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Radial Symmetry of Self-Similar Solutions for Semilinear Heat Equations

Yūki Naito

Department of Applied Mathematics, Faculty of Engineering, Kobe University, Kobe 657-8501, Japan

and

Takashi Suzuki

The symmetry properties of positive solutions of the equation

$$\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u + u^p = 0 \qquad \text{in } \mathbf{R}^n,$$

where $n \geqslant 2$, $p > (n+2)/n$, was studied. It was proved that u must be radially symmetric about the origin provided $u(x) = o(|x|^{-2/(p-1)})$ as $|x| \to \infty$, and that there exist non-radial solutions u satisfying $\limsup_{|x| \to \infty} |x|^{2/(p-1)} u(x) > 0$. © 2000 Academic Press

1. INTRODUCTION

In this paper we consider the symmetry properties of positive solutions of the equation

$$\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u + u^p = 0 \qquad \text{in } \mathbf{R}^n, \tag{1.1}$$

where $n \geqslant 2$ and $p > 1$. This equation arises in the study of (forward) self-similar solutions of the semilinear heat equation

$$w_t = \Delta w + w^p \qquad \text{in } \mathbf{R}^n \times (0, \infty). \tag{1.2}$$

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It is well known that if $w(x, t)$ satisfies (1.2), then, for $\mu > 0$ the rescaled functions

$$w_\mu(x, t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t)$$

define a one parameter family of solutions to (1.2). A solution w is said to be self-similar, when $w_\mu(x, t) = w(x, t)$ for all $\mu > 0$. It can be easily checked that w is a self-similar solution to (1.2) if and only if w has the form

$$w(x, t) = t^{-1/(p-1)} u(x/\sqrt{t}), \tag{1.3}$$

where u satisfies the elliptic Eq. (1.1). Moreover, if u has spherical symmetry, that is if $u = u(r)$, $r = |x|$, then u satisfies the ordinary differential equation

$$u'' + \left(\frac{n-1}{r} + \frac{r}{2}\right) u' + \frac{1}{p-1} u + u^p = 0, \qquad r > 0. \tag{1.4}$$

Such self-similar solutions are often used to describe the large time behavior of global solutions to the Cauchy problem, see, e.g., [11, 13, 3, 4, 14, 5, and 15], and to show nonuniqueness of solution to (1.2) with zero initial data in a certain functional space, see [12].

First we state the result concerning the symmetry properties of the solution of (1.1).

THEOREM 1.1. *Let* $u \in C^2(\mathbf{R}^n)$ *be a positive solution of* (1.1) *such that*

$$u(x) = o(|x|^{-2/(p-1)}) \qquad as \quad |x| \to \infty. \tag{1.5}$$

Then u must be radially symmetric about the origin.

The proof of Theorem 1.1 is based on the moving planes argument. This technique was developed by Serrin [18] in PDE theory, and extended and generalized by Gidas, Ni, and Nirenberg [9, 10]. We remark that with a change of variables we are still able to prove a radial symmetry result for Eq. (1.1).

Let us consider the problem

$$\begin{cases} u'' + \left(\dfrac{n-1}{r} + \dfrac{r}{2}\right) u' + \dfrac{1}{p-1} u + |u|^{p-1} u = 0, & r > 0, \\ u'(0) = 0 \quad \text{and} \quad u(0) = \alpha \in \mathbf{R}. \end{cases} \tag{1.6}$$

The problem (1.6) has been investigated extensively in [12, 16, 20, and 2]. We denote by $u(r; \alpha)$ the unique solution of (1.6). We recall that $u(r; \alpha)$ has the following properties:

(i) $\lim_{r \to \infty} r^{2/(p-1)} u(r; \alpha) = L(\alpha)$ exists and is finite for every $\alpha \in \mathbf{R}$ (see [12, Theorem 5]);

(ii) if $L(\alpha) = 0$, then there exists a constant $A \neq 0$ such that

$$u(r; \alpha) = A e^{-r^2/4} r^{2/(p-1)-n} \{1 + O(r^{-2})\} \qquad \text{as} \quad r \to \infty$$

(see [16, Theorem 1]);

(iii) if $p \geqslant (n+2)/(n-2)$, then $u(r; \alpha)$ is positive on $[0, \infty)$ and $L(\alpha) > 0$ for every $\alpha > 0$ (see [12, Theorem 5]);

(iv) if $(n+2)/n < p < (n+2)/(n-2)$, then there exists a unique $\alpha > 0$ such that $u(r; \alpha)$ is positive on $[0, \infty)$ and $L(\alpha) = 0$ (see [20, Theorem 1] and [2, Theorem 1.2 and Corollary 1.3]).

By virtue of Theorem 1.1 we obtain the following:

Corollary 1.1. (i) *Assume that* $p \geqslant (n+2)/(n-2)$. *Then there exists no positive solution* u *of* (1.1) *satisfying* (1.5).

(ii) *Assume that* $(n+2)/n < p < (n+2)/(n-2)$. *Then there exists a unique positive solution* $u(x)$ *satisfying* (1.5). *Moreover, the solution* u *is radially symmetric about the origin.*

Remark. The result (i) is differently proven by [3, Proposition 4.3] based on the Pohozaev identity.

Following the notations in [3] and [14], we define

$$L^2(K) = \left\{ u \colon \mathbf{R}^n \to \mathbf{R};\ \int_{\mathbf{R}^n} |u|^2 \, K(x) \, dx < \infty \right\} \qquad \text{and}$$

$$H^1(K) = \left\{ u \colon \mathbf{R}^n \to \mathbf{R};\ \int_{\mathbf{R}^n} (|u|^2 + |\nabla u|^2) \, K(x) \, dx < \infty \right\},$$

where $K(x) = \exp(|x|^2/4)$. Escobedo and Kavian have shown in [3, Proposition 3.5] that if $1 < p < (n+2)/(n-2)$ and if $u \in H^1(K)$ is a solution of (1.1), then $u \in C^2(\mathbf{R}^n)$ and satisfies $u(x) = O(\exp(-|x|^2/8))$ as $|x| \to \infty$. As a consequence of Corollary 1.1, we obtain the following:

Corollary 1.2. *Assume that* $(n+2)/n < p < (n+2)/(n-2)$. *Then the problem*

$$\begin{cases} \Delta u + \dfrac{1}{2} x \cdot \nabla u + \dfrac{1}{p-1} u + u^p = 0 & \text{in } \mathbf{R}^n, \\ u \in H^1(K) \quad \text{and} \quad u > 0 & \text{in } \mathbf{R}^n, \end{cases} \tag{1.7}$$

has a unique solution.

Let us consider the Cauchy problem

$$\begin{cases} w_t = \Delta w + w^p & \text{in } \mathbf{R}^n \times (0, \infty), \\ w(x, 0) = \tau w_0 & \text{in } \mathbf{R}^n, \end{cases} \tag{1.8}$$

where $w_0 \in L^2(K) \cap L^\infty(\mathbf{R}^n)$, $w_0 \geqslant 0$, and $\tau > 0$ is a parameter. We denote by $w(x, t; \tau)$ the unique solutions of (1.8) (see [15]). Combining the result by Kawanago [15, Theorem 1] and Corollary 1.2, we obtain the following, where the asymptotic behavior of $w(\cdot, t; \tau)$ as $t \to \infty$ becomes clearer.

COROLLARY 1.3. *Assume that $(n+2)/n < p < (n+2)/(n-2)$. Then there exists a unique $\tau_0 > 0$ such that the solution $w(x, t; \tau)$ is a global solution if $\tau \in (0, \tau_0]$, and $w(x, t; \tau)$ blows up in finite time if $\tau \in (\tau_0, \infty)$. Moreover, $w(x, t; \tau_0)$ satisfies*

$$\lim_{t \to \infty} \| t^{1/(p-1)} w(\cdot, t; \tau_0) - u_0(\cdot / \sqrt{t}) \|_{L^\infty(\mathbf{R}^n)} = 0,$$

where u_0 is a unique solution of the problem (1.7).

Next we consider the existence of nonradial solutions of (1.1). Let $p > (n+2)/n$ and let $U(r)$ be a positive solution of (1.4) satisfying

$$U'(0) = 0 \qquad \text{and} \qquad \lim_{r \to \infty} r^{2/(p-1)} U(r) > 0. \tag{1.9}$$

The existence of such U is obtained by [12, Theorem 5]. Define $\ell = \ell(U) > 0$ as

$$\ell = \lim_{r \to \infty} r^{2/(p-1)} U(r). \tag{1.10}$$

We investigate the Cauchy problem for Eq. (1.2) with

$$w(x, 0) = w_0 \in L^1_{\text{loc}}(\mathbf{R}^n), \tag{1.11}$$

where

$$0 \leqslant w_0(x) \leqslant \ell\, |x|^{-2/(p-1)}, \qquad w_0 \not\equiv 0, \qquad x \in \mathbf{R}^n \setminus \{0\}. \tag{1.12}$$

Relation (1.11) is taken in the sense of $L^1_{\text{loc}}(\mathbf{R}^n)$, that is,

$$\int_K |w(x, t) - w_0(x)|\, dx \to 0 \qquad \text{as} \quad t \to 0$$

for any compact subset K of $\mathbf{R}^n$. We note that $w_0 \in L^1_{\text{loc}}(\mathbf{R}^n)$ if (1.12) holds with $p > (n+2)/n$.

THEOREM 1.2. *Let $p>(n+2)/n$. Assume that* (1.12) *holds, where ℓ is the constant in* (1.10). *Then there exists a positive solution $w \in C^{2,1}(\mathbf{R}^n \times (0, \infty))$ of* (1.2) *and* (1.11). *Assume, furthermore, that $w_0 \in C(\mathbf{R}^n \setminus \{0\})$, then w satisfies*

$$w(x, t) \to w_0(x) \quad \text{as } t \to 0 \text{ uniformly in } |x| \geqslant r \quad \text{for every } r>0. \tag{1.13}$$

Moreover, w is self-similar if $\mu^{2/(p-1)} w_0(\mu x) = w_0(x)$ for every $\mu>0$.

COROLLARY 1.4. *Let $p>(n+2)/n$. Assume that $A: S^{n-1} \to \mathbf{R}$ is continuous and satisfies*

$$0 \leqslant A(\sigma) \leqslant \ell, \qquad A \not\equiv 0, \qquad \sigma \in S^{n-1}. \tag{1.14}$$

Then there exists a positive self-similar solution $w \in C^{2,1}(\mathbf{R}^n \times (0, \infty))$ of (1.2) *satisfying* (1.11) *and* (1.13) *with $w_0(x) = A(x/|x|)\,|x|^{-2/(p-1)}$.*

Recall that self-similar solutions w to (1.2) have the form (1.3) with u satisfying (1.1). Therefore, $w(\sigma, t) = r^{2/(p-1)} u(r\sigma)$ for $\sigma \in S^{n-1}$, where $r = 1/\sqrt{t}$. Then we obtain the following corollary, which shows that condition (1.5) in Theorem 1.1 is crucial.

COROLLARY 1.5. *Let $p>(n+2)/n$. Assume that $A: S^{n-1} \to \mathbf{R}$ is continuous and satisfies* (1.14). *Then there exists a positive non-radial solution u of* (1.1) *satisfying*

$$r^{2/(p-1)} u(r\sigma) \to A(\sigma) \quad \text{as } r \to \infty \quad \text{uniformly in } \sigma \in S^{n-1}.$$

Remark. (i) If $1<p \leqslant (n+2)/n$, no time global, non-negative, and nontrivial solution exists in (1.2) (see, e.g., [7, 19, and 14]). Therefore, (1.1) admits a positive solution only if $p>(n+2)/n$.

(ii) We find that the solution w of (1.2) and (1.11) obtained in Theorem 1.2 is a minimal solution of the integral equation

$$w(x, t) = \int_{\mathbf{R}^n} \Gamma(x-y: t)\, w_0(y)\, dy + \int_0^t \int_{\mathbf{R}^n} \Gamma(x-y: t-s)[w(y, s)]^p\, dy\, ds,$$

where $\Gamma(x: t) = (4\pi t)^{-n/2}\, e^{-|x|^2/4t}$. See the proof of Theorem 1.2 below.

(iii) Galaktionov and Vazquez [8] studied the Cauchy problem (1.2) and (1.11) with singular initial values for the case $p>n/(n-2)$.

This paper is organized as follows: in Section 2, we treat the symmetry properties of the solutions and prove Theorem 1.1. In Section 3, we state some propositions concerning the properties of solutions to the Cauchy problem with singular initial data, and prove Theorem 1.2. We prove these propositions in Section 4.

2. RADIAL SYMMETRY

In this section we investigate more general equation

$$\Delta u + \tfrac{1}{2} x \cdot \nabla u + ku + f(u) = 0 \qquad \text{in } \mathbf{R}^n, \tag{2.1}$$

where $n \geqslant 2$, k is a nonnegative constant, and $f \in C^1[0, \infty)$. Theorem 1.1 is a consequence of the following result.

THEOREM 2.1. *Assume that*

$$f(s) = O(s^{\sigma}) \quad \text{as } s \to 0 \quad \text{for some } \sigma > 1. \tag{2.2}$$

Let u be a positive solution of (2.1) *such that*

$$u(x) = o(|x|^{-2k}) \qquad \text{as} \quad |x| \to \infty. \tag{2.3}$$

Then u must be radially symmetric about the origin.

We obtain Theorem 2.1 by the following two propositions.

PROPOSITION 2.1. *Assume that* (2.2) *holds. Let u be a positive solution satisfying* (2.3). *Then, for every $m > 0$, $u(x) = o(|x|^{-m})$ as $|x| \to \infty$.*

PROPOSITION 2.2. *Let u be a positive solution such that*

$$u(x) = o(|x|^{-2\alpha}) \quad \text{as } |x| \to \infty \quad \text{for some } \alpha > 0. \tag{2.4}$$

Assume that

$$\alpha > k + \max\{|f'(s)| : 0 \leqslant s \leqslant \|u\|_{L^\infty(\mathbf{R}^n)}\}. \tag{2.5}$$

Then u must be radially symmetric.

To prove Proposition 2.1 we prepare the following lemma.

LEMMA 2.1. *Assume that* (2.2) *holds. Let u be a positive solution of* (2.1) *such that*

$$u(x) = o(|x|^{-\beta}) \quad \text{as } |x| \to \infty \quad \text{for some } \beta > 2k.$$

Then, for every $m > 0$, $u(x) = o(|x|^{-m})$ as $|x| \to \infty$.

Proof. Set $v(x) = |x|^\beta u(x)$. Then $v(x) \to 0$ as $|x| \to \infty$ and v satisfies

$$\Delta v + \frac{1}{2} x \cdot \nabla v - \frac{2\beta}{|x|^2} x \cdot \nabla v - \left(\frac{\beta}{2} - k\right) v + \frac{\beta(\beta + 2 - n)}{|x|^2} v + |x|^\beta f(|x|^{-\beta} v) = 0.$$

Defining

$$Lv \equiv \Delta v + \frac{1}{2} x \cdot \nabla v - \frac{2\beta}{|x|^2} x \cdot \nabla v,$$

we have

$$Lv = \left[\left(\frac{\beta}{2} - k\right) - \frac{\beta(\beta + 2 - n)}{|x|^2} - \frac{f(u)}{u}\right] v.$$

Note that $\beta > k/2$ and $f(s)/s \to 0$ as $s \to 0$ by (2.2). Then there exists a $R_1 > 0$ such that $Lv \geqslant 0$ for $|x| \geqslant R_1$.

Fix $m > 0$ and define $w(r) = r^{-m}$, $r = |x|$. Then it follows that

$$Lw = w_{rr} + \frac{n - 1 - 2\beta}{r} w_r + \frac{1}{2} r w_r = r^{-1} \left[-\frac{m}{2} + \frac{m(m+1) - m(n - 1 - 2\beta)}{r^2}\right].$$

Then there exists a $R_2 > 0$ such that $Lw \leqslant 0$ for $|x| \geqslant R_2$. Let $R_0 = \max\{R_1, R_2\}$. Take $C > 0$ so large that $Cw - v \geqslant 0$ on $|x| = R_0$. Then $Cw - v$ satisfies

$$L(Cw - v) \leqslant 0 \quad \text{for } |x| > R_0 \qquad \text{and} \qquad Cw - v \to 0 \quad \text{as } |x| \to \infty.$$

By the maximum principle, we obtain $Cw \geqslant v$ for $|x| \geqslant R_0$, i.e., $u(x) \leqslant C|x|^{-m-\beta}$ for $|x| \geqslant R_0$. Since $m > 0$ is arbitrary, we obtain the conclusion. ∎

Proof of Proposition 2.1. Set $v(x) = |x|^{2k} u(x)$. Then $v(x) \to 0$ as $|x| \to \infty$ and v satisfies

$$\Delta v + \frac{1}{2} x \cdot \nabla v - \frac{4k}{|x|^2} x \cdot \nabla v + \frac{2k(2k + 2 - n)}{|x|^2} v + |x|^{2k} f(|x|^{-2k} v) = 0.$$

Defining

$$Lv \equiv \Delta v + \frac{1}{2} x \cdot \nabla v - \frac{4k}{|x|^2} x \cdot \nabla v,$$

we have

$$Lv = -\frac{2k(2k+2-n)}{|x|^2} v - |x|^{2k} f(u).$$

We note here that

$$|x|^{2k} f(u) = \frac{f(u)}{u^\sigma} \frac{v^\sigma}{|x|^{2k(\sigma-1)}}.$$

Then, by (2.2), we obtain

$$Lv = o(|x|^{-\delta_0}) \qquad \text{as} \quad |x| \to \infty,$$

where $\delta_0 = \min\{2, 2k(\sigma-1)\}$. Let $w(x) = |x|^{-\delta}$, where $\delta = \delta_0/2$. Then we have

$$Lw = |x|^{-\delta} \left[-\frac{\delta}{2} + \frac{\delta(\delta+1) - \delta(n-1-2k)}{|x|^2} \right] \leqslant -\frac{\delta}{4} |x|^{-\delta}$$

for $|x|$ sufficiently large. Then there exists a $R_0 > 0$ such that $Lw - Lv \leqslant 0$ for $|x| \geqslant R_0$. Take $C \geqslant 1$ so large that $Cw \geqslant v$ on $|x| = R_0$. Then $Cw - v$ satisfies

$$L(Cw - v) \leqslant 0 \quad \text{for } |x| > R_0 \qquad \text{and} \qquad Cw - v \to 0 \quad \text{as } |x| \to \infty.$$

By the maximum principle, we obtain $Cw \geqslant v$ for $|x| \geqslant R_0$, i.e., $v(x) \leqslant C|x|^{-\delta}$ for $|x| \geqslant R_0$. This implies that $u(x) = o(|x|^{-\beta})$ as $|x| \to \infty$ for $\beta = 2k + \delta > 2k$. By Lemma 2.1, we conclude that $u(x) = o(|x|^{-m})$ as $|x| \to \infty$ for every $m > 0$. ∎

Next we prove Proposition 2.2. Let u be a positive solution of (2.1) satisfying (2.4). Define

$$w(x, t) = t^{-\alpha} u(x/\sqrt{t}), \qquad (x, t) \in \mathbf{R}^n \times (0, \infty). \tag{2.6}$$

LEMMA 2.2. (i) *For every* $T > 0$, $w(x, t) \to 0$ *as* $|x| \to \infty$ *uniformly in* $t \in (0, T]$;

(ii) *For every* $\lambda > 0$, $w(x, t) \to 0$ *as* $t \to 0$ *uniformly in* $|x| \geqslant \lambda$.

Proof. From (2.4), for any $\varepsilon > 0$, there exists a $R > 0$ such that

$$|x|^{2\alpha} u(x) < \varepsilon, \qquad |x| \geqslant R. \tag{2.7}$$

By virtue of (2.6) we have

$$|x|^{2\alpha} w(x, t) = |x/\sqrt{t}|^{2\alpha} u(x/\sqrt{t}). \tag{2.8}$$

(i) Fix $T > 0$. From (2.7) and (2.8) it follows that

$$|x|^{2\alpha} w(x, t) < \varepsilon \qquad \text{for} \quad |x| \geqslant RT^{1/2}, \quad t \in (0, T].$$

Since $\varepsilon > 0$ is arbitrary, we have $w(x, t) \to 0$ as $|x| \to \infty$ uniformly in $t \in (0, T]$.

(ii) From (2.7) and (2.8) we obtain

$$\lambda^{2\alpha} w(x, t) \leqslant |x|^{2\alpha} w(x, t) < \varepsilon, \qquad \text{for} \quad |x| \geqslant \lambda, \quad 0 < t < (\lambda/R)^2.$$

Then $w(x, t) \to 0$ as $t \to 0$ uniformly in $|x| \geqslant \lambda$. ∎

For $\lambda \in \mathbf{R}$, we define T_λ and Ω_λ as

$$T_\lambda = \{x = (x_1, ..., x_n) \in \mathbf{R}^n : x_1 = \lambda\} \qquad \text{and} \qquad \Omega_\lambda = \{x \in \mathbf{R}^n : x_1 < \lambda\},$$

respectively. For $x = (x_1, ..., x_n) \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$, let x^λ be the reflection of x with respect to T_λ, i.e., $x^\lambda = (2\lambda - x_1, x_2, ..., x_n)$. It is easy to see that, if $\lambda > 0$,

$$|x^\lambda| > |x| \quad \text{for } x \in \Omega_\lambda \qquad \text{and} \qquad \{x^\lambda : x \in \Omega_\lambda\} = \{x : x_1 > \lambda\} \subset \{x : |x| \geqslant \lambda\}.$$

By Lemma 2.2 we have the following:

LEMMA 2.3. *Let* $\lambda > 0$.

(i) *For every* $T > 0$, $w(x^\lambda, t) \to 0$ *as* $|x| \to \infty$, $x \in \Omega_\lambda$, *uniformly in* $t \in (0, T]$;

(ii) $w(x^\lambda, t) \to 0$ *as* $t \to 0$ *uniformly in* $x \in \Omega_\lambda$.

LEMMA 2.4. *Define* ϕ *as* $\phi(x, t) = w(x, t) - w(x^\lambda, t)$. *Then we have*

$$\phi_t = \Delta\phi + c(x, t)\,\phi \quad \text{in } \Omega_\lambda \times (0, \infty) \qquad \text{and} \qquad \phi = 0 \quad \text{on } T_\lambda \times (0, \infty),$$

where

$$c(x, t) = \left[-(\alpha - k) + \int_0^1 f'(u^\lambda - s(u^\lambda - u))\, ds \right] t^{-1} \tag{2.9}$$

with $u = u(x/\sqrt{t})$ *and* $u^\lambda = u(x^\lambda/\sqrt{t})$.

Proof. From (2.6) we have

$$w_t = \Delta w - (\alpha - k)\, t^{-1} w + t^{-\alpha-1} f(t^\alpha w), \qquad (x, t) \in \mathbf{R}^n \times (0, \infty).$$

Let $w^\lambda(x, t) = w(x^\lambda, t)$. Then w^λ satisfies

$$w_t^\lambda = \Delta w^\lambda - (\alpha - k)\, t^{-1} w^\lambda + t^{-\alpha-1} f(t^\alpha w^\lambda), \qquad (x, t) \in \Omega_\lambda \times (0, \infty).$$

Then we have $\phi_t = \Delta\phi + c\phi$, where c is the function in (2.9). ∎

LEMMA 2.5. *For $\lambda > 0$, we have $w(x, t) \geqslant w(x^\lambda, t)$ for $(x, t) \in \Omega_\lambda \times (0, \infty)$.*

Proof. Let $\phi(x, t) = w(x, t) - w(x^\lambda, t)$. We show that $\phi(x, t) \geqslant 0$ for $(x, t) \in \Omega_\lambda \times (0, \infty)$. Assume to the contrary that there exists a $(x_0, t_0) \in \Omega_\lambda \times (0, \infty)$ such that $\phi(x_0, t_0) < 0$. Take $\varepsilon > 0$ so small that $\phi(x_0, t_0) < -\varepsilon < 0$. By Lemma 2.3(ii), we can take $T_0 \in (0, t_0)$ so that $w(x^\lambda, T_0) < \varepsilon$ for $x \in \Omega_\lambda$. Then it follows that

$$\phi(x, T_0) > -\varepsilon, \qquad x \in \Omega_\lambda. \tag{2.10}$$

Fix $T > t_0$. By Lemma 2.3(i), we can take $R > |x_0|$ so large that $w(x^\lambda, t) < \varepsilon$ for $|x| \geqslant R$, $x \in \Omega_\lambda$, $t \in [T_0, T]$. Then we obtain

$$\phi(x, t) > -\varepsilon, \qquad |x| \geqslant R, \qquad x \in \Omega_\lambda, \qquad t \in [T_0, T]. \tag{2.11}$$

Define $Q = \{x \in \Omega_\lambda : |x| < R\}$. Let Σ be a parabolic boundary of $Q \times (T_0, T)$, i.e.,

$$\Sigma = (Q \times \{T_0\}) \cup (\partial Q \times (T_0, T)).$$

From Lemma 2.4, (2.10), and (2.11) we have

$$\phi_t = \Delta\phi + c\phi \quad \text{in } Q \times (T_0, T) \qquad \text{and} \qquad \phi \geqslant -\varepsilon \quad \text{on } \Sigma.$$

Note that $c(x, t) \leqslant 0$ from (2.5) and (2.9). Define ψ as $\psi(x, t) = \phi(x, t) + \varepsilon$. Then ψ satisfies

$$\psi_t \geqslant \Delta\psi + c\psi \quad \text{in } Q \times (T_0, T) \qquad \text{and} \qquad \psi \geqslant 0 \quad \text{on } \Sigma.$$

By the maximum principle ([17, Chapter 3, Theorem 7]), we have $\psi \geqslant 0$ on $\bar{Q} \times [T_0, T]$, which implies that

$$\phi(x, t) \geqslant -\varepsilon, \qquad (x, t) \in \bar{Q} \times [T_0, T]. \tag{2.12}$$

On the other hand, $(x_0, t_0) \in Q \times (T_0, T)$ and $\phi(x_0, t_0) < -\varepsilon$. This contradicts to (2.12). Hence $\phi(x, t) \geqslant 0$ for $(x, t) \in \Omega_\lambda \times (0, \infty)$. ∎

Proof of Proposition 2.2. By Lemma 2.5, we have $w(x, t) \geqslant w(x^\lambda, t)$ for $\lambda > 0$, $(x, t) \in \Omega_\lambda \times (0, \infty)$. From the continuity of w, we have $w(x, t) \geqslant w(x^0, t)$ for $(x, t) \in \Omega_0 \times (0, \infty)$.

We can repeat the previous arguments for the negative x_1-direction to conclude that $w(x, t) \leqslant w(x^0, t)$ for $(x, t) \in \Omega_0 \times (0, \infty)$. Hence, $w(x, t)$ must be symmetric about the plane $x_1 = 0$. Since the Eq.(2.1) is invariant under the rotation and conclude that $w(x, t)$ is symmetric in every direction. Therefore, u must be radially symmetric about the origin. ∎

3. PROOF OF THEOREM 1.2

We assume, henceforth, that $p > (n+2)/n$. Recall that $U(r)$ is a positive solution of (1.4) satisfying (1.9), and that ℓ is a positive constant defined by (1.10). Let $W = W(x, t)$ be a self-similar solution of (1.2) of the form

$$W(x, t) = t^{-1/(p-1)} U(|x|/\sqrt{t}). \tag{3.1}$$

To prove Theorem 1.2, we consider the Cauchy problem

$$\begin{cases} w_t = \Delta w + g & \text{in } \mathbf{R}^n \times (0, \infty), \\ w(x, 0) = w_0 \in L^1_{\text{loc}}(\mathbf{R}^n). \end{cases} \tag{3.2}$$

The initial condition in (3.2) is taken in the sense of $L^1_{\text{loc}}(\mathbf{R}^n)$. We always assume that w_0 satisfies (1.12), and that g satisfies

$$0 \leqslant g(x, t) \leqslant [W(x, t)]^p \qquad \text{in } \mathbf{R}^n \times (0, \infty). \tag{3.3}$$

We consider the function w satisfying the growth condition:

$$\begin{cases} \text{there exists positive constants } C, \alpha, \text{ and } r \text{ such that} \\ |w(x, t)| \leqslant C e^{\alpha |x|^2} \text{ for all } |x| \geqslant r \text{ and all } t \in [0, \infty). \end{cases} \tag{3.4}$$

We obtain the following propositions. Proofs will be given in the next section.

PROPOSITION 3.1. *The function*

$$\phi(x, t) = \int_0^t \int_{\mathbf{R}^n} \Gamma(x - y : t - s)\, g(y, s)\, dy\, ds$$

is well defined on $\mathbf{R}^n \times (0, \infty)$. *Assume, furthermore, that* g *is continuous on* $\mathbf{R}^n \times (0, \infty)$, *then* $\phi \in C^{1,0}(\mathbf{R}^n \times (0, \infty))$.

PROPOSITION 3.2. *Assume that*

$$\begin{cases} g \in C(\mathbf{R}^n \times (0, \infty)) \text{ and } x \mapsto g(x,t) \text{ is locally Hölder continuous} \\ \text{uniformly in } t \in [t_0, T] \text{ for every } t_0 \text{ and } T \text{ with } 0 < t_0 < T < \infty. \end{cases} \tag{3.5}$$

Then the function $w \in C^{2,1}(\mathbf{R}^n \times (0, \infty))$ *is a solution of the problem* (3.2) *satisfying the growth condition* (3.4) *if and only if*

$$w(x,t) = \int_{\mathbf{R}^n} \Gamma(x-y: t)\, w_0(y)\, dy + \int_0^t \int_{\mathbf{R}^n} \Gamma(x-y: t-s)\, g(y,s)\, dy\, ds. \tag{3.6}$$

Assume, furthermore, that $w_0 \in C(\mathbf{R}^n \backslash \{0\})$, *then* w *defined by* (3.6) *satisfies*

$$w(x,t) \to w_0(x) \quad \text{as } t \to 0 \quad \text{uniformly in } |x| \geqslant r \quad \text{for every } r > 0. \tag{3.7}$$

We show that W defined by (3.1) satisfies

$$W(x,0) = \ell\, |x|^{-2/(p-1)} \qquad \text{in the sense of } L^1_{\text{loc}}(\mathbf{R}^n). \tag{3.8}$$

In fact, we have $|x|^{2/(p-1)}\, W(x,t) = |x/\sqrt{t}\,|^{2/(p-1)}\, U(|x|/\sqrt{t})$. By virtue of (1.10), we obtain $0 \leqslant |x|^{2/(p-1)}\, W(x,t) \leqslant C$ with a constant C and

$$|x|^{2/(p-1)}\, W(x,t) \to \ell \qquad \text{as} \quad t \to 0, \quad |x| \neq 0.$$

Fix a compact set $K \subset \mathbf{R}^n$. By the Lebesgue dominated convergence theorem, we have

$$\int_K |W(x,t) - \ell\, |x|^{-2/(p-1)}|\, dx$$

$$= \int_K |x|^{-2/(p-1)}\, \big||x|^{2/(p-1)}\, W(x,t) - \ell\big|\, dx \to 0 \qquad \text{as} \quad t \to 0.$$

Hence, (3.8) holds. A quick check implies that W is a solution to the problem (3.2) with $g = W^p$ and $w_0(x) = \ell\, |x|^{-2/(p-1)}$. By Propositions 3.1 and 3.2, we have

$$\int_0^t \int_{\mathbf{R}^n} \Gamma(x-y: t-s)[W(y,s)]^p\, dy\, ds < \infty$$

and

$$W(x,t)=\int_{\mathbf{R}^n}\Gamma(x-y:t)[\ell\,|y|^{-2/(p-1)}]\,dy + \int_0^t\int_{\mathbf{R}^n}\Gamma(x-y:t-s)[W(y,s)]^p\,dy\,ds \qquad (3.9)$$

for $(x,t)\in\mathbf{R}^n\times(0,\infty)$.

Proof of Theorem 1.2. Define w_i, $i=1,2,...$, inductively by

$$\begin{cases} w_1(x,t)=\int_{\mathbf{R}^n}\Gamma(x-y:t)\,w_0(y)\,dy, \\ w_{i+1}(x,t)=w_1(x,t)+\int_0^t\int_{\mathbf{R}^n}\Gamma(x-y:t-s)[w_i(y,s)]^p\,dy\,ds, \quad i=1,2,.... \end{cases} \qquad (3.10)$$

By virtue of (1.12) and (3.9) we have

$$0\leqslant w_1(x,t)\leqslant\int_{\mathbf{R}^n}\Gamma(x-y:t)[\ell\,|y|^{-2/(p-1)}]\,dy \leqslant W(x,t) \qquad \text{for} \quad (x,t)\in\mathbf{R}^n\times(0,\infty).$$

Then, by induction, w_i is well defined and satisfies

$$0\leqslant w_1(x,t)\leqslant\cdots\leqslant w_i(x,t)\leqslant w_{i+1}(x,t)\leqslant\cdots\leqslant W(x,t) \qquad \text{in } \mathbf{R}^n\times(0,\infty).$$

Define $w(x,t)=\lim_{i\to\infty}w_i(x,t)$. Letting $i\to\infty$ in (3.10), by the Lebesgue dominated convergence theorem, we obtain

$$w(x,t)=\int_{\mathbf{R}^n}\Gamma(x-y:t)\,w_0(y)\,dy+\int_0^t\int_{\mathbf{R}^n}\Gamma(x-y:t-s)[w(y,s)]^p\,dy\,ds. \qquad (3.11)$$

Observe that w is continuous and satisfies $w\leqslant W$. Then $g=w^p$ is continuous and satisfies (3.3). By Proposition 3.1, we have $w\in C^{1,0}(\mathbf{R}^n\times(0,\infty))$, and then $g=w^p$ satisfies (3.5). Therefore, by Proposition 3.2, $w\in C^{2,1}(\mathbf{R}^n\times(0,\infty))$ and w is a solution of (3.2) with $g=w^p$, that is, a solution of (1.1) and (1.11). Moreover, if $w_0\in C(\mathbf{R}^n\backslash\{0\})$, then w satisfies (1.13). We easily see that w is a minimal solution of the integral equation (3.11).

Assume that $\mu^{2/(p-1)}w_0(\mu x)=w_0(x)$ for all $\mu>0$. Then we have $\mu^{2/(p-1)}w(\mu x,\mu^2 t)=w(x,t)$ for all $\mu>0$ by the uniqueness of the minimal solutions. This implies that w is a self-similar solution. ∎

4. PROOF OF PROPOSITIONS 3.1 AND 3.2

4.1. Define ψ by

$$\psi(x, t) = \int_{\mathbf{R}^n} \Gamma(x - y : t)\, w_0(y)\, dy,$$

where $w_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ satisfies (1.12). Then $\psi \in C^{2,1}(\mathbf{R}^n \times (0, \infty))$ and satisfies

$$\psi_t = \Delta\psi \quad \text{in } \mathbf{R}^n \times (0, \infty) \qquad \text{and} \qquad \psi(\cdot, t) \to w_0(\cdot) \quad \text{in } L^1_{\mathrm{loc}}(\mathbf{R}^n) \quad \text{as } t \to 0. \tag{4.1}$$

(See, e.g., [1, Chapter 5, Theorem 6.1].) We obtain the following:

PROPOSITION 4.1. *For every $r > 0$, there exists a constant $C = C(r) > 0$ such that*

$$0 \leqslant \psi(x, t) \leqslant C \qquad \textit{for} \quad |x| \geqslant r \quad \textit{and} \quad 0 < t < \infty. \tag{4.2}$$

Assume, furthermore, that $w_0 \in C(\mathbf{R}^n \backslash \{0\})$, then ψ satisfies

$$\psi(x, t) \to w_0(x) \quad \textit{as } t \to 0 \quad \textit{uniformly in } |x| \geqslant r \quad \textit{for every } r > 0. \tag{4.3}$$

Proof. Let $r > 0$ be arbitrary but fixed. We note that, by (1.12), there exists a constant $C_1 > 0$ such that

$$0 \leqslant w_0(x) \leqslant C_1 \qquad \text{for} \quad |x| \geqslant r. \tag{4.4}$$

First, we show that there exists a constant $C > 0$ satisfying

$$0 \leqslant \psi(x, t) \leqslant C \qquad \text{for} \quad |x| \geqslant 2r \quad \text{and} \quad 0 < t < \infty. \tag{4.5}$$

In fact, we write

$$\psi(x, t) = \int_{|y| \leqslant r} + \int_{|y| \geqslant r} \Gamma(x - y : t)\, w_0(y)\, dy \equiv I_1 + I_2.$$

Observe that $|x - y| \geqslant r$ for $|x| \geqslant 2r$ and $|y| \leqslant r$. Then, since $w_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$,

$$I_1 \leqslant \Gamma_r \int_{|y| \leqslant r} w_0(y)\, dy < \infty,$$

where $\Gamma_r = \sup\{\Gamma(x-y:t): |x-y| \geqslant r\} = \Gamma(r:r^2/2n)$. From (4.4) we have

$$I_2 \leqslant C_1 \int_{|y| \geqslant r} \Gamma(x-y:t)\, dy \leqslant C_1.$$

Therefore we obtain (4.5). Since $r>0$ is arbitrary, (4.2) holds.

Next we show (4.3). Since $\int_{\mathbf{R}^n} \Gamma(x-y:t)\, dy = 1$, we have

$$\psi(x,t) - w_0(x) = \int_{\mathbf{R}^n} \Gamma(x-y:t)[w_0(y) - w_0(x)]\, dy.$$

For $|x| \geqslant r$, we write

$$\begin{aligned} &|\psi(x,t) - w_0(x)| \\ &\quad\leqslant \int_{\substack{|x-y| \geqslant \rho \\ |y| \geqslant r}} + \int_{\substack{|x-y| \geqslant \rho \\ |y| \leqslant r}} + \int_{|x-y|<\rho} \Gamma(x-y:t)\, |w_0(y) - w_0(x)|\, dy \\ &\quad\equiv I_1 + I_2 + I_3, \end{aligned}$$

where $\rho \in (0,r)$ is arbitrary but be fixed. From (4.4) we have

$$I_1 \leqslant 2C_1 \int_{|x-y| \geqslant \rho} \Gamma(x-y:t)\, dy \to 0 \qquad \text{as} \quad t \to 0.$$

Since $w_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, we have

$$I_2 \leqslant \int_{|y| \leqslant r} (w_0(y) + C_1)\, dy \sup_{|x-y| \geqslant \rho} \Gamma(x-y:t) \to 0 \qquad \text{as} \quad t \to 0.$$

We estimate I_3 as

$$I_3 \leqslant \sup_{|x-y|<\rho} |w_0(y) - w_0(x)| \int_{\mathbf{R}^n} \Gamma(x-y:t)\, dy = \sup_{|x-y|<\rho} |w_0(y) - w_0(x)|.$$

Therefore, for arbitrary $\rho \in (0,r)$,

$$\lim_{t \to 0} \big(\sup_{|x| \geqslant r} |\psi(x,t) - w_0(x)| \big) \leqslant \sup_{\substack{|x| \geqslant r \\ |x-y|<\rho}} |w_0(y) - w_0(x)|.$$

We see that $w_0(x)$ is uniformly continuous in $|x| \geqslant r$ since $w_0(x) \to 0$ as $|x| \to \infty$. Hence, we obtain (4.3). ∎

4.2. In this subsection we show the following:

PROPOSITION 4.2. *Assume that* g *satisfies* (3.3). *Then the function* ϕ *defined by*

$$\phi(x, t) = \int_0^t \int_{\mathbf{R}^n} \Gamma(x - y : t - s)\, g(y, s)\, dy\, ds$$

is well defined on $\mathbf{R}^n \times (0, \infty)$ *and satisfies the following properties*:

(i) $\phi(x, t) \to 0$ *as* $t \to 0$ *uniformly in* $|x| \geqslant r$ *for every* $r > 0$;

(ii) *For every* $r > 0$, *there exists a constant* $C > 0$ *such that*

$$0 \leqslant \phi(x, t) \leqslant C \qquad \textit{for} \quad |x| \geqslant r \quad \textit{and} \quad 0 < t < \infty;$$

(iii) $\phi(\cdot, t) \to 0$ *in* $L^1_{\text{loc}}(\mathbf{R}^n)$ *as* $t \to 0$;

(iv) *Assume that* $g \in C(\mathbf{R}^n \times (0, \infty))$. *Then* $\phi \in C^{1,0}(\mathbf{R}^n \times (0, \infty))$;

(v) *Assume that* (3.5) *holds. Then* $\phi \in C^{2,1}(\mathbf{R}^n \times (0, \infty))$ *and satisfies*

$$\phi_t = \Delta\phi + g \qquad \textit{in } \mathbf{R}^n \times (0, \infty).$$

To prove Proposition 4.2, define J and Φ as

$$\begin{cases} J(x, t : s) = \displaystyle\int_{\mathbf{R}^n} \Gamma(x - y : t - s)[W(y, s)]^p\, dy \\[2ex] \Phi(x, t) = \displaystyle\int_0^t J(x, t : s)\, ds. \end{cases}$$

We note that $\phi(x, t) \leqslant \Phi(x, t)$ by (3.3). We recall that $[W(x, t)]^p = t^{-p/(p-1)}[U(x/\sqrt{t})]^p$. Since $\mu^{2/(p-1)} W(\mu x, \mu^2 t) = W(x, t)$ for all $\mu > 0$, we have

$$\mu^{2/(p-1)} \Phi(\mu x, \mu^2 t) = \Phi(x, t) \qquad \text{for all} \quad \mu > 0 \tag{4.6}$$

by direct calculation. Since $U(r)$ is bounded on $[0, \infty)$, there exists a constant $C_2 > 0$ such that, for $(x, t) \in \mathbf{R}^n \times (0, \infty)$,

$$J(x, t : s) \leqslant C_2 s^{-p/(p-1)}, \qquad 0 < s < t. \tag{4.7}$$

We obtain the following estimates of $J(x, t : s)$.

LEMMA 4.1. *For every* $t_0>0$, *there exist constants* $\sigma\in[0,1)$ *and* $C_3=C_3(t_0)>0$ *such that*

$$J(x,t:s)\leqslant C_3 s^{-\sigma}\qquad \text{for all}\quad (x,t,s)\in\mathbf{R}^n\times[t_0,\infty)\times(0,t_0/2]. \tag{4.8}$$

LEMMA 4.2. *There exist constants* $\sigma\in[0,1)$ *and* $C_4>0$ *such that*

$$J(x,t:s)\leqslant C_4 s^{-\sigma}\qquad \text{for}\quad 0<s<t\leqslant\tfrac{1}{4}\quad\text{and}\quad |x|\geqslant 1. \tag{4.9}$$

To prove Lemmas 4.1 and 4.2, we prepare the following lemma.

LEMMA 4.3. *Let* $p>(n+2)/n$. *Then there exists* $q\geqslant p$ *satisfying*

$$q>\frac{n}{n-2}\qquad\text{and}\qquad \frac{p}{p-1}-\frac{q}{q-1}<1. \tag{4.10}$$

Proof. If $p>n/(n-2)$, then $q=p$ satisfies (4.10). Assume that $(n+2)/n<p\leqslant n/(n-2)$. We see that $(n+2)/n<p$ implies $(n-2)/n>2-p$. Choose $q>0$ as $(n-2)/n>1/q>2-p$. Then we obtain $q>n/(n-2)$ and $pq-2q+1>0$. The latter implies the second of (4.10). ∎

Proof of Lemma 4.1. By Lemma 4.3 we can choose $q\geqslant p$ satisfying (4.10). By virtue of (1.9) there exists a constant $C_5>0$ such that

$$[U(r)]^p\leqslant C_5 r^{-2p/(p-1)}\qquad\text{for}\quad r\geqslant 1.$$

Since $2p/(p-1)$ is decreasing, we obtain

$$[U(r)]^p\leqslant C_5 r^{-2q/(q-1)}\qquad\text{for}\quad r\geqslant 1. \tag{4.11}$$

With no loss of generality, we may assume $t_0<1$. Fix $(x,t,s)\in\mathbf{R}^n\times[t_0,\infty)\times(0,t_0/2]$. Note that $0<\sqrt{s}<1$ for $s\in(0,t_0/2]$. We write

$$\begin{aligned}J(x,t:s)&=s^{-p/(p-1)}\left[\int_{|y|\geqslant 1}+\int_{\sqrt{s}\leqslant|y|\leqslant 1}\right.\\&\qquad\left.+\int_{|y|\leqslant\sqrt{s}}\Gamma(x-y:t-s)[U(|y|/\sqrt{s})]^p\,dy\right]\\&\equiv s^{-p/(p-1)}[I_1+I_2+I_3].\end{aligned}$$

From (4.11) we have

$$I_1\leqslant C_5 s^{q/(q-1)}\int_{|y|\geqslant 1}\Gamma(x-y:t-s)\,dy\leqslant C_5 s^{q/(q-1)}$$

and

$$I_2 \leqslant C_5 s^{q/(q-1)} \int_{\sqrt{s} \leqslant |y| \leqslant 1} \Gamma(x-y : t-s)\, |y|^{-2q/(q-1)}\, dy.$$

Since $\Gamma(x, t) \leqslant (4\pi t)^{-n/2}$ and $t - s \geqslant t_0/2$, we obtain

$$I_2 \leqslant C_5 (2\pi t_0)^{-n/2}\, s^{q/(q-1)} \int_{|y| \leqslant 1} |y|^{-2q/(q-1)}\, dy.$$

By virtue of $q > n/(n-2)$, the integral on the right-hand side is convergent. To estimate I_3, we perform the change of variable $z = y/\sqrt{s}$ to obtain

$$\begin{aligned} I_3 &\leqslant (2\pi t_0)^{-n/2} \int_{|y| \leqslant \sqrt{s}} [U(|y|/\sqrt{s})]^p\, dy \\ &\leqslant (2\pi t_0)^{-n/2}\, s^{n/2} \int_{|z| \leqslant 1} [U(|z|)]^p\, dz. \end{aligned}$$

Therefore, we obtain

$$J(x, t : s) \leqslant C[s^{-p/(p-1)+q/(q-1)} + s^{-p/(p-1)+n/2}] \qquad \text{for some} \quad C > 0.$$

Note that $p > (n+2)/n$ implies $p/(p-1) - n/2 < 1$. Hence, we obtain (4.8) with

$$\sigma = \min\left\{\frac{p}{p-1} - \frac{q}{q-1}, \frac{p}{p-1} - \frac{n}{2}\right\} < 1.$$

This completes the proof. ∎

Proof of Lemma 4.2. By the same argument in the proof of Lemma 4.1, we obtain (4.11). Let $|x| \geqslant 1$ and let $0 < s < t \leqslant 1/4$. Note that $\sqrt{s} < \sqrt{t} \leqslant 1/2$. We write

$$\begin{aligned} J(x, t : s) &= s^{-p/(p-1)} \left[\int_{|y| \geqslant 1/2} + \int_{\sqrt{s} \leqslant |y| \leqslant 1/2} \right. \\ &\qquad \left. + \int_{|y| \leqslant \sqrt{s}} \Gamma(x-y : t-s)[U(|y|/\sqrt{s})]^p\, dy \right] \\ &\equiv s^{-p/(p-1)}[I_1 + I_2 + I_3]. \end{aligned}$$

From (4.11) we have

$$I_1 \leqslant C_5 s^{q/(q-1)} \int_{|y| \geqslant 1/2} \Gamma(x-y : t-s)\, |y|^{-2q/(q-1)}\, dy \leqslant 2^{2q/(q-1)} C_5 s^{-q/(q-1)}$$

and

$$I_2 \leqslant C_5 s^{q/(q-1)} \int_{\sqrt{s} \leqslant |y| \leqslant 1/2} \Gamma(x-y : t-s)\, |y|^{-2q/(q-1)}\, dy.$$

Observe that $|x-y| \geqslant 1/2$ for $|x| \geqslant 1$ and $|y| \leqslant 1/2$. Then we obtain

$$I_2 \leqslant C_5 \Gamma_{1/2} s^{-q/(q-1)} \int_{|y| \leqslant 1/2} |y|^{-2q/(q-1)}\, dy,$$

where $\Gamma_{1/2} = \sup_{t>0} \{\Gamma(x-y : t) \colon |x-y| \geqslant 1/2\} = \Gamma(1/2 : n/8)$. Since $q > n/(n-2)$, the integral on the right-hand side is convergent. To estimate I_3, we perform the change of variable $z = y/\sqrt{s}$ to obtain

$$I_3 \leqslant \Gamma_{1/2} \int_{|y| \leqslant \sqrt{s}} [U(|y|/\sqrt{s})]^p\, dy \leqslant \Gamma_{1/2} s^{n/2} \int_{|z| \leqslant 1} [U(|z|)]^p\, dz.$$

Therefore, by the same argument of the proof of Lemma 4.1, we obtain (4.9). ∎

LEMMA 4.4. *The function Φ is well defined and continuous on $\mathbf{R}^n \times (0, \infty)$. Moreover, for every $t_0 > 0$, there exists a constant $C = C(t_0) > 0$ such that*

$$0 \leqslant \Phi(x, t) \leqslant C \qquad \textit{for} \quad x \in \mathbf{R}^n \quad \textit{and} \quad t \geqslant t_0. \tag{4.12}$$

Proof. For $t \geqslant t_0 > 0$, we write

$$\Phi(x, t) = \int_0^{t_0/2} J(x, t : s)\, ds + \int_{t_0/2}^{t} J(x, t : s)\, ds \equiv I_1 + I_2.$$

From (4.8), we have $I_1 \leqslant (1-\sigma)^{-1}\, 2^{\sigma-1} C_3 t_0^{1-\sigma}$. By (4.7) it follows that

$$I_2 \leqslant \int_{t_0/2}^{\infty} J(x, t : s)\, ds \leqslant (p-1)\, 2^{1/(p-1)} C_2 t_0^{-1/(p-1)}.$$

Thus we obtain (4.12). Since $t_0 > 0$ is arbitrarily, Φ is well defined on $\mathbf{R}^n \times (0, \infty)$. ∎

LEMMA 4.5. (i) *$\Phi(x, t) \to 0$ as $t \to 0$ uniformly in $|x| \geqslant r$ for every $r > 0$;*

(ii) *There exists a constant $C > 0$ such that*

$$|x|^{2/(p-1)} \Phi(x, t) \leqslant C \qquad \text{for} \quad x \in \mathbf{R}^n \setminus \{0\}, \quad 0 < t < \infty;$$

(iii) *$\Phi(\cdot, t) \to 0$ in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ as $t \to 0$.*

Proof. First, we show that

$$\Phi(x, t) \to 0 \quad \text{as } t \to 0 \quad \text{uniformly on } |x| \geqslant 1, \tag{4.13}$$

and that there exists a constant $C_6 > 0$ satisfying

$$\Phi(x, t) \leqslant C_6 \qquad \text{for} \quad |x| \geqslant 1, \quad 0 < t < \infty. \tag{4.14}$$

In fact, by Lemma 4.2 we have

$$\Phi(x, t) \leqslant \frac{C_4}{1-\sigma} t^{1-\sigma} \qquad \text{for} \quad 0 < t \leqslant \frac{1}{4} \quad \text{and} \quad |x| \geqslant 1.$$

This implies (4.13). From (4.12), we obtain (4.14).

By virtue of (4.6), we find that (4.13) implies (i). Moreover,

$$|\mu x|^{2/(p-1)} \Phi(\mu x, \mu^2 t) = |x|^{2/(p-1)} \Phi(x, t) \qquad \text{for all} \quad \mu > 0.$$

Let $|x| = 1$. Then, by (4.14), we obtain

$$|\mu x|^{2/(p-1)} \Phi(\mu x, \mu^2 t) \leqslant C_6 \qquad \text{for all} \quad \mu > 0, \quad |x| = 1, \quad 0 < t < \infty.$$

This implies (ii). Fix a compact set $K \subset \mathbf{R}^n$. By the Lebesgue dominated convergence theorem, we observe that

$$\int_K \Phi(x, t)\, dx = \int_K |x|^{-2/(p-1)} \left| |x|^{2/(p-1)} \Phi(x, t) \right| dx \to 0 \qquad \text{as} \quad t \to 0,$$

which implies (iii). ∎

Proof of Proposition 4.2. Since $\phi \leqslant \Phi$, the function ϕ is well defined on $\mathbf{R}^n \times (0, \infty)$ and satisfies (i)–(iii) by Lemmas 4.4 and 4.5(i)–(iii). To show (iv) and (v), fix t_0 and T with $0 < t_0 < T$. By the Fubini theorem and the property of the heat kernel Γ, we obtain

$$\phi(x, t) = \int_{\mathbf{R}^n} \Gamma(x - y : t - t_0)\, \phi(y, t_0)\, dy + \int_{t_0}^{t} \int_{\mathbf{R}^n} \Gamma(x - y; t - s)\, g(y, s)\, dy\, ds$$

for $(x, t) \in \mathbf{R}^n \times [t_0, T]$. If g is continuous, then $\phi \in C^{1,0}(\mathbf{R}^n \times [t_0, T])$, and if (3.5) holds, then $\phi \in C^{2,1}(\mathbf{R}^n \times [t_0, T])$ and satisfies $\phi_t = \Delta \phi + g$ in

$\mathbf{R}^n \times [t_0, T]$. (See, e.g., [6, Chapter 1, Theorem 9].) Since t_0 and T are arbitrarily, (iv) and (v) hold. ∎

4.3. Observe that Proposition 3.1 is included by Proposition 4.2. We prove Proposition 3.2 by employing Propositions 4.1 and 4.2.

Proof of Proposition 3.2. Assume that w is defined by (3.6), that is, $w = \psi + \phi$. Then, $w \in C^{2,1}(\mathbf{R}^n \times (0, \infty))$ and is a solution to the problem of (3.2) by (4.1) and Proposition 4.2(iii) and (v). By (4.2) and Proposition 4.2(ii), w satisfies the growth condition (3.4).

Conversely, assume that w is a solution of (3.2) satisfying (3.4). Define $\tilde{w}$ as

$$\tilde{w}(x, t) = \int_{\mathbf{R}^n} \Gamma(x - y : t)\, w_0(y)\, dy + \int_0^t \int_{\mathbf{R}^n} \Gamma(x - y : t - s)\, g(y, s)\, dy\, ds.$$

Then, by the argument above, $\tilde{w}$ is a solution to the problem (3.2) satisfying (3.4). Let $v = w - \tilde{w}$. Then v satisfies the growth condition (3.4) and is a solution of the problem

$$v_t = \Delta v \quad \text{in } \mathbf{R}^n \times (0, \infty) \qquad \text{and} \qquad v(x, 0) = 0 \quad \text{in the sense of } L^1_{\text{loc}}(\mathbf{R}^n).$$

By the uniqueness theorem ([1, Chapter 5, Theorem 6.1]), we have $v \equiv 0$, that is, $w \equiv \tilde{w}$. This implies (3.6).

Moreover, if $w_0 \in C(\mathbf{R}^n \backslash \{0\})$, then, by (4.3) and Proposition 4.2(i), we obtain (3.7). ∎

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