \mid n \in \mathbb{N}\} \ll_u \psi$.*

Proof. In view of Theorem 9 and Lemma 3, observe that the sequence (T_n^{**}) is strong operator convergent in $B(A^{**}, X^{**})$. Furthermore, Proposition 5 implies that each T_n^{**} is completely additive on $P(A^{**})$. Since $\{T_n^{**} \mid n \in \mathbb{N}\} \ll_p \psi^{**}$, we can invoke [8, Theorems 2.1] to deduce that $\{T_n^{**} \mid n \in \mathbb{N}\} \ll_u \psi^{**}$. □

The next theorem provides a structural characterization of C^*-algebras with the Brooks–Jewett property; they are precisely those which are Grothendieck and for which the irreducible representations are finite-dimensional. The same characterization was given in a quite different line of research; namely that concerning the Dunford–Pettis property for operator algebras. As a consequence, we obtain that for Grothendieck C^*-algebras, the Brooks–Jewett property and the Dunford–Pettis property are equivalent. So we can see the interplay of seemingly different properties. Let us recall that a Banach space X has the *Dunford–Pettis property* if each weakly compact operator from X to other Banach space is completely continuous in the sense that it carries weakly convergent sequences to norm convergent sequences. Alternatively, X has the Dunford–Pettis property if, and only if, $\varphi_n(x_n) \to 0$ whenever (φ_n) and (x_n) are weakly null sequences in X^* and X, respectively. It was established by Chu, Iochum and Watanabe [9,10] based on the results by Hamana [14] that a C^*-algebra has the Dunford–Pettis property if, and only if, all irreducible representations of this algebra act on a finite-dimensional Hilbert space.

Theorem 11. *Let A be a C^*-algebra. The following conditions are equivalent*:

(1) *A has the Brooks–Jewett property*;
(2) *A is Grothendieck and has the Dunford–Pettis property*;
(3) *A is Grothendieck and every irreducible representation of A is finite-dimensional*;
(4) *A is Grothendieck and A^{**} is a finite type I von Neumann algebra.*

Proof. (4) implies (1) by Theorem 9.

If A^{**} is finite then any irreducible representation of A is finite-dimensional, for otherwise A^{**} would have an infinite direct summand. Hence (1) ⇒ (3) by Theorem 9.

It has been proved in [14, Lemma 5] that the bidual of a C^*-algebra A is finite type I if, and only if, all irreducible representations of A are finite-dimensional. It gives immediately the implication (3) ⇒ (4).

The equivalence of (2) and (3) follows from the deep fact mentioned above that a C^*-algebra has the Dunford–Pettis property if, and only if, every irreducible representation is finite-dimensional. □

Since abelian C^*-algebras have only one-dimensional irreducible representations, Theorem 11 implies that the Grothendieck property and the Brooks–Jewett property coalesce in the commutative case. However, the Brooks–Jewett property is much stronger than the Grothendieck property in general. For instance, every von Neumann factor is Grothendieck but for it to satisfy that every irreducible representation is finite-dimensional, the factor itself must be finite-dimensional.

The next theorem provides a structural description of von Neumann algebras having the Brooks–Jewett property. The proof is based on the equivalence of the following two statements: (a) M^{**} is a finite type I von Neumann algebra, and (b) M is a finite direct sum of finite type I homogeneous algebras. Notwithstanding the fact that this follows from the work of M. Hamana [14] (see also [22, p. 358]) we here prefer to give an alternative simpler proof to aid clarity.

Theorem 12. *A von Neumann algebra M has the Brooks–Jewett property if, and only if, M is a finite sum of finite type I homogeneous algebras.*

Proof. As every von Neumann algebra is Grothendieck we know by Theorem 11 that M has the Brooks–Jewett property if, and only if, M^{**} is of finite type I. So M itself must be of finite type I. Suppose, for a contradiction, that there is a strictly increasing sequence of integers (n_i) and a sequence X_{n_i} of hyperstonean spaces such that M is $*$-isomorphic to $\sum_i C(X_{n_i}) \otimes B(H_{n_i})$. (Here $C(X_{n_i})$ is the algebra of continuous functions on X_{n_i} and H_{n_i} is a Hilbert space of dimension n_i.) First of all we show that $(\sum B(H_{n_i}))^{**}$ is not finite.

To this end, for each $i \in \mathbb{N}$, let $\{\xi^i_j \mid 0 \leqslant j \leqslant n_i - 1\}$ be an orthonormal basis of H_{n_i}. For every $k \in \mathbb{N}$ let $e^i_{k \bmod n_i}$ be the projection in $B(H_{n_i})$ mapping H_{n_i} onto span$\{\xi^i_{k \bmod n_i}\}$ and define $e_k = \sum_i e^i_{k \bmod n_i} \in \sum_i B(H_{n_i})$. The set $\{e_k \mid k \in \mathbb{N}\}$ consists of mutually equivalent projections in $\sum_i B(H_{n_i})$ and $e_k e_{k'}$ is a finite-dimensional projection when $k \neq k'$. Since the algebra $N = \sum_i B(H_{n_i})$ is finite there is a canonical central-valued trace T on N. Let ϱ be a pure state on the centre of N such that $\varrho(T(e_1)) > 0$ and such that ϱ is zero on any finite-dimensional central projection in N. Then $\varphi = \varrho \circ T$ is a tracial state vanishing on all finite-dimensional projections and such that $\varphi(e_1) > 0$. Therefore $\varphi(e_k) = \varphi(e_1) \neq 0$ for every $k \in \mathbb{N}$. Let (π_φ, H_φ) be the GNS representation associated with φ. Let us observe that $\pi_\varphi(e_k) \perp \pi_\varphi(e_{k'})$ whenever $k \neq k'$. Summing it up the set $\{\pi_\varphi(e_k) \mid k \in \mathbb{N}\}$ consists of mutually equivalent nonzero orthogonal projections in $\pi_\varphi(N)$. Consequently, N^{**} is not finite. As there is a $*$-homomorphism mapping M onto N, we see that M has a representation which is not finite and so M^{**} cannot be a finite algebra either. This implies that M^{**} is not finite. □

The content of Theorem 12 is that the von Neumann algebras enjoying the Brooks–Jewett property can be obtained from classical function algebras $C(X)$ on hyperstonean spaces by forming tensor products with matrix algebras and taking finite sums. In [14, Theorem 2] it was shown that such algebras are precisely those isomorphic to abelian C^*-algebras as Banach spaces.

Corollary 13. *A von Neumann algebra enjoys the Brooks–Jewett property if, and only if, it is topologically linearly isomorphic to an abelian C^*-algebra.*

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Appendix A

The foregoing results of the paper reveal that a bona-fide noncommutative analogue for the Brooks–Jewett Theorem is not possible unless one imposes relatively strong restriction on the underlying algebra. In order to obtain Brooks–Jewett type theorems for general C^*-algebras one has to consider a stronger form of absolute continuity. Following [5], $T \in B(A, X)$ is said to be strongly absolutely continuous with respect to $\psi \in A^*_+$ if for each $f \in X^*$ we have that $[f \circ T]$ is absolutely continuous with respect to ψ. The following theorem was proved first in [4] for a sequence of functionals on von Neumann algebras. A far reaching generalization was then obtained in [6] for monotone σ-complete C^*-algebras. We generalize the result further to Grothendieck C^*-algebras.

Theorem 14. *Let A be a Grothendieck C^*-algebra and X a normed space. Let (T_n) be a sequence of weakly compact operators from A to X such that (T_n) converges in the strong operator topology. If each T_n is strongly absolutely continuous with respect to ψ, then $\{T_n \mid n \in \mathbb{N}\}$ is uniformly absolutely continuous with respect to ψ.*

Proof. By Proposition 6 we know that the set $K = \{f \circ T_n \mid f \in X^*_1,\ n \in \mathbb{N}\}$ is weakly relatively compact. Now the results follow from [5, Theorem 5.5.]. □

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