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The estimates of approximation by using a new type of weighted modulus of continuity

A.D. Gadjiev[a], A. Aral[b,*]

[a] Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 9 F. Agayev Str., Bakü, Azerbaijan
[b] Kırıkkale University, Department of Mathematics, 71450 Yahşihan, Kırıkkale, Turkey

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Abstract

In this paper, we introduce a new type modulus of continuity for function f belonging to a particular weighted subspace of $C[0,\infty)$ and show that it has some properties of ordinary modulus of continuity. We obtain some estimates of approximation of functions with respect to a suitable weighted norm via the new type moduli of continuity. Finally, we give some examples.

Keywords: Modulus of continuity; Weighted spaces; Beurling classes; Positive linear operators

1. Introduction and preliminaries

As it is well known, the modulus of continuity ω of the uniformly continuous function f on the finite (or infinite) interval I is given by the formula

$$\omega(f;\delta) = \sup_{|t-x|\le\delta} |f(t)-f(x)|, \quad t,x\in I.$$

Note that the function ω has the property that $\lim_{\delta\to 0}\omega(f;\delta)=0$. Because of this property, the modulus of continuity is used to estimate the order of approximation to the function f by polynomials, by entire functions or, in general, by the sequences $L_n(f;x)$, where L_n are positive linear operators (see, for example, [1,2]).

Recall that uniform continuity of a function on unbounded sets restricts its growth: at infinity it cannot grow faster than $(1+|x|)$. Therefore, it is impossible to use the modulus of continuity ω for estimating the rate of approximation of functions having high order of growth at infinity. However, the weighted modulus of continuity, the definition of which includes the growth of function at infinity, may be used for this purpose. For instance, let's consider the weighted modulus of continuity $\widetilde{\omega}$ introduced in [3] for the classes of continuous function f defined on $\mathbb{R}$ that grows, at infinity, no faster than $(1+|x|)^{\sigma}$, $\sigma>1$,

$$\widetilde{\omega}(f;\delta) = \sup_{|t-x|\le\delta} \frac{|f(t)-f(x)|}{(1+|t-x|)^{\sigma}(1+|x|)^{\sigma}}, \quad t,x\in\mathbb{R}.$$

* Corresponding author.
E-mail addresses: frteb@aas.ab.az (A.D. Gadjiev), aral@science.ankara.edu.tr, aliaral73@yahoo.com (A. Aral).

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This modulus of continuity has the property $\lim_{\delta\to 0}\widetilde{\omega}(f;\delta) = 0$ on subclass of functions for which $\lim_{|x|\to\infty}\frac{f(x)}{(1+|x|)^{\sigma}}$ exists and is finite (see, for example, [4]).

For continuous functions defined on $\mathbb{R}$ or $\mathbb{R}^+ = [0,\infty)$ where the growth at infinity is not faster than some weight function $\rho(x)$ which does not need to be a power function, taking sup on $|\rho(t)-\rho(x)| \le \delta$ rather than $|t-x| \le \delta$ is convenient for our purposes in the definition of the new type of modulus of continuity.

Recall that any function f from Beurling class [5] contracted with respect to the function φ satisfies the inequality:

$$|f(t)-f(x)| \le |\varphi(t)-\varphi(x)|$$

for any $t, x \in \mathbb{R}$. It is obvious that

$$\sup_{|\varphi(t)-\varphi(x)|\le\delta} |f(t)-f(x)| \le \delta.$$

We consider the weighted spaces of the functions which are defined on the semi-axis $\mathbb{R}^+ = [0,\infty)$ and satisfy the inequality $|f(x)| \le M_f\rho(x)$. Here, $\rho(x)$ is a weight function and M_f is a positive constant. We denote the set of functions that satisfy this inequality by B_ρ to obtain:

$$B_\rho\left(\mathbb{R}^+\right) := \left\{f : |f(x)| \le M_f\rho(x)\right\}.$$

Then we define

$$C_\rho\left(\mathbb{R}^+\right) := \left\{f : f \in B_\rho \text{ and } f \text{ is continuous}\right\},$$
$$C_\rho^k\left(\mathbb{R}^+\right) := \left\{f : f \in C_\rho\left(\mathbb{R}^+\right) \text{ and } \lim_{x\to\infty}\frac{f(x)}{\rho(x)} = K_f, \text{ a constant}\right\}$$

and

$$C_\rho^0\left(\mathbb{R}^+\right) := \left\{f : f \in C_\rho^k\left(\mathbb{R}^+\right) \text{ and } \lim_{x\to\infty}\frac{f(x)}{\rho(x)} = 0\right\}.$$

It is obvious that $C_\rho^k\left(\mathbb{R}^+\right) \subset C_\rho\left(\mathbb{R}^+\right) \subset B_\rho\left(\mathbb{R}^+\right)$. We define the norm of f belonging to B_ρ by

$$\|f\|_\rho = \sup_{x\ge 0}\frac{|f(x)|}{\rho(x)}.$$

Thus, the following results on the sequence of positive linear operators in these spaces are given [6,7].[1]

Lemma A. *In order that the sequence of positive linear operators $(L_n)_{n\ge1}$ act from $C_\rho\left(\mathbb{R}^+\right)$ to $B_\rho\left(\mathbb{R}^+\right)$, it is necessary and sufficient that the inequality is fulfilled*

$$L_n(\rho; x) \le K\rho(x)$$

with some positive constant K.

Theorem A. *If a sequence of positive linear operators (L_n) satisfies the conditions*

$$\lim_{n\to\infty}\left\|L_n(\rho^{m/2}; x) - \rho^{m/2}(x)\right\|_\rho = 0, \quad m = 0, 1, 2,$$

then for any function $f \in C_\rho^k\left(\mathbb{R}^+\right)$

$$\lim_{n\to\infty}\|L_n f - f\|_\rho = 0.$$

As it is seen in Theorem A, the weight function $\rho(x)$ not only characterizes the growth of f at infinity but also defines the test functions in Korovkin type theorem. Other theorems of this type are given in [8,9].

[1] In translation, papers [6] and [7] Gadžiev = Gadzhiev = Gadjiev.

The rest of the paper is organized as follows. In Section 2 we construct the weighted modulus of continuity connected with the function ρ and reflecting the dependence of the difference $|f(t) - f(x)|$ on $|\rho(t) - \rho(x)| \leq \delta$. We also study the characteristic properties of this modulus of continuity. In Section 3 we prove the estimates of approximation of functions by linear positive operators in ρ-norms by means of the new type of modulus of continuity. Some examples are given in Section 4.

2. A new type of weighted modulus of continuity

We now introduce the weighted modulus of continuity. Consider the weight function ρ satisfying the following assumptions:

(i) ρ is a continuously differentiable function on $\mathbb{R}^+$ and $\rho(0) = 1$.
(ii) $\inf_{x\geq 0} \rho'(x) \geq 1$.

From now on, we consider the spaces $C_\rho^k(\mathbb{R}^+)$, $C_\rho(\mathbb{R}^+)$ and $B_\rho(\mathbb{R}^+)$ having the assumptions (i) and (ii).

For each $f \in C_\rho(\mathbb{R}^+)$ and for every $\delta > 0$ we set

$$\Omega_\rho(f;\ \delta) := \Omega_\rho(f;\ \delta)_{\mathbb{R}^+} = \sup_{\substack{x,t\in\mathbb{R}^+ \\ |\rho(t)-\rho(x)|\leq\delta}} \frac{|f(t) - f(x)|}{[|\rho(t) - \rho(x)| + 1]\,\rho(x)}. \tag{2.1}$$

We call the function $\Omega_\rho(f;\ \delta)$ as weighted modulus of continuity. We observe that $\Omega_\rho(f;\ 0) = 0$ for every $f \in C_\rho(\mathbb{R}^+)$ and that the function $\Omega_\rho(f;\ \delta)$ is nonnegative and nondecreasing with respect to δ for $f \in C_\rho(\mathbb{R}^+)$.

Weighted modulus of continuity Ω_ρ has some properties that are similar to the properties of the classical modulus of continuity. We give these properties in the following lemmas. Note that for different weighted modulus of smoothness similar properties are discussed in [2].

First, we show the Ω_ρ is bounded.

Lemma 1. *If* $f \in C_\rho(\mathbb{R}^+)$ *then*

$$\Omega_\rho(f;\ \delta) \leq 2\,\|f\|_\rho\,.$$

Proof. Since $f \in C_\rho(\mathbb{R}^+)$ we have

$$|f(t) - f(x)| \leq \|f\|_\rho\,(|\rho(t) - \rho(x)| + 2\rho(x)) \leq 2\,\|f\|_\rho\,(|\rho(t) - \rho(x)| + 1)\,\rho(x)\,. \quad \square$$

Lemma 2. *If the function* ρ *satisfies the conditions* (i) *and* (ii) *then*

$$\lim_{\delta\to 0} \Omega_\rho(\rho;\ \delta) = 0.$$

Proof. The statement follows immediately from the inequality $\Omega_\rho(\rho;\ \delta) \leq \delta$. $\square$

Lemma 3. *If* $f \in C_\rho(\mathbb{R}^+)$ *and* $\lambda > 0$ *then*

$$\Omega_\rho(f;\ \lambda\delta) \leq (1 + \lambda)(1 + \delta)\,\Omega_\rho(f;\ \delta)$$

holds for every $\delta > 0$.

Proof. Let $x, t \in R^+$ be fixed points. Consider a partition of the interval $[x, t]$ such that $0 < x = x_0 < x_1 < x_2 < \cdots < x_m = t$, $m \in N$. By the uniform continuity of ρ on $[x, t]$, for every δ there exists a number μ such that

$$|x_k - x_{k-1}| < \mu \quad \text{implies } |\rho(x_k) - \rho(x_{k-1})| < \delta, k = 1, 2, \ldots m.$$

Thus we can write

$$|\rho(t) - \rho(x)| < m\delta. \tag{2.2}$$

On the other hand, we have

$$\begin{aligned} |f(t)-f(x)| &\leq \sum_{k=1}^{m} \frac{|f(x_k)-f(x_{k-1})|}{\left[|\rho(x_k)-\rho(x_{k-1})|+1\right]\rho(x_{k-1})}\left[|\rho(x_k)-\rho(x_{k-1})|+1\right]\rho(x_{k-1}) \\ &\leq (1+\delta)\,\Omega_\rho(f;\ \delta)\sum_{k=1}^{m}\rho(x_{k-1}) \\ &= m(1+\delta)\,\Omega_\rho(f;\ \delta)\left[|\rho(t)-\rho(x)|+1\right]\rho(x). \end{aligned}$$

This inequality together with (2.1) and (2.2) imply that for $m \in N$

$$\Omega_\rho(f;\ m\delta) \leq m(1+\delta)\,\Omega_\rho(f;\ \delta).$$

Since $\Omega_\rho(f;\ \delta)$ is a nondecreasing function of δ the inequality

$$\Omega_\rho(f;\ \lambda\delta) \leq (1+\lambda)(1+\delta)\,\Omega_\rho(f;\ \delta)$$

holds for $\lambda > 0$. □

Corollary 1. *For $f \in C_\rho\left(\mathbb{R}^+\right)$ and any positive $\delta < 1$, the inequality*

$$\delta \leq \frac{4}{\Omega_\rho(f;\ 1)}\Omega_\rho(f;\ \delta)$$

holds.

Proof. Using Lemma 3 we write

$$\Omega_\rho(f;\ 1) \leq \left(1+\frac{1}{\delta}\right)(1+\delta)\,\Omega_\rho(f;\ \delta) \leq \frac{4}{\delta}\Omega_\rho(f;\ \delta). \quad \square$$

Lemma 4. *For any $f \in C_\rho^k\left(\mathbb{R}^+\right)$*

$$\lim_{\delta\to 0}\Omega_\rho(f,\delta) = 0.$$

Proof. Let $f \in C_\rho^k\left(\mathbb{R}^+\right)$. We can write

$$\begin{aligned} \frac{|f(t)-f(x)|}{(|\rho(t)-\rho(x)|+1)\,\rho(x)} &= \frac{\left|\frac{f(t)}{\rho(t)}\frac{\rho(t)}{\rho(x)}-\frac{f(x)}{\rho(x)}\right|}{|\rho(t)-\rho(x)|+1} \\ &= \frac{\left|\frac{f(t)}{\rho(t)}\left(\frac{\rho(t)}{\rho(x)}-1\right)+\left(\frac{f(t)}{\rho(t)}-\frac{f(x)}{\rho(x)}\right)\right|}{|\rho(t)-\rho(x)|+1} \\ &\leq \frac{\|f\|_\rho\,|\rho(t)-\rho(x)|+\omega\left(\frac{f}{\rho},\ |t-x|\right)}{|\rho(t)-\rho(x)|+1} \\ &\leq \|f\|_\rho\,|\rho(t)-\rho(x)|+\omega\left(\frac{f}{\rho},\ |t-x|\right), \end{aligned}$$

where ω is defined as in Section 1.

By the assumption (ii) and the mean value theorem we have $|\rho(t)-\rho(x)| \geq |t-x|$. Thus we have

$$\Omega_\rho(f,\delta) \leq \|f\|_\rho\,\delta + \omega\left(\frac{f}{\rho},\delta\right).$$

Since $\frac{f}{\rho}$ uniformly continuous, for every $t, x \geq 0$ with $|\rho(t)-\rho(x)| \leq \delta$, we have $\lim_{\delta\to 0}\omega\left(\frac{f}{\rho},\delta\right) = 0$. Thus the proof is completed. □

Lemma 5. *For each* $f \in C_\rho^k\left(\mathbb{R}^+\right)$ *and for each* $x, t \in [0, \infty)$ *the inequality*

$$|f(t) - f(x)| \leq 2\rho(x)(1+\delta)^2 \left(1 + \frac{(\rho(t) - \rho(x))^2}{\delta^2}\right) \Omega_\rho(f;\ \delta) \tag{2.3}$$

holds, where δ *is any fixed positive number.*

Proof. Directly from the definition of $\Omega_\rho(f;\ \delta)$ and Lemma 3 we have

$$\begin{aligned} |f(t) - f(x)| &\leq \rho(x)(|\rho(t) - \rho(x)| + 1)\, \Omega_\rho(f;\ |\rho(t) - \rho(x)|) \\ &\leq \rho(x)(|\rho(t) - \rho(x)| + 1)(1+\delta)\left(1 + \frac{|\rho(t) - \rho(x)|}{\delta}\right) \Omega_\rho(f;\ \delta). \end{aligned}$$

Therefore,

$$\begin{aligned} |f(t) - f(x)| &\leq 2\rho(x)(1+\delta)^2\, \Omega_\rho(f;\ \delta) \quad \text{if } |\rho(t) - \rho(x)| \leq \delta, \\ |f(t) - f(x)| &\leq 2\rho(x)(1+\delta)^2 \frac{(\rho(t) - \rho(x))^2}{\delta^2} \Omega_\rho(f;\ \delta) \quad \text{if } |\rho(t) - \rho(x)| > \delta. \end{aligned}$$

This proves the lemma. □

Now we introduce an analogy of the classical Lipschitz space $\mathrm{Lip}_M \alpha$.

Definition 1. Let $\rho(x)$ satisfy the conditions (i) and (ii), $0 < \alpha \leq 1$ and $M > 0$. Denote by $\mathrm{Lip}_M(\rho(x);\ \alpha)$ the set of all functions f satisfying the inequality

$$|f(t) - f(x)| \leq M\,|\rho(t) - \rho(x)|^\alpha, \quad x, t \geq 0.$$

It is obvious that we have

$$\mathrm{Lip}_M \alpha \subset \mathrm{Lip}_M(\rho(x);\ \alpha) \quad \text{and} \quad \mathrm{Lip}_M \alpha = \mathrm{Lip}_M(1 + x;\ \alpha).$$

We give following examples:

(a) For $x \geq 0$, $x \in \mathrm{Lip}_1(e^x;\ \alpha)$.
(b) $e^x \in \mathrm{Lip}_{1/2}\left(e^{2x};\ 1\right)$, but $e^x \notin \mathrm{Lip}_M 1$.

Using (2.1) and Definition 1, we have

$$\Omega_\rho(f;\ \delta) \leq M\delta^\alpha \tag{2.4}$$

for any function $f \in \mathrm{Lip}_M(\rho(x);\ \alpha)$

The class $\mathrm{Lip}_1(\rho(x);\ 1)$ coincides with Beurling class of contractive functions

$$|f(t) - f(x)| \leq |\rho(t) - \rho(x)|$$

for any points $t, x \geq 0$.

The detailed discussion and more properties of the Beurling class can be found in [10–12].

3. The estimate of approximation by modulus of Ω_ρ

Theorem 1. *Let* $(L_n)_{n \geq 1}$ *be a sequence of positive linear operators satisfying the conditions*

$$\|L_n 1 - 1\|_\rho = \alpha_n, \tag{3.1}$$

$$\|L_n \rho - \rho\|_\rho = \beta_n, \tag{3.2}$$

$$\left\|L_n \rho^2 - \rho^2\right\|_{\rho^2} = \gamma_n, \tag{3.3}$$

where α_n, β_n and γ_n tend to zero as $n \to \infty$. Then

$$\|L_n f - f\|_{\rho^4} \leq 16\Omega_\rho\left(f;\ \sqrt{\alpha_n + 2\beta_n + \gamma_n}\right) + \|f\|_\rho \alpha_n$$

for all $f \in C_\rho^k(\mathbb{R}^+)$ and n large enough.

Proof. First, observe that as a consequence of (3.2) and (3.3) and Lemma A the operators $(L_n)_{n\geq 1}$ is a mapping from $C_\rho(\mathbb{R}^+)$ into $B_\rho(\mathbb{R}^+)$ and also a mapping from $C_{\rho^2}(\mathbb{R}^+)$ into $B_{\rho^2}(\mathbb{R}^+)$.

Let us note that by virtue of (2.3) the following pointwise estimate

$$\frac{|L_n(f;\ x) - f(x)|}{\rho(x)} \leq 4(1+\delta_n)^2 \frac{1}{\delta_n^2} L_n\left((\rho(t)-\rho(x))^2;\ x\right)\Omega_\rho(f;\ \delta_n) + \frac{|f(x)|}{\rho(x)}|L_n(1;\ x) - 1| \quad (3.4)$$

holds for all $f \in C_\rho^k(\mathbb{R}^+)$, where δ_n is any sequence of positive numbers (which have been chosen below).

On the other hand,

$$L_n\left((\rho(t)-\rho(x))^2;\ x\right) \leq \left|L_n\left(\rho^2;\ x\right) - \rho^2(x)\right| + 2\rho(x)|L_n(\rho;\ x) - \rho(x)| + \rho^2(x)|L_n(1;\ x) - 1|.$$

By Eqs. (3.1)–(3.3), for any $n \geq 1$ and $x \geq 0$, we have

$$L_n\left((\rho(t)-\rho(x))^2;\ x\right) \leq \gamma_n\rho^2(x) + 2\beta_n\rho^2(x) + \alpha_n\rho^3(x) \leq \rho^3(x)(\alpha_n + 2\beta_n + \gamma_n). \quad (3.5)$$

Substituting this inequality into (3.4) and choosing $\delta_n = \sqrt{\alpha_n + 2\beta_n + \gamma_n}$ we see that $\delta_n < 1$ for large n. Therefore, (3.4) and (3.5) give

$$\frac{|L_n(f;\ x) - f(x)|}{\rho^4(x)} \leq 16\Omega_\rho\left(f;\ \sqrt{\alpha_n + 2\beta_n + \gamma_n}\right) + \|f\|_\rho \alpha_n$$

for n large enough. This proves Theorem 1. □

Using (2.4) we can give the following result.

Corollary 2. *Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators, satisfying the conditions (3.1)–(3.3). If $f \in \mathrm{Lip}_M(\rho(x);\ \alpha)$ for some $0 < \alpha \leq 1$, then*

$$\|L_n f - f\|_{\rho^4} \leq 16M(\alpha_n + 2\beta_n + \gamma_n)^{\frac{\alpha}{2}} + \|f\|_\rho \alpha_n,$$

for n large enough, where M is a positive constant independent of n.

The following theorem gives the convergence of sequences of positive linear operators in weighted space $C_\rho^k(\mathbb{R}^+)$, where the interval of convergence expands as $n \to \infty$. Results of this type were first obtained in [8].

Theorem 2. *Under the assumptions of Theorem 1, if the sequence of positive real numbers η_n satisfies the conditions*

$$\lim_{n\to\infty} \eta_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \rho^{\frac{3}{2}}(\eta_n)\delta_n = 0,$$

then for each $f \in C_\rho^k(\mathbb{R}^+)$ and n large enough

$$\sup_{0\leq x\leq \eta_n} \frac{|L_n(f;\ x) - f(x)|}{\rho(x)} \leq 16\Omega_\rho\left(f;\ \rho^{\frac{3}{2}}(\eta_n)\delta_n\right) + \|f\|_\rho \rho^{\frac{3}{2}}(\eta_n)\delta_n.$$

Proof. By replacing the values δ_n in (3.4) with $\rho^{\frac{3}{2}}(\eta_n)\delta_n$, where as above $\delta_n = \sqrt{\alpha_n + 2\beta_n + \gamma_n}$, we can write

$$|L_n(f;\ x) - f(x)| \leq 16\rho(x)\frac{1}{\rho^3(\eta_n)\delta_n^2} L_n\left((\rho(t)-\rho(x))^2;\ x\right)\Omega_\rho\left(f;\ \rho^{\frac{3}{2}}(\eta_n)\delta_n\right) + \|f\|_\rho \alpha_n \rho^2(x).$$

Since $\rho(x)$ is an increasing function, by (3.5) the above inequality immediately gives

$$\frac{|L_n(f;\ x) - f(x)|}{\rho(x)} \leq 8\Omega_\rho\left(f;\ \rho^{\frac{3}{2}}(\eta_n)\,\delta_n\right) + \|f\|_\rho\,\rho^{\frac{3}{2}}(\eta_n)\,\delta_n$$

for all $x \in [0, \eta_n]$ and n large enough. □

The proof of Theorem 1 allow us to obtain a more general result. We will denote $\psi(x) = \max(\psi_1(x), \psi_2(x))$ for a given functions ψ_1 and ψ_2 if for all $x \geq 0$ $\psi_1(x) \leq \psi(x)$ and $\psi_2(x) \leq \psi(x)$.

Theorem 3. *Let $\rho(x) \leq \psi_k(x)$, $k = 1, 2, 3$ and the sequences of the positive linear operators $(L_n)_{n\geq 1}$ satisfies the following conditions:*

$$\|L_n 1 - 1\|_{\psi_1} = \alpha_n,$$
$$\|L_n\rho - \rho\|_{\psi_2} = \beta_n,$$
$$\left\|L_n\rho^2 - \rho^2\right\|_{\psi_3} = \gamma_n,$$

where α_n, β_n and γ_n tends to zero as $n \to \infty$ and $\psi(x) = \max\{\psi_1(x), \psi_2(x), \psi_3(x)\}$. Then for any function $f \in C_\rho^k(\mathbb{R}^+)$ the inequality

$$\|L_n(f;\ x) - f\|_{\psi\rho^2} \leq 16\Omega_\rho\left(f;\ \sqrt{\alpha_n + 2\beta_n + \gamma_n}\right) + \|f\|_\rho\,\alpha_n$$

holds for n large enough.

Proof. From the inequality (3.5), we get

$$L_n\left((\rho(t) - \rho(x))^2;\ x\right) \leq \gamma_n\psi_3(x) + 2\beta_n\rho(x)\,\psi_2(x) + \alpha_n\psi_1(x)\,\rho^2(x)$$
$$\leq \psi(x)\,\rho^2(x)\,(\alpha_n + 2\beta_n + \gamma_n)$$

and the proof may be completed as in Theorem 1. □

Remark 1. Using Corollary 1, we can write the inequalities of Theorems 1 and 3 in the form

$$\|L_n(f;\ x) - f\|_{\rho^4} \leq C(f)\,\Omega_\rho\left(f;\ \sqrt{\alpha_n + 2\beta_n + \gamma_n}\right),$$
$$\|L_n(f;\ x) - f\|_{\psi\rho^2} \leq C(f)\,\Omega_\rho\left(f;\ \sqrt{\alpha_n + 2\beta_n + \gamma_n}\right)$$

where $C(f)$ is the constant depending on f.

4. Examples and applications

Consider now a sequences of positive linear operators, satisfying the assumptions of the above theorems. First we recall a weighted Korovkin type theorem, which was proved (in a more general case) in [7].

Theorem B. *Let $\omega(x) = 1 + x^2$ and let B_n be a sequence of positive linear operators from $C_\omega(\mathbb{R}^+)$ to $B_\omega(\mathbb{R}^+)$. If*

$$\left\|B_n(t^m; x) - x^m\right\|_\omega \to 0 \quad \text{as } n \to \infty \text{ for } m = 0, 1, 2,$$

then for all functions $f \in C_\omega^k(\mathbb{R}^+)$

$$\|B_n f - f\|_\omega \to 0 \quad \text{as } n \to \infty.$$

Define a sequence of positive linear operators $(L_n)_{n\geq 1}$ by

$$L_n(f;\ x) = \rho^2(x)\sum_{k=0}^{\infty}\frac{f\left(\frac{k}{n}\right)}{\rho^2\left(\frac{k}{n}\right)}a_{k,n}(x), \tag{4.1}$$

where the nonnegative functions $a_{k,n}(x)$, $k = 0, 1, 2, \ldots, n = 1, 2, \ldots, x \in \mathbb{R}^+$ satisfy the following conditions:

(a) $\sum_{k=0}^{\infty} a_{k,n}(x) = 1$,

(b) $\lim_{n\to\infty} \left\| \sum_{k=0}^{\infty} \left(\frac{k}{n}\right)^m a_{k,n}(x) - x^m \right\|_{\omega} = 0$ for $m = 1, 2$.

A simple calculation shows that

$$L_n(1, x) - 1 = \rho^2(x) \left[\sum_{k=0}^{\infty} \frac{1}{\rho^2\left(\frac{k}{n}\right)} a_{k,n}(x) - \frac{1}{\rho^2(x)} \right],$$

$$L_n(\rho, x) - \rho(x) = \rho^2(x) \left[\sum_{k=0}^{\infty} \frac{1}{\rho\left(\frac{k}{n}\right)} a_{k,n}(x) - \frac{1}{\rho(x)} \right],$$

$$L_n\left(\rho^2, x\right) - \rho^2(x) = 0.$$

Since the functions $\frac{1}{\rho(x)}$ and $\frac{1}{\rho^2(x)}$ are bounded, Theorem B gives

$$\left\| \sum_{k=0}^{\infty} \frac{1}{\rho^2\left(\frac{k}{n}\right)} a_{k,n}(x) - \frac{1}{\rho^2(x)} \right\|_{\omega} \to 0,$$

$$\left\| \sum_{k=0}^{\infty} \frac{1}{\rho\left(\frac{k}{n}\right)} a_{k,n}(x) - \frac{1}{\rho(x)} \right\|_{\omega} \to 0 \quad \text{as } n \to \infty.$$

Therefore

$$\alpha_n = \|L_n(1, x) - 1\|_{\rho^2\omega} \to 0,$$

$$\beta_n = \|L_n(\rho, x) - \rho\|_{\rho^2\omega} \to 0,$$

$$\gamma_n = \left\| L_n\left(\rho^2, x\right) - \rho^2 \right\|_{\rho^2\omega} \to 0 \quad \text{as } n \to \infty.$$

Thus the assumptions of Theorem 3 are satisfied for the operators (4.1). Using Remark 1, we have

$$\|L_n(f, x) - f\|_{\rho^4\omega} \le C(f)\, \Omega_\rho\left(f;\ \sqrt{\alpha_n + 2\beta_n + \gamma_n}\right)$$

for each $f \in C_\rho^k\left(\mathbb{R}^+\right)$.

In conclusion, we shall state some consequences of Theorem 1.

Proposition 1. *Let $(L_n)_{n\ge 1}$ be a sequence of positive linear operators from $C_{\rho^2}\left(\mathbb{R}^+\right)$ into $B_{\rho^2}\left(\mathbb{R}^+\right)$, satisfying the conditions (3.1)–(3.3). Then*

$$\left\| L_n\left((\rho(t) - \rho(x))^2;\ x\right) \right\|_{\rho^3} \to 0, \tag{4.2}$$

$$\|L_n(|\rho(t) - \rho(x)|;\ x)\|_{\rho^2} \to 0, \quad \textit{as } n \to \infty. \tag{4.3}$$

Proof. (4.2) is a direct consequence of the conditions (3.1)–(3.3) and (3.5). (4.3) follows from (4.2), since

$$L_n(|\rho(t) - \rho(x)|;\ x) \le \sqrt{L_n\left((\rho(t) - \rho(x))^2;\ x\right) L_n(1;\ x)}. \quad \square$$

Proposition 2. *Under the conditions of Proposition 1, we get*

$$\left\| L_n \frac{1}{\rho} - \frac{1}{\rho} \right\|_{\rho} \to 0, \qquad \left\| L_n \frac{1}{\sqrt{\rho}} - \frac{1}{\sqrt{\rho}} \right\|_{\rho} \to 0 \quad \textit{and} \quad \left\| L_n \frac{1}{\rho^2} - \frac{1}{\rho^2} \right\|_{\rho} \to 0 \quad \textit{as } n \to \infty.$$

Proof. A simple calculation shows that

$$\left|L_n\left(\frac{1}{\rho(t)};\ x\right)-\frac{1}{\rho(x)}\right|\leq\frac{L_n(|\rho(t)-\rho(x)|;\ x)}{\rho(x)}+\frac{|L_n(1;\ x)-1|}{\rho(x)},$$

$$\left|L_n\left(\frac{1}{\sqrt{\rho(t)}};\ x\right)-\frac{1}{\sqrt{\rho(x)}}\right|\leq\frac{L_n(|\rho(t)-\rho(x)|;\ x)}{\rho(x)}+\frac{|L_n(1;\ x)-1|}{\sqrt{\rho(x)}},$$

and

$$\left|L_n\left(\frac{1}{\rho^2(t)};\ x\right)-\frac{1}{\rho^2(x)}\right|\leq\frac{L_n\left(|\rho(t)-\rho(x)|\left(\frac{\rho(t)}{\rho(x)\rho^2(t)}+\frac{\rho(x)}{\rho(x)\rho^2(t)}\right);\ x\right)}{\rho(x)}+\frac{|L_n(1;\ x)-1|}{\rho^2(x)}$$
$$\leq 2\frac{L_n(|\rho(t)-\rho(x)|;\ x)}{\rho(x)}+\frac{|L_n(1;\ x)-1|}{\rho^2(x)}.$$

Therefore, by taking $g_0(t)=\frac{1}{\sqrt{\rho(t)}}$, $g_1(t)=\frac{1}{\rho(t)}$, $g_3(t)=\frac{1}{\rho^2(t)}$ we can write

$$\|L_n(g_k)-g_k\|_{\rho}\leq 2\,\|L_n(|\rho(t)-\rho(x)|;\ x)\|_{\rho^2}+\|L_n(1;\ x)-1\|_{\rho}$$

and the proof follows from the conditions (3.1) and (4.3). □

Remark 2. The Propositions 1 and 2 may be used for construction of a sequences of operators, satisfying the conditions (3.1)–(3.3) by replacing test functions 1, ρ, ρ^2 with 1, $\sqrt{\rho}$, ρ. For example, if B_n is the sequence of positive linear operators satisfying the conditions

$$\lim_{n\to\infty}\|B_n(t^m;x)-x^m\|_{\omega}=0,\quad m=0,1,2,$$

where $\omega(x)=1+x^2$, then

$$D_n(f;x)=\frac{\rho(x)}{\omega(x)}B_n\left(\omega(t)\frac{f(t)}{\rho(t)};x\right)$$

is the desired construction of linear positive operators, acting from $C_{\rho}\left(\mathbb{R}^+\right)$ into $B_{\rho}\left(\mathbb{R}^+\right)$.

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