MATHEMATICS

A NOTE ON SMOOTH FUNCTIONS[1])

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§ 1. The present note arose out of an attempt to understand better the meaning and significance of the following theorem of SALEM [1] (see also [2]) in which $S[f]$ denotes the Fourier series of f, and $S_n(x)$, or $S_n[f]$ its partial sums; by periodic functions we mean functions of period 2π.

Theorem A. *Suppose that $f(x)$ is periodic, integrable, and satisfies the condition*

$$(1)\qquad \frac{1}{2h}\int_0^h [f(x+t)-f(x-t)]\,dt = o\left\{\frac{1}{\log h}\right\} \quad (h \to +0)$$

uniformly in x. Then

(i) *$S[f]$ converges almost everywhere;*

(ii) *the convergence is uniform over every closed interval of points of continuity of f;*

(iii) *if f is in L^p, $p>1$, the function*

$$(2)\qquad s^*(x) = \sup_n |S_n(x)|$$

belongs to L^p.

The main result of this section is that condition (1) alone implies that f is in L^p for *every* p, and the result is primarily a theorem about smooth functions (see below). This is a special case of the following theorem.

Theorem 1. *If $F(x)$ is periodic and for some $\beta > \frac{1}{2}$ satisfies*

$$(3)\qquad \Delta^2 F(x,h) = F(x+h)+F(x-h)-2F(x) = O\left\{\frac{h}{|\log h|^\beta}\right\}$$

uniformly in x, then F is the indefinite integral of an f belonging to every L^p.

Functions F satisfying the condition $\Delta^2 F(x,h) = o(h)$ for each x and $h \to 0$ are called *smooth*; (3) is a strengthening of the condition of smoothness. For the theory of smooth functions and some of their properties see e.g. [4], or [3_I], pp. 42 and 114.

It is of interest that Theorem 1 is false for $\beta = \frac{1}{2}$. For example, the Weierstrass type function

$$(4)\qquad F(x) = \sum_{n=1}^{\infty} \frac{\cos 2^n x}{2^n n^{\frac{1}{2}}}$$

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satisfies, as can easily be seen (see e.g. [3_{I}], p. 47) condition (3) with $\beta=\frac{1}{2}$ and is at the same time differentiable almost nowhere, since the last series when differentiated termwise is lacunary but not in L^2 (see [3_{I}], p. 206).

The following results makes possible an application of Theorem 1 to the proof of Theorem A.

Theorem 2. *Suppose that a periodic and continuous F satisfies the condition*

$$\Delta^2 F(x, h) = o\left\{\frac{h}{\log h}\right\} \tag{5}$$

and let s_n and σ_n be respectively the partial sums and $(C, 1)$ means of $S'[F]$ (i.e. $S[F]$ differentiated termwise). Then

$$s_n - \sigma_n \to 0 \tag{6}$$

uniformly in x.

In view of Theorem 1, $\sigma_n(x)$ converges almost everywhere to $f=F'$, and the convergence is uniform over every closed interval of points of continuity of f. This implies parts (i) and (ii) of Theorem A. It is also well known that if $f \in L^p$, then the function

$$\sigma^*(x) = \sup |\sigma_n(x)|$$

is also in L^p, so that part (iii) of Theorem A is immediate.

We now pass to the proof of Theorem 1.

Let $E_n[F]$ be the best approximation of F by trigonometric polynomials of order n. The hypothesis (3) implies that

$$E_n[F] = O\,\{n^{-1} (\log n)^{-\beta}\}.$$

This follows immediately if we e.g. consider Jackson's polynomials

$$J_n(x, F) = \frac{1}{k_n} \int_{-\pi}^{\pi} F(x+t) \left(\frac{\sin nt}{\sin t}\right)^4 dt,$$

where

$$k_n = \int_{-\pi}^{\pi} \left(\frac{\sin nt}{\sin t}\right)^4 dt \simeq A\, n^3,$$

and observe that

$$J_n(x, F) - F(x) = \frac{1}{k_n} \int_0^{\pi} \Delta^2 F(x, t) \left(\frac{\sin nt}{\sin t}\right)^4 dt =$$

$$= \frac{1}{k_n} \int_0^{1/n} O\left\{\frac{1}{n(\log n)^\beta}\right\} O(n^4)\, dt + \frac{1}{k_n} \int_{1/n}^{\pi} O\left\{\frac{t}{(\log t)^\beta}\right\} O\left(\frac{1}{t^4}\right) dt = O\left(\frac{1}{n \log^\beta n}\right).$$

On the other hand, it is also known that if S_n are the partial sums of $S[F]$ then the "delayed means"

$$\tau_n(x) = \frac{S_n + S_{n+1} + \ldots + S_{2n-1}}{n}$$

differ from $F(x)$ by not more than $4\,E_n[F]$ (see [3_{I}], p. 115). Hence, with the hypothesis (3),

$$F(x)-\tau_n(x) = O\{n^{-1} (\log n)^{-\beta}\}.$$

Write

$$F(x)=\tau_1+(\tau_2-\tau_1)+(\tau_4-\tau_2)+\ldots+(\tau_{2^n}-\tau_{2^{n-1}})+\ldots=\sum_{n=0}^{\infty} U_n,$$

say, so that $U_n=\tau_{2^n}-\tau_{2^{n-1}}$ for $n=1, 2, \ldots$. Observe now that $\tau_m(x)$ is obtained by multiplying the k-th term of $S[F]$ by λ_k, where $\lambda_k=1$ for $k\leqslant m$ and decreases linearly to 0 as k increases from m to $2m$. Hence the non-zero terms of $U_n=\tau_{2^n}-\tau_{2^{n-1}}$ are of ranks k satisfying the condition $2^{n-1}<k<2^{n+1}$. It follows that the two series

$$U_1+U_3+U_5+\ \ldots,\ U_2+U_4+U_6+\ \ldots$$

are non-overlapping. We show that if we differentiate these two series termwise we obtain Fourier series of functions belonging to every L^p. In view of the theorem of LITTLEWOOD and PALEY [1]) (see e.g. [3_{II}], p. 233) it is enough to prove that the two functions

$$(U_1'^2+U_3'^2+U_5'^2+\ \ldots)^{\frac{1}{2}} \quad \text{and} \quad (U_2'^2+U_4'^2+U_6'^2+\ \ldots)^{\frac{1}{2}}$$

are in every L^p. We shall show that under our hypotheses they are bounded. For by Bernstein's theorem on the derivatives of trigonometric polynomials,

$$\max_x |U_n'(x)|\leqslant 2^{n+1} \max_x |U_n(x)|=o(n^{-\beta}),$$

and since a series with terms $O(n^{-2\beta})$ converges, Theorem 1 is established.

We now pass to the proof of Theorem 2 and first show that under its hypotheses,

(7) $$F-S_n(x) = o\Big(\frac{1}{n}\Big)$$

uniformly in x. To see this we write

$$F = \tau_m+\varrho_m, \text{ where } m = [\tfrac{1}{2}n].$$

Then

$$\begin{aligned} F-S_n[F] &= F-S_n[\tau_m]-S_n[\varrho_m] \\ &= (F-\tau_m)-S_n[\varrho_m] = \varrho_m-S_n[\varrho_m]. \end{aligned}$$

Since $\varrho_m=o(1/m\log m)=o(1/n\log n)$ and, using Lebesgue constants,

$$|S_n[\varrho_m]| = O\,(\log n) \max_n |\varrho_m(x)| = o(1/n),$$

(7) follows.

Write

$$F' = f\sim\sum_1^{\infty} (a_k \cos kx+b_k \sin kx) = \sum_1^{\infty} A_k(x)$$

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[1]) The theorem asserts that if $f\in L^p$, $p>1$, and if we decompose $S[f]$ into a series of blocks, $S[f]=\Sigma\Delta_k$, where Δ_k consists of the terms of rank ν satisfying $n_k\leqslant n<n_{k+1}$, with $1<\alpha<n_{k+1}/n_k<\beta<\infty$, then the ratio of $\|f\|_p$ and $\|(\Sigma\Delta_k^2)^{\frac{1}{2}}\|_p$ is contained between two constants depending on α and β only.

and let $B_k = a_k \sin kx - b_k \cos kx$. Then (7) can be written

$$\sum_{k=n}^{\infty} \frac{B_k(x)}{k} = o\left(\frac{1}{n}\right).$$

In particular

$$\sum_{\frac{1}{2}N<k\leqslant N} k^{-1}\, B_k(x) = o(N^{-1})$$

for any positive number N, not necessarily an integer and, by Berstein's theorem,

$$\sum_{\frac{1}{2}N<k\leqslant N} A_k(x) = o(1).$$

Using Bernstein's inequality for conjugate trigonometric polynomials of order n ($\max|\widetilde{T}'(x)|\leqslant n \max|T(x)|$) we have

$$\sum_{\frac{1}{2}N<k\leqslant N} k\, A_k(x) = o(N),$$

whence, replacing N by $\frac{1}{2}N, \frac{1}{8}N, \frac{1}{8}N, \ldots$ and adding,

$$\sum_{k=1}^{N} k\, A_k(x) = o(N),$$

which is (6), with N for n.

§ 2. In this section we consider a generalization of Theorem 2. A periodic $F(x) \in L^p$ will be said to satisfy condition Λ_p^* if

$$(8) \qquad \|\Delta^2 F(x,h)\|_p = \Big(\int_0^{2\pi} |F(x+h)+F(x-h)-2F(x)|^p\, dx\Big)^{1/p} = o(h).$$

Replacing here 'O' by 'o' we obtain condition λ_*^p. It is well known that Λ_p^* and λ_p^* are the classes of functions which in the metric L^p can be approximated to by polynomials of order n with an error $O(1/n)$ and $o(1/n)$ respectively. While functions in Λ_*^1 can be discontinuous and even unbounded (for example the function equal to $\log|x|$ for $|x|\leqslant\pi$ and continued periodically is in Λ_*^1), the functions from Λ_*^p, $p>1$, are essentially continuous and even have absolutely convergent Fourier series. In addition to Λ_*^p we shall also consider the classes $\Lambda_{*,\beta}^p$ of functions satisfying the condition

$$\|\Delta^2 F(x,h)\|_p = O\left\{\frac{h}{|\log h|^\beta}\right\}.$$

Theorem 3. (i) *If $F \in \Lambda_{*,\beta}^p$, $1\leqslant p\leqslant 2$, $\beta>1/p$, then F is absolutely continuous and $F' \in L^p$. The result is false if $\beta = 1/p$.*

(ii) *If $F \in \Lambda_{*,\beta}^p$, $2\leqslant p<\infty$, $\beta>\frac{1}{2}$, then F is absolutely continuous and $F' \in L^p$. The result is false if $\beta=\frac{1}{2}$.*

(i) Let S_n and s_n be the partial sums of $S[F]$ and $S'[F]$ respectively and let (*assuming that the constant term of $S[F]$ is* 0)

$$S[F] = \sum_{k=1}^{\infty} k^{-1}\, B_k(x).$$

The hypothesis implies that the best approximation of F in the metric L^p by trigonometric polynomials of order n is $O(1/n \log^\beta n)$ (we may use Jackson's polynomials for the proof) and so, if $p>1$, leaving the case $p=1$ temporarily aside,

$$(9) \qquad \{\int_0^{2\pi} |F-S_n|^p\, dx\}^{1/p} = O\{n^{-1} (\log n)^{-\beta}\}.$$

Let Δ_n and δ_n denote the blocks of terms with indices $2^{n-1} \leqslant k < 2^n$ $(n=1, 2, \dots)$ for $S[F]$ and $S'[F]$ respectively. From (9) we deduce that

$$(10) \qquad ||\Delta_n||_p = O(2^{-n} n^{-\beta}),$$

and so, using Bernstein's inequality for the metric L^p,

$$(11) \qquad ||\delta_n||_p = O(n^{-\beta}).$$

Applying the theorem of Littlewood and Paley, we see that the positive assertion of (i) will follow if we show that

$$\int_0^{2\pi} (\sum \delta_n^2)^{\frac{1}{2}p}\, dx$$

is finite. Since $p \leqslant 2$, the last integral is majorized by

$$\int_0^{2\pi} (\sum |\delta_n|^p)\, dx = \sum \|\delta_n\|_p^p = \sum O(n^{-\beta p}) < \infty,$$

if $\beta p > 1$.

The case $p=1$ must be treated slightly differently since it is no longer true that the approximation to F by the $S_n[F]$ is of the same order as the best approximation (in the metric L). The required modification has already been used in § 1. Let τ_n be the delayed means of $S[F]$. Then

$$F = \tau_1 + (\tau_2 - \tau_1) + (\tau_4 - \tau_2) + \dots = U_0 + U_1 + U_2 + \dots.$$

The two series $U_0 + U_2 + U_4 + \dots$ and $U_1 + U_3 + U_5 + \dots$ are non-overlapping and are respectively $S[F_1]$ and $S[F_2]$. It is enough to show that $S'[F_1]$ and $S'[F_2]$ are both Fourier series. Now

$$||U_n||_1 = O(2^{-n} n^{-\beta}), \quad ||U_n'||_1 = O(n^{-\beta}),$$

and it is enough to observe that, say,

$$\sum \int_0^{2\pi} |U_{2k}'|\, dx = \sum O(k^{-\beta}) < \infty$$

if $\beta > 1$.

As regards counterexamples, suppose first that $1 < p \leqslant 2$ and consider the periodic function

$$F(x) = |x|^{1/p'} \left(\log \frac{2\pi}{|x|}\right)^{-1/p} \quad (|x| \leqslant \pi),$$

where p' is the index conjugate to p. A simple computation which we omit shows that F is in $\Lambda^{p}_{*,1/p}$ and that $F'(x)$, which is asymptotically equal to $x^{-1/p}\,(\log 1/x)^{-1/p}$ as $x \to +0$, is not in L^p.

Similarly the periodic function equal to

$$\log\log \frac{\pi}{|x|}$$

for $|x| \leqslant \pi$ is in $\Lambda^{1}_{*,1}$, but its derivative is not in L.

(ii) We pass to the case $2 \leqslant p < \infty$, and suppose that $F \in \Lambda^{p}_{*,\beta}$ where $\beta > \frac{1}{2}$. Defining the blocks Δ_n and δ_n as before, we may write $S[F] = \sum \Delta_n$, $S'[F] = \sum \delta_n$.

We again have (9), (10) and (11). Since $p \geqslant 2$, Minkowski's inequality gives

$$\Big\{\int_0^{2\pi} (\textstyle\sum \delta_n^2)^{\frac{1}{2}p}\, dx\Big\}^{2/p} \leqslant \sum \Big\{\int_0^{2\pi} (\delta_n^2)^{\frac{1}{2}p}\, dx\Big\}^{2/p} = \sum \|\delta_n\|_p^2.$$

Since the terms of the last series are $O(n^{-2\beta})$, and $\beta > \frac{1}{2}$, the series converges, the integral on the left is finite and the Littlewood–Paley theorem shows that $S'[F] = \sum \delta_n$ is the Fourier series of a function in L^p. This completes the proof of the positive assertion in (ii). That the result is false for $\beta = \frac{1}{2}$ is seen by the example of the function (4), for which

$$\Delta^2 F(x, h) = O\,\{h \log^{-\frac{1}{2}} (1/h)\},$$

so that $F \in \Lambda^{p}_{*,\frac{1}{2}}$, and which is differentiable only in a set of measure 0.

Remarks. a. Salem localized his theorem to a subinterval (a, b) of a period. We can likewise generalize Theorem 2:

Theorem 2′. *If F is periodic, integrable, continuous in an interval (a, b), satisfies uniformly in that interval condition* (5), *and has Fourier coefficients $o(1/n)$, then $s_n - \sigma_n$ tends uniformly to 0 in every $(a+\varepsilon, b-\varepsilon)$, $\varepsilon > 0$, where s_n and σ_n are the partial sums and $(C, 1)$ means of $S'[F]$.*

The proof might, in principle, imitate that of Theorem 2, but since then a few non-trivial details would have to be attended to, we prefer to reduce Theorem 2′ to Theorem 2. If we could represent the F in Theorem 2′ as a sum $F_1 + F_2$ of two periodic functions such that F_1 is everywhere continuous and satisfies a condition analogous to (5), and F_2 is integrable and zero in (a, b), the reduction would be immediate. For then, if $s_{1,n}$, $s_{2,n}$, $\sigma_{1,n}$, $\sigma_{2,n}$ are respectively the partial sums and $(C, 1)$ means of $S'[F_1]$ and $S'[F_2]$, we would have

$$s_n - \sigma_n = (s_{1,n} - \sigma_{1,n}) + (s_{2,n} - \sigma_{2,n}),$$

and since $s_{1,n} - \sigma_{1,n}$ tends uniformly to 0 it would be enough to show that $s_{2,n} - \sigma_{2,n}$ tends uniformly to 0 in $(a+\varepsilon, b-\varepsilon)$. But $s_{2,n} - \sigma_{2,n}$ is the n-th partial sum divided by $(n+1)$ of $\tilde{S}''[F_2]$, and since the coefficients of $\tilde{S}''[F_2]$ are $o(n)$, and $F_2 = 0$ in (a, b), the partial sums of $\tilde{S}''[F_2]$ would be $o(n)$ uniformly in $(a+\varepsilon, b-\varepsilon)$ ([3], p. 367) and the assertion would follow.

Whether a decomposition of the kind just described is possible we do not know, and the problem of extending a smooth function outside the initial interval of definition is not obvious though possibly not difficult. For our purposes however it is enough to show that in (a, b) we can find points a_1, b_1 arbitrarily close to a and b respectively and such that F can be continued outside (a_1, b_1) with the preservation of (5). First we show that if F satisfies condition (5) in (a, b) then there is a dense set of points $\xi \in (a, b)$ such that $F'(\xi)$ exists and

$$\frac{F(\xi+h)-F(\xi)}{h} - F'(\xi) = o\left(\frac{1}{\log|h|}\right).$$

This is certainly true, with $F'(\xi)=0$, if ξ is an extremum of F, and subtracting from F linear functions we obtain a dense set – even one of the power of the continuum – of the points ξ, and if we take for a_1 and b_1 points ξ, and define F_1 as equal to F in (a_1, b_1) and equal to an arbitrary function of the class C' elsewhere, provided F_1 is continuous and differrentiable at a_1 and b_1, then it is not difficult to see that F_1 is a required extension of F. (On a similar argument we might base an extension from (a_1, b_1) interior to (a, b) of a function which satisfies in (a, b) the condition $\Delta^2F(x, h)=o(h)$ or $=O(h)$, not necessarily uniformly in x.)

b) There is an analogue of Theorem 2 for the metric L^p, $1 \leqslant p < \infty$. The cases $p=1$ and $1<p<\infty$ are slightly different. If $F \in \Lambda^1_{*,1}$, then

$$||s_n - \sigma_n||_1 \to 0,$$

and, slightly more generally, if $F \in \Lambda^1_{*,\beta}$, then $||s_n-\sigma_n||_p = o\,\{(\log n)^{1-\beta}\}$. If $1<p<\infty$, and if $F \in \Lambda^p_{*,\beta}$, then $||s_n-\sigma_n|| = o\,\{(\log n)^{-\beta}\}$. The '$o$' can be replaced by '$O$' throughout.

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