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Topological Fluid Dynamics: Theory and Applications

On the long time behavior of fluid flows

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Abstract

This work is devoted to the study of the long time asymptotics of solutions of the Euler equations in a bounded 2-dimensional domain. Experiments and numerical simulations indicate the presence of an attracting set in the space of incompressible velocity fields. In this work this attractor is described, and its attracting property is established in an extended dynamics where the time is replaced by the 'long time' taking values in the Alexandroff line. The attracting property in the usual sense remains a conjecture.

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Keywords: Euler equations, vorticity mixing, Alexandroff line, pseudoevolution

1. Introduction

Consider the flow of ideal incompressible fluid in a bounded domain $M \subset \mathbf{R}^2$ or on a compact 2D Riemannian manifold M, possibly with boundary. The flow is described by the Euler equations

$$\frac{\partial u}{\partial t} + (u, \nabla)u + \nabla p = 0\,, \quad \nabla \cdot u = 0. \tag{1.1}$$

The boundary condition is

$$u_n|_{\partial M} = 0, \tag{1.2}$$

and the initial condition is

$$u(x, 0) = u_0(x). \tag{1.3}$$

It is known [2] that a unique solution for this problem exists for infinite time provided the initial velocity u_0 is regular enough (say, if $u_0 \in H^s$, $s > 2$). So, the natural question is, what can be said about the asymptotic behaviour of the solution $u(x, t)$ for $t \to \infty$? In this problem, intuition is a poor guide, and we should turn to physical and computer experiments. The results are striking and counter-intuitive.

In an experiment performed by Maasen et al.[5], motion of water in a shallow tank was studied. The initial motion was produced by dragging a grid made of thin rods through the water layer, several cm. deep, in a 1m–square container.

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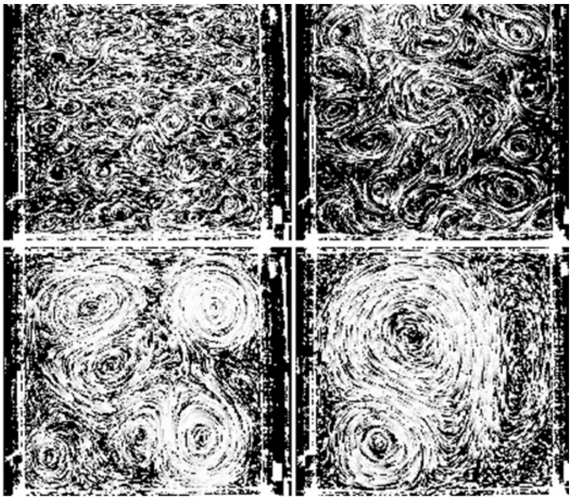

Fig. 1. Grid generated flow in a shallow tank at four instants (Maassen et al. [5]; reproduced with the authors' permission)

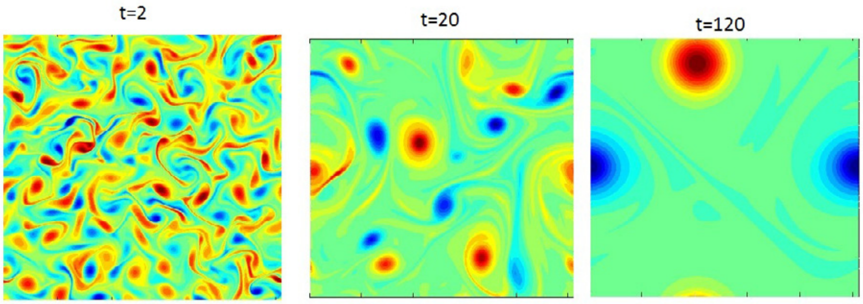

Fig. 2. Vorticity field at different instants (numerical simulation)

The flow field at four instants is shown in figure 1. The first picture is taken 10s after the motion was initiated; the flow looks like a collection of several hundred small vortices of either sign. In the course of the flow small vortices merged and formed larger ones which merged again and produced even larger vortices. The result of this inverse cascade process is shown in the last picture taken 50m later, where we see just two large vortices filling the whole container.

Similar results are seen in numerical simulation (figure 2). Such simulations have been performed since the 1980s (Segre & Kida [14], and references therein). In our simulation, the Navier-Stokes equations on a 2D torus were solved using the standard pseudo-spectral method combined with 4th-order Runge-Kutta discretisation. We used a basis of $2^{10} \times 2^{10}$ harmonics; the viscosity $\nu = 10^{-6}$ was enough for numerical stability, and reduced the energy loss to less than 1%. The initial vorticity $\omega_0(x)$ was chosen in the form of a sum of plane waves with random directions, amplitudes and phases with wavelength $\lambda \sim 0.05$.

We see again that small vortices merge and form bigger ones, and finally we see just two big vortices of opposite signs slowly moving around. Similar result can be seen for any initial velocity (see, for example, the work of Segre & Kida [14] where a rich variety of initial conditions is studied). This is a robust phenomenon, and it requires explanation from first principles.

2. The statistical approach

The problem of long-time behaviour of flows of ideal fluid has been approached by a group of theories named 'statistical hydrodynamics' (SHD) (Onsager [11], Miller, Weichman &Cross [6], Robert [13], Robert & Sommeria [12], and others). In these theories, the fluid is approximated by a finite-dimensional, or even discrete system. It may be a system of point vortices (Onsager), or a Galerkin approximation of the Euler equations (Robert), or permutations of small cells carrying vorticity (Miller, Robert, Sommeria). In all these theories the infinite-dimensional phase space of the fluid (denote it $\mathcal{H}$) is replaced by a finite-dimensional, or even finite, space $\mathcal{H}_N$. Then an approximate conservative dynamics is defined in this space (preserving energy E and phase volume, and possibly some other quantities). For a given initial velocity u_0, let $\mathcal{H}_0$ be the set of admissible states having the same energy and other integrals as u_0. Then it is possible to define for the finite-dimensional system a microcanonical ensemble, i.e. an invariant measure μ_0 on $\mathcal{H}_0$, and then to study the resulting probability distribution in the space of the flows. In the typical cases, the microcanonical measure concentrates around a single velocity field $u_*(x)$ which depends only on the energy and vorticity distribution of the initial velocity $u_0(x)$ (this is an instance of 'large deviation theory').

The ergodic hypothesis then implies that the finite-dimensional system spends most of the time in a domain of the phase space where the velocity field is close to u_*. Then, by a leap of logic, it is claimed that the original system, i.e. the flow described by the Euler equations, also has the property that the trajectory starting at u_0 spends most of the time near u_*. This is an illegal transposition of two limit passages, $t \to \infty$ and $N \to \infty$ where N is the dimension of the phase space of the approximating system. Such transposition is admissible for intrinsically discrete systems like a gas of elastic spheres (where the ergodicity is proved [1]). However, it is dubious for a continuous system like a fluid. Any of the above-mentioned approximations are accurate for some time $|t| < t_N$, but soon after this time the approximate solution $u_N(t)$ and the exact solution $u(t)$ become completely unrelated. In particular, small cells M_i are distorted by the flow and eventually for $t > t_*$ their images have a linear size of order unity. After this time, the approximation of the flow by permutations of the cells M_i is completely senseless. Moreover, SHD predicts that the flow $u(t)$ asymptotically approaches, as $t \to \infty$, some steady solution $\tilde{u}$ which is completely determined by the energy E and the vorticity distribution function $\sigma(\omega)$ of the initial velocity u_0. This prediction contradicts the experiments and computer simulations which show that the flow $u(t)$ can asymptotically approach various non-stationary regimes which are different for different initial velocities with the same energy and vorticity distribution.

The failure of SHD to predict the long-time behaviour of the flow has a deep reason. The fluid (regarded as a continuous system) cannot be in a state of statistical equilibrium. In fact, this system is extremely far from equilibrium, and never approaches it. There are various evidences of the absence of equilibrium for the fluid, such as the existence of Liapunov functions [17] and wandering domains [10] in the phase space. This problem will be discussed in more detail elsewhere.

3. The mixing theory

Our approach is based on the mixing property of the flow that makes SHD so unrealistic. The flow g_t distorts the vorticity ω and, as $t \to \infty$, effectively mixes it. This mixing is, in general, irreversible. It is natural to conjecture that eventually the vorticity is maximally mixed, so that any further mixing is prohibited by some conservation law. This possibility was studied in [15], and here are some details. First we have to give a rigorous definition of mixing.

Definition 3.1 *Consider a class of linear operators in $L^2(M)$ having the form $Kf(x) = \int_M K(x,y)f(y)dy$ where the kernel $K(x,y)$ satisfies the following conditions:*

$$(i) K(x,y) \geq 0; \quad (ii) \int\limits_M K(x,y)dx \equiv 1; \quad (iii) \int\limits_M K(x,y)dy \equiv 1. \tag{3.1}$$

Such operators are called **mixing***, or bi-stochastic operators. The set of all mixing operators is denoted by $\mathcal{K}$.*

Examples: (1) $K(x,y) = \delta(y - g^{-1}(x))$ where g is any element of the group $\mathbf{D}$ of area-preserving diffeomorphisms of M (i.e. $Kf(x) = f(g^{-1}(x))$);
(2) $K(x,y) \equiv 1$ (this means that the operator K is a complete mixing, i.e. $Kf(x) \equiv \int_M f(x)dx$).

$\mathcal{K}$ is a convex, weakly compact semigroup of contractions in $L^2(M)$. Thus, it defines a partial order relation $\prec$ in L^2:

Definition 3.2 *Let $f, g \in L^2(M)$; we say that $f \prec g$ if there exists $K \in \mathcal{K}$ such that $f = Kg$. We say that $f \sim g$ if $f \prec g$ and $g \prec f$.*

Let us denote by V^s the space of vector fields $u \in H^s$ such that $\nabla \cdot u = 0$, and $u_n|_{\partial M} = 0$.

Definition 3.3 *Let $u, v \in V^s$; we say that $u \prec v$ if* $\text{curl}\, u \prec \text{curl}\, v$. *We say that $u \sim v$ if* $\text{curl}\, u \sim \text{curl}\, v$.

For any $u \in V^1$ we define its energy $E(u) = \frac{1}{2}||u||^2_{L^2}$. Given a vector field $u_0 \in V^1$, we define

$$\Omega_{u_0} = \{u \mid u \prec u_0, \text{ and } E(u) = E(u_0)\} \tag{3.2}$$

The set $\Omega_{u_0} \subset V^1$ inherits the partial order relation $\prec$. An element $v \in \Omega_{u_0}$ is called *minimal* if for any $w \in \Omega_{u_0}$, $w \prec v$ implies $v \sim w$. The existence of minimal elements in Ω_{u_0} for any $u_0 \in V^1$ is proved in [15]; the proof is based on the Zorn Lemma.

The property of $v \in V^1$ to be a minimal element of Ω_{u_0} for some u_0 does not depend on u_0 because it is equivalent to the fact that v is a minimal element of Ω_v. In this case we call v a *minimal flow*. The set of all minimal flows is denoted by $\mathcal{M}$. The significance of minimal elements is expressed in the following

Theorem 3.1 *(i) Any minimal flow $v \in V^1$ is a stationary solution of the Euler equations.*

(ii) There are three classes of minimal flows denoted by $\mathcal{M}_+$, $\mathcal{M}_-$, and $\mathcal{M}_0$. The class $\mathcal{M}_+$ ($\mathcal{M}_-$) consists of the fields u such that if $v \prec u$ than $E(v) \le E(u)$ ($E(v) \ge E(u)$), and if $E(v) = E(u)$ than $v \sim u$. The class $\mathcal{M}_0$ consists of the fields $w(x)$ such that $\text{curl}\, u = \text{const}$. *The classes $\mathcal{M}_+$ and $\mathcal{M}_-$ are called respectively* **energy-excessive** *and* **energy-deficients** *minimal flows.*

For any $v \in V^1$ we can define a set Γ_v of vector fields which are 'equirotated' with v, namely

$$\Gamma_v = \{w \mid \text{there exists } \xi \in \mathbf{D} \text{ such that } \text{curl}\, w = \text{curl}\, v \circ \xi\}. \tag{3.3}$$

The set Γ_v is a (generally non-smooth) manifold in V^1. Any stationary flow v is a critical point of the energy E on Γ_v. A stationary flow v is called **Arnold stable** if v is a local maximum or a local minimum of E on Γ_v. If $u(x)$ is Arnold stable, then the streamfunction $\psi(x)$ satisfies a relation of the form $\psi = F(\Delta\psi)$, and the quadratic form

$$\mathcal{H}_\psi(\varphi) = \frac{1}{2}\int_M \left[-\delta\psi \cdot \delta\omega + \frac{\nabla\psi}{\nabla\Delta\psi}(\delta\omega)^2\right] dx, \quad \varphi|_{\partial M} = 0, \tag{3.4}$$

where $\delta\omega = \nabla\omega \wedge \nabla\varphi$, $\delta\psi = \Delta^{-1}\delta\omega$ is either positive-definite (for energy-excessive flows), or negative-definite (for energy-deficient flows).

Theorem 3.2 *(i) If $v \in \mathcal{M}_+$ ($v \in \mathcal{M}_-$) then v is a point of global maximum (global minimum) of the energy E on Γ_v; if $v \in \mathcal{M}_0$ then $\Gamma_v = \{v\}$.*

(ii) If $v \in \mathcal{M}_-$ then v is Arnold stable, i.e. v is a point of a strict global minimum of E on Γ_v.

(iii) If $v \in \mathcal{M}_+$ then the quadratic form $\mathcal{H}_\psi$ is nonpositive, and its null-space, as well as the set $\{w \in \Gamma_v \mid E(w) = E(v)\}$ is at most finite-dimensional.

It is quite natural to conjecture that in a 'generic' flow (i.e. a solution $u(x,t)$ of the Euler equations), the mixing of vorticity continues until any further mixing becomes impossible because of energy conservation (i.e. any further mixing would change the energy). In other words, the solution $u(x,t)$ tends to some $w \in \mathcal{M}$ as $t \to \infty$. However, this statement contradicts numerical simulations. Accurate simulations show that starting from any initial velocity u_0, the flow approaches, after a short 'turbulent period', some stationary, time-periodic, or quasi-periodic flow $u_1(x,t)$ (figure 3).

Close observation of the flows $u_1(x,t)$ obtained by the numerical simulation reveals the following properties of such flows.

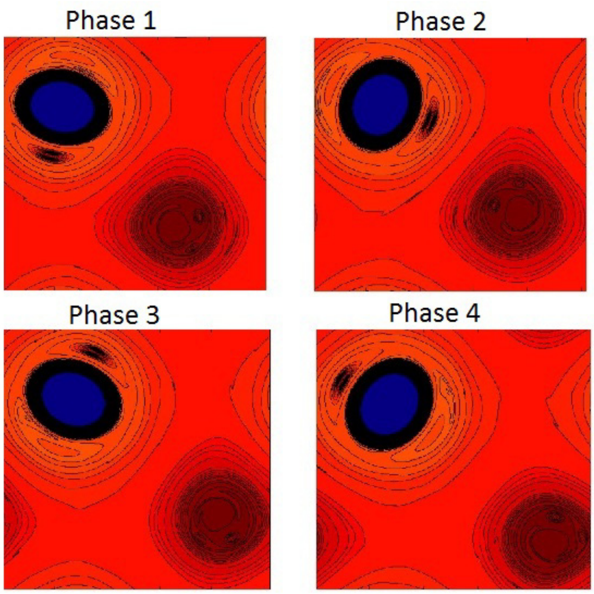

Fig. 3. Four phases of a nearly periodic solution

(1) Consider the set $S_c(t) = \{x \mid \omega_1(x,t) < c\}$ where $\omega_1 = \operatorname{curl} u_1$. This is an open set having a finite or countable number of components $S_{c,i}(t)$. Each component is a time-dependent domain such that its boundary $\Gamma_{c,i}(t)$ is a union of a finite or countable number of closed curves $\Gamma_{c,i,j}(t)$. For all c, i, j the family $\Gamma_{c,i,j}(t),\quad t \in \mathbf{R}$, is compact in the space of close curves (in the sense that their lengths are bounded uniformly for all t). We express this property by naming the $S_{c,i}(t)$ *non-mixing domains*.

(2) Every domain $S_{c,i}(t)$ depends on t quasi-periodically; this means, for example, that for any function $\varphi(x) \in C^\infty(M)$, the function $f_c^{(i)}(t) = \int_{S_{c,i}(t)} \varphi(x)dx$ is quasi-periodic.

(3) The relative configuration of different domains $S_{c,i}(t)$ can be complicated; they form 'islands', 'archipelagos', 'lakes', they can look like 'satellites' of larger 'islands', which in their turn are satellites of larger islands within some lake, and this hierarchy can have an indefinite number of levels. For example, in the flow shown in figure 3, we see two large islands; the left one has two large satellites of different signs so that each of them has two lakes inside, while the right island has at least two lakes inside. For the actual inviscid fluid the number of embedded non-mixing domains can be infinite, and the flow becomes transcendentally complex.

This regime is stable in the following sense: any small change of the velocity field u gives rise to a new 'turbulent period' of intensive mixing of vorticity. However, at the end of this mixing, the flow again becomes quasi-periodic and close to u_1.

Thus, the numerical experiments show that there exists an invariant and attracting set in V^1 consisting of stationary, periodic, and quasi-periodic flows. In the next section we give its characterisation and establish its attraction property.

4. Generalized minimal flows and pseudo-evolution

For any $u \in V^1$ let $\mathcal{O}(u)$ be the orbit of u under the action of the Euler equations, i.e. $\mathcal{O}(u) = S_{\mathbf{R}_+}$ where S_t is the 1-parameter group of transformations in V^1 generated by the Euler equations. Consider the set $\mathcal{C}(u)$, the closure of $\mathcal{O}(u)$ in V^0.

Definition 4.1 *A vector field $u \in V^1$ is called a* **generalized minimal flow** *(GMF) if for any $v \in \mathcal{CO}(u_0)$, $||\operatorname{curl} v||_{L^2} = ||\operatorname{curl} u||_{L^2}$. The set of all GMF is denoted by $\mathcal{N}$.*

Any stationary, time-periodic or quasi-periodic solution (in particular, any minimal flow) is a GMF.

Conjecture 4.1 *The set $\mathcal{N}$ is a global attractor for the Euler equations in V^1.*

In fact, this conjecture is true if we change the meaning of the word 'attractor' by some modification of dynamics. First of all, we describe a method of constructing the GMFs starting from an arbitrary vector field $u_0 \in V^1$ with $a_0 = ||\text{curl}\, u_0||_{L^2}$. If $u_0 \notin \mathcal{N}$, then there exists a field $u_1 \in \mathcal{C}(\mathcal{O}(u_1))$ such that $a_1 = ||\text{curl}\, u_1||_{L^2} < a_0 = ||\text{curl}\, u_0||_{L^2}$. Otherwise, if for any $v \in \mathcal{C}(\mathcal{O}(u_0))$, $||\text{curl}\, v||_{L^2} = ||\text{curl}\, u_0||_{L^2}$, we define $u_1 = u_0$. If $u_1 \notin \mathcal{N}$, we find $u_2 \in \mathcal{C}(\mathcal{O}(u_1))$ such that $a_2 = ||\text{curl}\, u_2||_{L^2} < a_1$; otherwise we define $u_2 = u_1$. Continuing this process, we define for every $n \in \mathbf{N}$ a field u_n and a positive number $a_n = ||\text{curl}\, u_n||_{L^2}$.

The sequence $\{u_n\}$ is compact in V^0 (because $||u_n||_{V^1} = a_n \leq a_0$). Consider the limit points of the sequence $\{u_n\}$ in V^0. If v is such a limit point then $||\text{curl}\, v||_{V^0} \leq \liminf_{n<\omega} a_n$ (by the semicontinuity of $||\cdot||_{V^1}$ in $||\cdot||_{V^0}$). If among the limit points there exists some v_* such that $||\text{curl}\, v_*||_{L^2} < \liminf_{n<\omega_0} a_n$ then we define $u_{\omega_0} = v_*$, $a_{\omega_0} = ||\text{curl}\, u_{\omega_0}||_{L^2}$. Otherwise we define $u_{\omega_0} = v$, $a_{\omega_0} = ||\text{curl}\, u_{\omega_0}||_{L^2}$ where v is any limit point of $\{u_n \mid n < \omega_0\}$. Then we continue the above construction for $n = \omega + 1, \omega + 2, \ldots$; after it is done for all $n = \omega + k,\ \ k \in \mathbf{N}$, we define $u_{2\omega}$ as a $\liminf$ point of $\{u_n \mid n < 2\omega\}$.

Continuing this way, we define u_n for all countable ordinals n by the following rules:

(i) If u_n is already defined, and $u_n \notin \mathcal{N}$, then u_{n+1} is a limit point of $\mathcal{O}(u_n)$ such that $a_{n+1} = ||\text{curl}\, u_{n+1}||_{L^2} < a_n$; otherwise $u_{n+1} = u_n$, $a_{n+1} = a_n$.

(ii) Suppose n is a limit ordinal, i.e. there is no k such that $n = k + 1$. Then consider all the limit points v of the sequence $\{u_k \mid k < n\}$. If there is among them some v_* such that $||\text{curl}\, v_*||_{L^2}|| < \liminf_{k<n} a_k$ then we define $u_n = v_*$, $a_n = ||\text{curl}\, u_n||_{L^2}$; otherwise define $u_n = v,\ \ a_n = ||\text{curl}\, u_n||_{L^2}$ where v is arbitrary limit point.

The principle of transfinite recursion says that by this method we construct the sequences $\{u_n\}, \{a_n\}$ for all countable ordinals $n < \omega_1$ where ω_1 is the least uncountable ordinal. Thus, we have constructed a non-increasing sequence of positive numbers a_n where n runs over all the countable ordinals.

The following lemma is certainly well known; however we give its proof because of its central role in this work.

Lemma 4.1 *For any non-increasing sequence $\{a_n\}$ of positive numbers defined for all $n < \omega_1$ there exists $n_0 < \omega_1$ such that $a_n = a_{n_0}$ for all $n \geq n_0$.*

In other words, the sequence $\{a_n\}$ stabilizes after some (countable) index n_0.

Proof. Let A be the set of indices $n < \omega_1$ such that $a_n < a_{n+1}$. This set is countable. In fact, for every $n < \omega_1$, if $a_n < a_{n+1}$, choose a rational number r_n such that $a_n < r_n < a_{n+1}$. The numbers r_n are different for different n because the sequence $\{a_n\}$ is non-increasing. So, the set of chosen rational numbers is not more than countable, and therefore the set of n such that $a_n > a_{n+1}$ is also not more than countable.

Similarly, let B be the set of all countable ordinals k such that $a_k > \sup_{p<k} a_p$. By the same reason, the set B is also countable.

Any countable set of countable ordinals $n_1, n_2, \ldots$ is bounded from above by some countable ordinal m. In fact, if we identify each ordinal k with the set of ordinals $s_k = \{p \mid p \leq k\}$, then $s = \bigcup_i s_{n_i}$ is countable, while $\{n < \omega_1\}$ is **uncountable**. Hence, there exists $n_0 < \omega_1$ such that $A \bigcup B < n_0$. By the definition of A and B, $u_n = u_{n_0}$ for all $n > n_0$.

Q. E. D.

Thus, u_{n_0} is a GMF; otherwise by our construction, $a_{n_0+1} < a_{n_0}$.

5. Long time ontology

The above construction may look like a formal existence proof of some class of flows. If so, we don't need it at all, because there exist, say, stationary flows which are already GMFs. Actually, the meaning of the construction is different. We regard it as a short description of a sort of generalized dynamics. Let us describe it in more detail.

Let us first give a definition of an important space [18] which is named 'Long', or 'Alexandroff line' AL and extends the time axis.

Definition 5.1 *The space AL is the direct product of the smallest uncountable ordinal ω_1 and the semisegment $[0, 1)$ endowed with the lexicographic order and the order topology. If $\tau = (n, x) \in$ AL, we use the notation $\tau = n + x$.*

In other words, elements of AL are the pairs $\tau = (n, x)$ such that $n < \omega_0$ is a countable ordinal, and $0 \leq x < 1$. If $\tau_1 = (n_1, x_1),\ \tau_2 = (n_2, x_2)$, we say that $\tau_1 < \tau_2$ if $n_1 < n_2$, or, in the case $n_1 = n_2$, if $x_1 < x_2$. A point $\tau_{n+1} = (n + 1, 0)$ is a limit point of the points $\tau_n = (n, x)$ (when $x \to 1$). If k is a limit ordinal (i.e. k has no predecessors), then $\tau_k = (k, 0)$ is a limit point of the elements $\tau = (n, x)$ for $n < k$. So, we can think of AL as a result of filling the gaps between all ordinals n and $n + 1$ by an interval $(n, n + 1)$, and attaching, for every limit ordinal k, of everything which is less than k, to k. The space AL includes the real line $\mathbf{R}$ as its initial part, but it is 'much longer', and has radically different properties. In particular,

Lemma 5.1 *For any monotone decreasing function $f(\tau)$, $\tau \in$ AL, there is a $\tau_0 \in$ AL such that for all $\tau \geq \tau_0$, $f(\tau) = f(\tau_0)$.*

This lemma follows immediately from Lemma 2.1.

Consider now any 'process' which is described by some function $x(t)$ satisfying some equation $dx/dt = f(x)$ where $x \in X$, the phase space of the process. Suppose there exists a parameter $h(x)$ characterizing the 'degree of degeneration' of the state (it can be energy, entropy, temperature, or some more refined quantity). Suppose that $h(t)$ is monotonely non-increasing along any trajectory (i.e. h is a Liapunov function). Suppose further that if $h(x(t))$ is constant for all $t \geq t_0$, then the system has reached the 'bottom', i.e. it is in a most degenerate state (note that such a state need not be stationary, i.e. $x(t)$ may be non-constant).

If the time is the 'short time', i.e. the time axis is the ordinary number line $\mathbf{R}$, then some processes do not reach the bottom: they simply don't have enough time for it. Now suppose that the process can be extended to the 'long time' described by the time parameter $\tau \in$ AL, so that the function h remains the Liapunov function in this long time too (in particular, this means that as the index n crosses a limit ordinal k, the value of $h(x(n + \kappa))$ does not jump up). Then by Lemma 3.1 such a process will definitely reach the bottom by some moment $\tau_0 \in$ AL.

Let us call **evolution** the process $x(t)$ considered in the usual short time $\mathbf{R}$, and **pseudo-evolution** the extension of the same process in the long time AL.

In the problem considered in this work we use the functional $h(u) = ||\text{curl } u||_{L^2}$ which is constant along $\mathcal{O}(u)$. So, if there exists some $v \in \mathcal{C}(\mathcal{O}(u))$ such that $h(v) < h(u)$, and the process continues for $\tau \geq \omega_0$, there should be some infinitesimal jump at $\tau = \omega_0$ from $\mathcal{O}(u)$ to v. It is natural to name such a jump **clinamen**, the term used by Epicurus and Lucretius [4] for infinitesimal unpredictable swerves of atoms in the Universe which make determinism impossible. Note that the concept of **clinamina** (plural of **clinamen**) has nothing to do with the issue of stability of stationary flows; if $u(x)$ is a stationary flow, however unstable, our constructions yields the same flow even for the long time AL. It also has nothing to do with the concept of the nonstandard analysis (NSA) with its infinitesimals, because we remain in the domain of standard objects (velocities, vorticities, etc.), and only the time variable belongs to the extended real line AL; however, this extension is principally different from the one used in the NSA (in particular, elements like $\omega_0,\ \omega_1$, etc. don't exist in the NSA).

So, in our extended dynamics in the long time with possible **clinamina** the statement that the set $\mathcal{N}$ of GMFs is a global attractor is straightforward. However, it is natural to ask, whether the set $\mathcal{N}$ is an attractor in the usual sense, namely in our short time $\mathbf{R}$ and without any clinamina (or with just one corresponding to the passage $t \to \infty$ in the usual sense). This problem is much harder, and even simple concrete questions are still waiting for an answer (see below).

6. Generalized minimal flows and Landau damping

There is one more question which is crucial for the above theory. We must prove that the set $\mathcal{N}$ of generalized minimal flows is nontrivial, i.e. it does not coincide with the whole velocity space V (this set is certainly nonempty, for there exist plenty stationary flows which all are GMFs). So, we have to find a vector field $u \in V^1$ such that for some limit element $v \in \mathcal{C}(\mathcal{O}(u))$, $||\text{curl } v||_{L^2} < ||\text{curl } u||_{L^2}$. Such a field can look like a small and smooth perturbation of

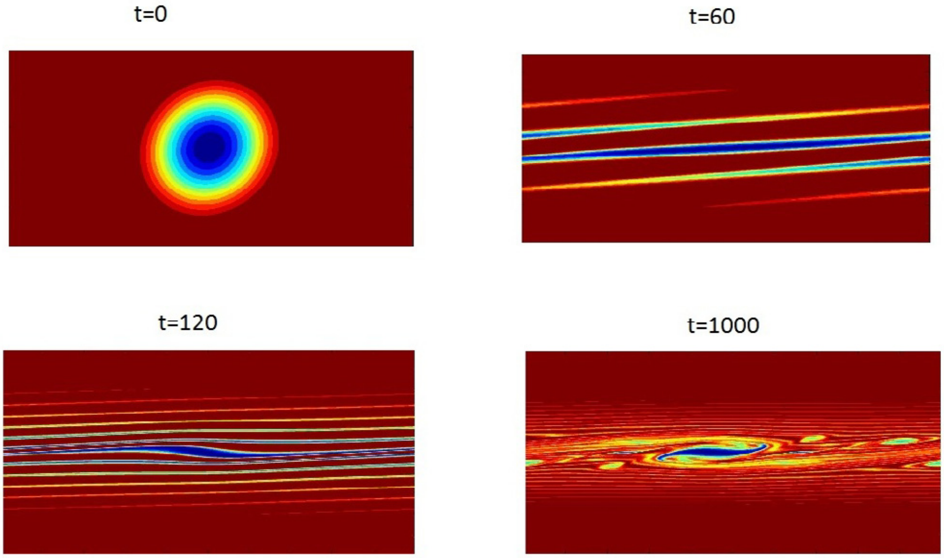

Fig. 4. Perturbation of the Couette flow at different moments

some Arnold-stable stationary solution w_0, i.e. $u = w_0 + \varepsilon w_1$. Consider the vorticity of this flow: $\omega = \omega_0 + \varepsilon\omega_1$. Then the vorticity of the perturbed solution $v(x,t)$, curl $v(x,t) = \omega_0(x) + \varepsilon\omega_1(x,t)$. The linearized equation is

$$\frac{\partial\omega_1}{\partial t} + (w_0, \nabla)\omega_1 + (\text{curl}^{-1}\omega_1, \nabla)\omega_0 = 0. \tag{6.1}$$

For the Arnold stable flows w_0 the last term in this equation is subordinate to the second one (and if $\text{curl}\, w_0 = 0$ it is absent altogether). So, the main effect described by the linearised equation is the passive transport of the vorticity perturbation ω_1 by the unperturbed flow w_0. As a result, the perturbation ω_1 becomes an oscillating function as $t \to \infty$, with characteristic frequency growing proportional to t. It follows that the amplitude of the perturbation w_1 itself decreases as t^{-1} as $t \to \infty$.

Similar behaviour occurs in the Vlasov-Poisson equation (VPE) where small and smooth initial perturbation of the probability density $f(x,p)$ becomes oscillatory, and tends weakly to zero as $t \to \infty$. This phenomenon is called Landau damping. The effect itself was discovered by Landau for the linearized VPE. The effect for the full nonlinear equation and finite smooth perturbations was proved recently by Mouhot & Villani [8], and required considerable effort and advanced analytical techniques.

The perturbation problem for parallel flows of the ideal incompressible fluid looks superficially similar to the perturbation problem for the 1D VPE, especially if the basic flow has a linear profile (Couette flow of an ideal incompressible fluid). However, the self-interaction term in this case is much stronger than a similar term for the VPE, and this makes the analogue of Landau damping quite unlikely. To see what happens here, consider the following numerical experiment. The basic flow was the Couette flow in the strip $-1/4 \le y \le 1/4$ periodic with period 1 along the x-axis, and the slip condition on the sides $y = -1/4$ and $y = 1/4$. The velocity was $w_0(x,y) = (y,0)$, and the vorticity $\omega_0 = 1$. The initial vorticity perturbation has a form of a round blob of amplitude 0.1. The vorticity perturbation is twisted by the basic flow, and becomes quite oscillatory in the y-direction. However, the self-action of the perturbation results in the 'overthrowing' of this structure, and the further development results in a complex structure very different from that resulting from Landau damping (fig. 4). This is analogous to the 'echo' in the VPE which was the main difficulty in the proof of Landau damping [8].

The naive hope is, that if ε is small enough, the echo will not occur, and the flow will eventually tend to some parallel flow close to the basic flow w_0. However, this is quite unlikely, because the vorticity perturbation ω_1 is a much more 'active' scalar than the perturbation f_1 of the vorticity distribution function $f_0(p)$ in the case of the VPE.

However, we can look for **some** solution ω_1 for which the Landau damping holds true. This is quite a different problem, and we hope that the following conjecture is true.

Conjecture 6.1 *Suppose* $u_0 = (U_0(y), 0)$ *is a parallel flow in the periodic strip (or on the 2-d torus). Then there exists a smooth perturbation* $u_1(x, y)$ *such that if* $u(x, y, t)$ *is a solution of the Euler equations with* $u(x, y, 0) = u_0(x, y) + u_1(x, y)$*, then (i)* $u(x, y, t) \to u_*(x, y) = V(y), 0)$*, and (ii)* $\operatorname{curl} u(x, y, t) \to \operatorname{curl} u_*(x, y)$ *as* $t \to \infty$.

There is another possible way to show that the set $\mathcal{N}$ is nontrivial. It has been proved recently ([16], [9], [3]) that for any stationary solution $u(x)$ of the Euler equations the level lines of vorticity $\omega = \operatorname{curl} u$ are analytic curves. We conjecture that this property holds true for periodic and quasiperiodic solutions as well:

Conjecture 6.2 *Suppose* $u(x, t)$ *is a time-periodic or quasi-periodic solution of the Euler equations. Then the lines* $\omega(x, t) = \text{const}$ *are analytic curves for any moment* t.

Our final conjecture relates to the generic GMF $u \in \mathcal{N}$:

Conjecture 6.3 *Any vector field* $u(x) \in \mathcal{N}$ *has the property that all the lines* $\omega = \operatorname{curl} u = \text{const}$ *are analytic curves.*

If the last conjecture is true than it is easy to produce a vector field $u \in V^1$ such that $u \notin \mathcal{N}$. Take any u such that all the lines $\omega = \operatorname{curl} u = \text{const}$ are **not** analytic.

7. Conclusions

1. The 2D ideal incompressible fluid is a system extremely far from statistical equilibrium. The formalism of equilibrium statistical mechanics cannot be applied to it, even if it produces a meaningful result. A more comprehensive study of the non-equilibrium aspect of the fluid will be published elsewhere.

2. The evolution of a flow is extremely complicated, and includes an infinite number of events. The temporal structure of the set of events during a long time is approximated by an infinite time interval. For example, if the events occur at the moments $t_1, t_2, \ldots$ forming an arithmetic progression, this sequence of events can be approximated by the real axis $\mathbf{R}$. However, the event sequence can have a more complicated structure, and may be modelled on different ordered structures including the Alexandroff line.This creates a new paradigm of dynamics.

3. It is proved that in the hydrodynamics with the extended time modelled by the Alexandroff line with infinitesimal swerves, or clinamina, there exists an attracting set $\mathcal{N}$ of generalized minimal flows. The stronger statement that $\mathcal{N}$ is an attractor for the same system in the usual (short) time $\mathbf{R}$ remains open.

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