1. INTRODUCTION

In [2], Greville gives a representation for the \((1, 2, 3, 4)-\)inverse\(^8\) \(M^\dagger\) of a complex matrix \(M\) partitioned as \(M = [A : C]\), where \(C\) is a single column. In [1], Cline obtains representations for \(M^\dagger\) where \(C\) may have more than one column. However, the latter representations are somewhat complicated. Moreover, it is often the case in applications of generalized inverses that one need only obtain a \((1)\)- or \((1, 2)\)-inverse. Thus there is a need to have simple representations for \((1)\)- and \((1, 2)\)-inverses of partitioned matrices.

In this paper we show how to derive general forms for \((1)\)-inverses of partitioned matrices and then we obtain some simple representations for \((1)\)- and \((1, 2)\)-inverses of partitioned matrices. We derive an algorithm for obtaining \((1, 2)\)-inverses similar to the one for obtaining the \((1, 2, 3, 4)\)-inverse given in [2]. Finally, representations for \((1)\)- and \((1, 2)\)-inverses of block triangular matrices are derived.

2. THE DERIVATION

Let \(A\) be an \(m \times p\) matrix and \(C\) be an \(m \times q\) matrix. Write

\[
M = [A : C].
\]

By definition, the matrix

\[
M^- = \begin{bmatrix} X \\ Z \end{bmatrix}
\]

\(^8\) By an \((i, j, k)\)-inverse of a complex matrix \(A\), we mean a matrix \(X\) satisfying the \(i\)th, \(j\)th, and \(k\)th equations of the four Penrose equations: (1) \(AXA = A\), (2) \(XAX = X\), (3) \((AX)^* = AX\), (4) \((XA)^* =XA\).
is a (1)-inverse for $M$ if and only if
\[ AXA = A - CZA \]  
(2.1)
and
\[ AXC = C - CZC. \]  
(2.2)

Let
\[ V = (I - AA^-)C, \]
where $(\cdot)^-$ denotes any (1)-inverse. Applying $I - AA^-$ to the left of (2.1) and (2.2) yields
\[ VZA = 0 \]
(2.3)
and
\[ VZC = V, \]
(2.4)
respectively. Because (1)-inverses always exist, the system defined by (2.1) and (2.2) is consistent, and hence (2.3) and (2.4) always possess at least one common solution.

Suppose $Z_0$ is any solution of (2.3) and (2.4). Then
\[ AXA = A - CZ_0A \]  
(2.1')
and
\[ AXC = C - CZ_0C \]  
(2.2')
are each consistent and, moreover, they possess a common solution.

To see this, observe that
\[ VZ_0A = 0 \text{ if and only if } CZ_0A = AA^-CZ_0A \]
and
\[ VZ_0C = V \text{ if and only if } AA^-C = AA^-CZ_0C = C - CZ_0C. \]

From Penrose's solvability theorem [5], (2.1') is consistent if and only if
\[ AA^- (A - CZ_0A)A^-A = A - CZ_0A, \]
or, equivalently,
\[ CZ_0A = AA^-CZ_0A. \]  
(2.5)
Likewise, (2.2') is consistent if and only if

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or, equivalently,

$$AA^{-}(C - CZ_{0}C)C^{-}C = C - CZ_{0}C.$$ 

Thus (2.1') and (2.2') are each consistent equations.

To see that (2.1') and (2.2') always possess a common solution, note that the general solution of (2.1') can be written in the form

$$X = A^{-} - A^{-}CZ_{0}AA^{-} + W - A^{-}AWAA^{-}$$ (2.6)

because, from (2.5), $A^{-} - A^{-}CZ_{0}AA^{-}$ is a particular solution of (2.1') and because $W - A^{-}AWAA^{-}$ is always the general solution for the homogeneous equation $AXA = 0$. (For complete details see [6].)

By using (2.5), we see that the expression given in (2.6) will satisfy (2.2') if and only if

$$AWV = C - CZ_{0}C + AA^{-}CZ_{0}AA^{-} - AA^{-}C$$

$$= V - CZ_{0}V.$$ (2.7)

Using Penrose's solvability theorem again, we know there exists a $W$ which satisfies (2.7) if and only if

$$AA^{-}(V - CZ_{0}V)V^{-}V = V - CZ_{0}V,$$ (2.8)

or, equivalently (since $AA^{-}V = 0$),

$$VZ_{0}V = V.$$ 

However, for any $Z_{0}$ satisfying (2.3) and (2.4), it is always true that

$$VZ_{0}V = VZ_{0}C - VZ_{0}AA^{-}C = V.$$ 

Thus, for each $Z_{0}$ satisfying (2.3) and (2.4), there exists a $W$ so that the expression in (2.6) is a solution of (2.1') and (2.2').

We summarize our results in the following.

**Theorem 2.1.** For each common solution $Z_{0}$ of

$$\text{VZA} = 0 \quad \text{and} \quad \text{VZC} = V,$$ (2.9)

there always exists a common solution $X_{0}$ of

$$AXA = A - CZ_{0}A \quad \text{and} \quad AXC = C - CZ_{0}C$$ (2.10)

and hence

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is a (1)-inverse for \([A : C]\). Moreover, it is clear that every (1)-inverse of 
\([A : C]\) may be obtained in this manner, i.e., by first solving the system

(2.9) and then solving the system (2.10).

3. General Forms for \([A : C]^{-}\)

Using the results of Section 2, it is easy to derive a general form for
(1)-inverses of matrices partitioned as \([A : C]\).

The general solution of \(VZA = 0\) is of the form

\[ Z = H - V^{-1}VHA^{-1}, \]  

so that \(VZC = V\) becomes

\[ VH = V. \]  

Let \(E = I - AA^{-1}\) so that \(V = EC\). The general solution of (3.2) is
of the form

\[ H = V^{-1}E + G - V^{-1}VGVV^{-1}. \]  

because \(V^{-1}E\) is a particular solution and \(G - V^{-1}VGVV^{-1}\) is the general
solution of the homogeneous equation. (For any \(G, G - V^{-1}VGVV^{-1}\) is clearly a solution of \(VHV = 0\) and moreover, if \(VH_0V = 0\), we can
write \(H_0 = H_0 - V^{-1}VH_0VV^{-1}\). Using (3.3) in (3.1), we see the general
solution of system (2.9) can be written in the form

\[ Z = V^{-1}E + G - V^{-1}VGVV^{-1} - V^{-1}VGA^{-1}, \]  

where \(G\) is an arbitrary conformable matrix.

The general solution of the system (2.10) can be written in the form
(2.6), where \(W\) represents the general solution of (2.7) and \(Z_0\) is given by
(3.4). Using (3.4) in (2.7), we get

\[ AWV = V - CV - CV + CV - VGV. \]  

By using the fact that \(AA^{-1}C = C - V\), it is easy to see that a particular
solution of (3.5) is

\[ -A^{-1}CV - A^{-1}CGVV^{-1} + A^{-1}CV - VGVV^{-1}. \]

The general solution of the homogeneous equation \(AWV = 0\) can be
written in the form

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so that we can write the general solution of (3.5) as
\[ W = -A^{-1}CV-E - A^{-1}CGVV-E + A^{-1}CV-VGVV-E + H - A^{-1}AHVV-E. \] (3.6)

Now, using (3.4) and (3.6) in (2.6), we obtain the general solution of (2.10) as
\[ X = A^{-1} - A^{-1}CGAA^{-1} + A^{-1}CV-VGAA^{-1} - A^{-1}CV-E - A^{-1}CGVV-E + A^{-1}CV-VGVV-E + H - A^{-1}AHVV-E - A^{-1}AHAA^{-1}. \] (3.7)

Thus the general form for (1)-inverses of $[A : C]$ can be written as
\[ [A : C]^{-1} = \begin{bmatrix} X \\ Z \end{bmatrix}, \]
where $X$ and $Z$ are given by (3.7) and (3.4), respectively. By direct computation, it can be verified that
\[ X = A^{-1}(I - CV-E) + A^{-1}(CV-VG - CG - AH)(VV-E + AA^{-1}) + H \]
and
\[ Z = VEV - V-VG(VV-E + AA^{-1}) + G, \]
where $H$ and $G$ are arbitrary conformable matrices.

If we take $H = 0$ and $G = 0$, we obtain that
\[ \begin{bmatrix} A^{-1}(I - CV-E) \\ V^{-E} \end{bmatrix} \]
is a particular (1)-inverse for $[A : C]$. Moreover, if we interpret $(\cdot)^{-1}$ as meaning any $(1, 2)$-inverse, then it is easily verified that this matrix is also a $(1, 2)$-inverse for $[A : C]$. We summarize our results in

**Theorem 3.1.** The general form for (1)-inverses of matrices partitioned as $[A : C]$ may be written as
\[
\begin{bmatrix}
A^{-1}(I - CV-E) \\
V^{-E}
\end{bmatrix} + \begin{bmatrix}
A^{-1}(CV-VG - CG - AH) \\
-V-VG
\end{bmatrix} [VV-E + AA^{-1}] + \begin{bmatrix}
H \\
G
\end{bmatrix},
\]

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where $G$ and $H$ are arbitrary conformable matrices and $E = I - AA^- \text{ and } V = EC$. The matrix

\[
\begin{bmatrix}
A^-(I - CV-E) \\
V-E
\end{bmatrix}
\]

is always a $(1)$- or $(1, 2)$-inverse for $[A: C]$, depending on whether $(\cdot)^-$ is interpreted as any $(1)$- or $(1, 2)$-inverse.

Once one has the expression (3.8) at his disposal, it is possible to give a simple proof to show that (3.8) is indeed the general form for $(1)$-inverses of $[A: C]$ without relying on Theorem 2.1. However, this proof gives no indication where (3.8) comes from and does not give one a true feeling for the general form for $[A: C]^\sim$. The proof follows.

**Proof.** The fact that (3.8) is a $(1)$-inverse of $[A: C]$ for every conformable $G$ and $H$ follows from direct computation by using $AA^- C = C - V$.

If

\[
\begin{bmatrix}
X_0 \\
Z_0
\end{bmatrix}
\]

is a $(1)$-inverse for $[A: C]$, then

\[
\begin{bmatrix}
X_0 \\
Z_0
\end{bmatrix}
\]

can be written in the form (3.8) by taking $H = X_0 - A^- + A^- CV^- E$ and $G = Z_0 - V^- E$. To see this, let $G$ and $H$ be as given above so that (3.8) becomes

\[
\begin{bmatrix}
\bar{X} \\
\bar{Z}
\end{bmatrix},
\]

where

\[
\begin{align*}
\bar{X} &= A^-CVZ_0VV^-E - A^-CVVV^-E - A^-CZ_0VV^-E \\
&\quad + A^-CVVV^-E - A^-AX_0VV^-E + A^-AA^-VV^-E \\
&\quad - A^-AA^-CVVV^-E + A^-CVVZ_0AA^- - A^-CZ_0AA^- \\
&\quad - A^-AX_0AA^- + A^-AA^- + X_0.
\end{align*}
\]

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Using \( VZ_0V = V \), \( VZ_0A = 0 \), \( AA^{-}V = 0 \), and \( AX_0A = A - CZ_0A \), we have

\[
\hat{X} = -A^{-}CZ_0VV^{-}E + A^{-}CVVV^{-}E - A^{-}AX_0VV^{-}E + X_0
\]

Starting with \( CZ_0C = C - AX_0C \) and subtracting \( CZ_0AA^{-}C \) from both sides, we see that \( CZ_0V = V - AX_0V \). Thus \( \hat{X} = X_0 \).

Also,

\[
\hat{Z} = -V-VZ_0VV^{-}E - V-VZ_0AA^{-} + V-VV^{-}E + Z_0
\]

because \( VZ_0V = V \) and \( VZ_0A = 0 \).

In a manner similar to that above, one can prove the following.

**Theorem 3.2.** Let \( \hat{E} = I - CC^{-} \) and \( \hat{V} = \hat{E}A \). Then the general form for (1)-inverses for \( [A : C] \) can be written as

\[
\begin{bmatrix}
\hat{V}-\hat{E} \\
C-(I - \hat{A}\hat{V}-\hat{E})
\end{bmatrix}
+ \begin{bmatrix}
-\hat{V}-\hat{V}H \\
C-(A\hat{V}-\hat{V}H - AH - CG)
\end{bmatrix} \begin{bmatrix}
\hat{V}\hat{V}-\hat{E} + CC^{-}
\end{bmatrix} + \begin{bmatrix}
H \\
G
\end{bmatrix},
\]

where \( H \) and \( G \) are arbitrary conformable matrices.

**Corollary 3.1.** The matrix

\[
\begin{bmatrix}
\hat{V}-\hat{E} \\
C-(I - \hat{A}\hat{V}-\hat{E})
\end{bmatrix}
\]

is a (1)- or (1, 2)-inverse for \( [A : C] \), depending on whether \((-\cdot)\) is interpreted as a (1)- or (1, 2)-inverse.

**Proof:** This can be verified by direct computation.

4. **General Forms for \([A : R]\)**

General forms for (1)-inverses of matrices partitioned as

\[
M = \begin{bmatrix}
A \\
R
\end{bmatrix}
\]
can be derived in a manner analogous to that used in Section 3. The results are given in the following.

**Theorem 4.1.** Let $A$ and $R$ be $m \times p$ and $n \times p$ complex matrices, respectively. Let $F = I - A^*A$, $\hat{F} = I - R^*R$, $U = RF$, and $\hat{U} = A\hat{F}$. Then the general form for $(1)$-inverses of

\[
\begin{bmatrix}
A \\
R
\end{bmatrix}
\]

\[
\text{can be written as}
\[
[(I - FU - R)A^* : FU^*] + \left[ (FU - U + A^*A) \left[ (GUU - R - GR - HA)A^* : GUU - H \right] + H : G \right]
\]

\[
\text{or as}
\]

\[
[\hat{F}U^* : (I - \hat{F}U - A)R^*] + \left[ \hat{F}U^* - (I - \hat{F}U - A) \right. \left. (R - R) \left[ - H\hat{U}U^* : (H\hat{U}U^* - HA - GR)R^* \right] + H : G \right],
\]

where $H$ and $G$ represent arbitrary conformable matrices.

**Corollary 4.1.** The matrices

\[
[(I - FU - R)A^* : FU^*] \tag{4.1}
\]

and

\[
[\hat{F}U^* : (I - \hat{F}U - A)R^*]
\]

are $(1)$- or $(1, 2)$-inverses for

\[
\begin{bmatrix}
A \\
R
\end{bmatrix},
\]

depending on whether $(\cdot)^*$ is interpreted as a $(1)$- or $(1, 2)$-inverse.

**Proof.** The validity of each of the above is easily established by direct computation.

5. COMPUTATIONS FOR COLUMN PARTITIONS

We now specialize the representation given by (3.9) so as to obtain an algorithm for computing $(1, 2)$-inverses by successively adjoining columns. Write

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\[ A_{k+1} = [A_k : c_{k+1}] \] and \[ E_k = I - A_k A_k^-, \]

where \( c_{k+1} \) is a single column. Then

\[
A_{k+1}^- = \begin{bmatrix}
A_k^- (I - c_{k+1} (E_k c_{k+1})^- E_k) \\
E_k c_{k+1}^- E_k
\end{bmatrix}
\] (5.1)

is a \((1)\)- or \((1, 2)\)-inverse for \( A_{k+1} \). If \( E_k c_{k+1} = 0 \), then we can take \((E_k c_{k+1})^- = 0\) and

\[
A_{k+1}^- = \begin{bmatrix}
A_k^- \\
0
\end{bmatrix}.
\]

Assume \( E_k c_{k+1} \neq 0 \). Then a \((1, 2, 4)\)-inverse for \( E_k c_{k+1} \) is always given by

\[
(E_k c_{k+1})^- = \frac{1}{x_k E_k c_{k+1}} x_k E_k,
\] (5.2)

where \( x_k \) is an arbitrary row vector such that \( x_k E_k c_{k+1} \neq 0 \). This follows from the fact that \( E_k \) is always idempotent. If \( E_k c_{k+1} = 0 \), then clearly there is always at least one \( x_k \) such that \( x_k E_k c_{k+1} \neq 0 \), namely, \( x_k = (E_k c_{k+1})^* \).

However, if the \( j \)th component of \( E_k c_{k+1} \) is nonzero, then

\[ x_k = [\xi_1 \xi_2 \ldots \xi_m], \]

where \( \xi_i = \alpha \delta_{ij} \), for any \( \alpha \neq 0 \), is also always a vector such that \( x_k E_k c_{k+1} \neq 0 \). Now define

\[
r_{k+1} = \begin{cases} 
\frac{1}{x_k E_k c_{k+1}} x_k E_k & \text{if } E_k c_{k+1} \neq 0, \\
0 & \text{if } E_k c_{k+1} = 0,
\end{cases}
\]

where \( x_k \) is any row vector such that \( x_k E_k c_k \neq 0 \). Then, from (5.1) and (5.2), we have

\[
A_{k+1}^- = \begin{bmatrix}
A_k^- (I - c_{k+1} r_{k+1}) \\
E_k c_{k+1}^- E_k
\end{bmatrix}
\] (5.3)

is a \((1)\)- or \((1, 2)\)-inverse for \( A_{k+1} \), depending on how \(( \cdot )^- \) is interpreted. Furthermore, using (5.3), we obtain

\[
E_{k+1} = E_k (I - c_{k+1} r_{k+1}).
\] (5.4)

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Hence (5.3) together with (5.4) provides a simple method for obtaining (1)- or (1, 2)-inverses.

It is often convenient to start with

\[ A_1 = [a_{11} \ a_{21} \ \ldots \ a_{m1}]^T \]

as the first column (assumed to be nonzero) of an \( m \times n \) matrix \( A \) and take

\[ A_1^- = [0 \ 0 \ \ldots \ 0 \ a_{j1}^{-1} \ 0 \ \ldots \ 0] \]

where \( a_{j1} \) is any nonzero component of \( A_1 \). However, every (1)-inverse for a nonzero column vector is always a (1, 2)-inverse so that, if we start with \( A_1 \) as the first column (assumed nonzero), then using (5.3) and (5.4) always produces a (1, 2)-inverse.

Because one has the freedom of picking \( x_k \), the computation can somewhat be controlled by proper choices of the \( x_k \)'s.

6. Computations for Row Partitions

Similar results may be obtained if \( A_{k+1} \) is partitioned as

\[ A_{k+1} = \begin{bmatrix} A_k \\ b_{k+1} \end{bmatrix} \]

where \( b_{k+1} \) is a row vector. Using the fact that

\[ A_{k+1}^- = [A_k \ -F_k(b_{k+1}F_k) \ b_{k+1}A_k : F_k(b_{k+1}F_k)] \]

is always a (1)- or (1, 2)-inverse for \( A_{k+1} \), where \( F_k = I - A_k^-A_k \), it easily follows that

\[ A_{k+1}^- = [(I - u_{k+1}b_{k+1})A_k^- : u_{k+1}] \]

is a (1)- or (1, 2)-inverse for \( A_{k+1} \), where

\[ u_{k+1} = \begin{cases} \frac{1}{b_{k+1}F_k y_k} \ F_k y_k & \text{if } b_{k+1}F_k \neq 0, \\ 0 & \text{if } b_{k+1}F_k = 0 \end{cases} \]

for any column vector \( y_k \) such that \( b_{k+1}F_k y_k \neq 0 \). Furthermore,

\[ F_{k+1} = (I - u_{k+1}b_{k+1})F_k. \]
7. BLOCK TRIANGULAR MATRICES

We now derive representations for (1)- and (1, 2)-inverses of block triangular matrices. Let $T_{m \times n}$ be an upper block triangular matrix of the form

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},$$

where $T_{11}$ and $T_{22}$ need not be square. In what follows, let $E = I - T_{11}T_{11}^-$, $F = I - T_{22}^-T_{22}$, and $Q = F(ET_{12}F)^-E$, where $(\cdot)^-$ can be interpreted as a (1)- or (1, 2)-inverse. Partition $T$ as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$$

and apply Theorem 2.1 to obtain

$$T^- = \begin{bmatrix} [T_{11} : 0] - T_{11}T_{12} & ET_{12} \hat{E} & E & 0 \\ T_{22}^- & 0 & I \end{bmatrix}.$$

(7.1)

Now use (4.1) of Corollary 4.1 to write

$$\begin{bmatrix} ET_{12}^- \\ T_{22}^- \end{bmatrix} = [F(ET_{12}F)^- : T_{22}^--QT_{12}T_{22}^-].$$

Thus (7.1) becomes

$$T^- = \begin{bmatrix} [T_{11} : 0] - T_{11}T_{12}[Q : T_{22}^- - QT_{12}T_{22}^-] \\ [Q : T_{22}^- - QT_{12}T_{22}^-] \end{bmatrix}$$

$$= \begin{bmatrix} T_{11}^- - T_{11}T_{12}Q & -T_{11}T_{12}T_{22}^- + T_{11}T_{12}QT_{12}T_{22}^- \\ 0 & T_{22}^- - QT_{12}T_{22}^- \end{bmatrix}.$$ 

We have now established

**Theorem 7.1.** If

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$$

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and $E = I - T_{11}T_{11}^{-}, \quad F = I - T_{22}T_{22}^{-}, \quad Q = F(ET_{12}F)^{-}E$, then

$$T^{-} = \begin{bmatrix} T_{11}^{-} & -T_{11}^{-}T_{12}T_{22}^{-} \\ 0 & T_{22}^{-} \end{bmatrix} + \begin{bmatrix} -T_{11}^{-}T_{12} \\ I \end{bmatrix} Q[I - T_{12}T_{22}^{-}]$$

is a $(1)$- or $(1,2)$-inverse for $T$, depending on whether $(\cdot)^{-}$ is interpreted as a $(1)$- or $(1,2)$-inverse.

By using Corollary 3.1 and Corollary 4.1 one can derive a similar representation for lower block triangular matrices. If

$$T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}$$

and $\hat{E} = I - T_{22}T_{22}^{-}, \quad \hat{F} = I - T_{11}^{-}T_{11}, \quad \hat{Q} = \hat{F}(\hat{E}T_{12}\hat{F})^{-}\hat{E}$, then

$$T^{-} = \begin{bmatrix} T_{11}^{-} & 0 \\ -T_{22}^{-}T_{21}T_{11}^{-} & T_{22}^{-} \end{bmatrix} + \begin{bmatrix} I \\ -T_{22}^{-}T_{21} \end{bmatrix} \hat{Q}[I - T_{21}T_{11}^{-}:I]$$

is a $(1)$- or $(1,2)$-inverse for $T$.

It is often the case that $Q$ or $\hat{Q}$ is zero in the representations above. This and other questions regarding generalized inverses of block triangular matrices are studied in [3]. Representations for $(1,3)$-inverses of partitioned matrices are given in [4].

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Received July 22, 1970