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Existence and Uniqueness of Nonnegative Solutions of Quasilinear Equations in R^n

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0. INTRODUCTION

We shall consider the quasilinear elliptic equation

$$\operatorname{div}\{A(|Du|)\, Du\} + f(u) = 0, \qquad x \in R^n, \quad n \geqslant 2, \tag{1}$$

where $A(p)$ is a real positive continuous function defined for all $p > 0$ and $f(u)$ is a real continuous function defined for all $u \geqslant 0$. Our principal concern will be with ground states of (1), namely non-negative non-trivial continuously differentiable solutions $u = u(x)$ which tend to zero as the point x approaches infinity.

The special case where $A \equiv 1$, corresponding to the Laplace operator, is well-known and takes the classical form

$$\Delta u + f(u) = 0. \tag{2}$$

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When the function $f(u)$ is of class $C^{1+\varepsilon}$ for u near zero, and when $f(0)=0$, $f'(0)<0$, it has been shown by Gidas, Ni & Nirenberg that ground states of (2) are necessarily radially symmetric. In addition, under further appropriate conditions on f, the existence of radially symmetric ground states has been established by Strauss, Coleman, Glazer & Martin, Berestycki & Lions, Atkinson & Peletier, and Kaper & Kwong, while the uniqueness of such ground states is further treated in papers of Coffman, Cortázar *et al.*, Peletier & Serrin, McLeod & Serrin, Kaper & Kwong, Kwong, Kwong & Zhang and Yanagida.

Ground states for the general equation (1) can in most cases be expected to be radially symmetric, and we shall accordingly restrict our discussion to that case. More specifically, we shall establish both existence and uniqueness of *radially symmetric ground states of* (1) under appropriate conditions on the operator $A(p)$ and the function $f(u)$, and we shall also obtain a number of qualitative results concerning the behavior of solutions. Finally, occasionally at the expense of somewhat longer proofs, we shall always deal with the weakest hypotheses which we can treat.

Our results apply in particular to the important cases of the degenerate Laplace operator $A(p)=p^{m-2}$, $m>1$, the mean curvature operator $A(p)=(1+p^2)^{-1/2}$, and more generally to $A(p)=(1+p^2)^{-s/2}\,p^{m-2}$, where m and s are positive constants which satisfy suitable conditions (see below). Equation (1) can also be viewed as the Euler–Lagrange equation corresponding to the variational problem

$$\delta \int \{G(|Du|)-F(u)\}\ dx=0, \tag{3}$$

where

$$f(u)=\frac{dF}{du}, \qquad F(u)=\int_0^u f(t)\,dt,$$

and

$$A(p)=\frac{G_p(p)}{p}, \qquad G(p)=\int_0^p \rho A(\rho)\,d\rho.$$

Section 1 deals with general properties of radial solutions and radial ground states of (1), using ideas introduced by Peletier & Serrin in [PS1] and [PS2]. In Section 2, we give our existence results. The approach here is based on shooting methods, partially following work of Berestycki, Lions & Peletier and of Peletier & Serrin. It is also possible to develop variational methods to obtain existence, as in the works of Berestycki & Lions and of

Strauss; for several results in this direction, see the works of Citti and Kichenassamy & Smoller. Further work on existence and non-existence of solutions of (1), particularly for the case of the mean curvature operator, can be found in [NS1], [NS2], [PS3], and [APS]. In addition some related existence results for Monge–Ampère operators are proved in [F].

Section 3 is concerned with the uniqueness question for radial ground states. The essential tools for this purpose are the monotone separation theorem (see [PS1] and [PS2] for the case $A \equiv 1$) and a new identity for radial solutions of (1). In general form, this identity appears in Lemma 3.1.3. For the mean curvature operator $A(p) = 1/\sqrt{1+p^2}$, in particular, it reduces to the striking relation

$$\frac{d}{dr}\left\{r\left(\frac{p^2}{1+p^2}+\frac{2F(u)}{\sqrt{1+p^2}}-F^2(u)\right)^{1/2(n-1)}\right\}$$
$$=\left(\frac{p^2}{1+p^2}+\frac{2F(u)}{\sqrt{1+p^2}}-F^2(u)\right)^{-(n-3/2)/(n-1)}\left(\frac{p^2+2}{\sqrt{1+p^2}}F(u)-F^2(u)\right),$$

where $u = u(r)$ is a radial solution of (1) for this operator, and $p = |u'(r)|$. Another remarkable formula occurs for the degenerate Laplace operator $A(p) = p^{m-2}$; namely for this case, radial solutions of (1) satisfy the identity

$$\frac{d}{dr}\left\{r\left(p^m+\frac{m}{m-1}F(u)\right)^{(m-1)/m(n-1)}\right\}$$
$$=\frac{m}{m-1}\left(p^m+\frac{m}{m-1}F(u)\right)^{-(n-2+1/m)/(n-1)}F(u).$$

(Of course the above formulas are applicable only when they are meaningful, that is when $(1+p^2)^{-1/2} - F(u) \in (0, 1)$ for the first, and $p^m + (m/(m-1))\,F(u) > 0$ for the second.)

We turn now to a more detailed statement of our assumptions and main conclusions. Throughout the paper we shall assume, without further comment, the following hypotheses concerning the function $f(u)$ and the operator $A(p)$:

(H1) f is continuous on $[0, \infty)$ and $f(0) = 0$,

(H2) A is continuous on $(0, \infty)$ and $pA(p) \to$ as $p \to 0$,

(H3) $pA(p)$ is strictly increasing for $p > 0$.

The continuity assumptions in (H1) and (H2) are essentially minimal, and indeed one of the purposes of the paper is just that, to work with minimal hypotheses. The conditions on f at $u = 0$ and on A as $p \to 0$ are required in order for a ground state to exist. Assumption (H3) is simply the condition that (1) be weakly elliptic or alternately that the integrand in (3) be strictly convex in Du. Note also that (H2) and (H3) imply $A>0$ for $p>0$.

It is easy to check that conditions (H2) and (H3) are satisfied both for the Laplace operator and for the degenerate Laplace operator $A=p^{m-2}$, $m>1$. They also hold for the mean curvature operator and even for the general operator $(1+p^2)^{-s/2}p^{m-2}$ when $s\geqslant 0$, $m>1$, $m\geqslant s+1$.

In order to state our main *existence theorems*, we first define

$$H(p)=p^2A(p)-\int_0^p \rho A(\rho)\,d\rho \quad \text{for} \quad \rho>0, \quad H(0)=0, \tag{4}$$

or, equivalently, with

$$\Omega(p)=pA(p) \quad \text{for} \quad p>0, \quad \Omega(0)=0,$$

and with $p(\Omega)$ denoting the inverse of $\Omega(p)$,

$$H(p)=\int_0^{\Omega(p)} p(\Omega)\,d\Omega. \tag{5}$$

If $A(p)$ arises from the Euler–Lagrange equation of the variational integrand $G(p)$, as in (3), then $H(p)$ becomes the Legendre transform of $G(p)$, that is

$$H(p)=pG'(p)-G(p). \tag{6}$$

It is clear from (5) and (H3) that $\Omega(p)$ and $H(p)$ are strictly increasing and positive for $p>0$. In turn we can define

$$\Omega(\infty)=\lim_{p\to\infty}\Omega(p), \qquad H(\infty)=\lim_{p\to\infty}H(p) \quad \text{(possibly infinite)}.$$

For the operator $A=p^{m-2}$ we have in particular $H=((m-1)/m)\,p^m$ and $\Omega(\infty)=H(\infty)=\infty$, while, for the mean curvature operator, $H=1-1/\sqrt{1+p^2}$ and $H(\infty)=1$.

Our principal conclusion concerning existence, Theorem 1 in Section 2, can now be stated.

THEOREM A. *Suppose that there exist constants $\beta>0$ and $\gamma>\beta$ (γ possibly infinite) such that the following conditions hold*:

(a) $F(u)<0$ *for* $0<u<\beta$; $\quad F(\beta)=0$,

(b) $f(u)>0$ *for* $\beta\leqslant u<\gamma$; $\quad f(\gamma)=0$ *if* $\gamma<\infty$,

(c) $\max_{[0,\beta]} |F(u)| + F(\gamma) < H(\infty)$ *if* $H(\infty) < \infty$,

(d) $\liminf_{u\to\infty} \dfrac{H^{-1}(F(u))}{u} = 0$ *if* $H(\infty) = F(\gamma) = \infty$, $\Omega(\infty) < \infty$,

(e) $\liminf_{u\to\infty} \dfrac{H^{-1}(F(u))}{u} < \infty$ *if* $H(\infty) = \Omega(\infty) = F(\gamma) = \infty$.

Then there exists a radially symmetric ground state $u = u(r)$ *of* (1), *with central value* $u(0) \in (\beta, \gamma]$ *if* $\gamma < \infty$, *or* $u(0) \in (\beta, \infty)$ *if* $\gamma = \infty$. *Moreover* $u(r) \leqslant 0$ *for all* $r \geqslant 0$.

In the statement of this theorem for the case $\gamma = \infty$ we define $F(\gamma) = \lim_{u\to\infty} F(u)$, which certainly exists though possibly being infinite. Clearly $F(\gamma) = \infty$ can occur only in the case $\gamma = \infty$. Note also that H^{-1} is well-defined since H is strictly increasing, and that $H(\infty) = \infty$ whenever $\Omega(\infty) = \infty$.

As we shall see from Proposition 1 below, conditions (a), (b), and (c), while not quite necessary for solutions to exist, are nevertheless nearly necessary. On the other hand, neither smoothness nor even Lipschitz continuity is required for the function $f(u)$ in this proposition. Clearly, if $F(\gamma) < \infty$ and $H(\infty) = \infty$ then only the conditions (a), (b) are needed. The appearance of the value $H(\infty)$ in (c) and (d), and $\Omega(\infty)$ in (e) is of interest. For the Laplace operator or the degenerate Laplace operator $H(\infty)$ and $\Omega(\infty)$ are both infinite, so conditions (c), (d) and (e) are not needed if $\gamma < \infty$, though, of course, (e) must be used when $F(\gamma) = \infty$. On the other hand, for the mean curvature operator we have $H(\infty) = 1$. Therefore, in consequence of (c), the function $F(u)$ must be quite restricted in order to obtain existence; see also Proposition 1 below.

For the case of the Laplace operator condition (e) takes the form

$$\liminf_{u\to\infty} \frac{F(u)}{u^2} < \infty$$

and for the degenerate Laplace operator

$$\liminf_{u\to\infty} \frac{F(u)}{u^m} < \infty.$$

Even for the Laplace operator this condition for existence when $\gamma = F(\infty) = \infty$ seems to be new. In fact, for the Laplace operator in the case

$$\liminf_{u\to\infty} f(u) > 0,$$

(so also $\gamma = F(\gamma) = \infty$) it was shown in [KK4] that existence holds if (a), (b) and

$$\text{(e}'\text{)} \quad \frac{f(u)}{u-\beta} \text{ is nonincreasing for } u > \beta$$

are satisfied, and f is Lipschitz continuous on $(0, \infty)$. This result clearly falls as a special case under (e) in Theorem A.

Various properties of ground state solutions of (1) are derived in Section 1, several of which are worth noting here. We first observe that radial solutions $u = u(r)$ of (1) satisfy the differential equation

$$(A(|u'|)\, u')' + \frac{n-1}{r} A(|u'|)\, u' + f(u) = 0, \qquad r > 0.$$

PROPOSITION 1. *Let $u = u(r)$ be a radially symmetric ground state of* (1), *under the hypotheses* (H1)–(H3) *on $f(u)$ and $A(p)$. Then*

$$\begin{aligned} &\text{(i)} \quad f(u(0)) \geqslant 0 \\ &\text{(ii)} \quad F(u(0)) > 0 \\ &\text{(iii)} \quad u'(r) \leqslant 0 \qquad \textit{for} \quad 0 < r < \infty \\ &\text{(iv)} \quad u'(r) \to 0 \qquad \textit{as} \quad r \to \infty. \end{aligned}$$

If $F(u) \leqslant 0$ for $0 < u < \beta$ then also

$$\text{(v)} \quad \max_{[0,\beta]} |F| < H(\infty).$$

If $F(u) \leqslant 0$ at every point u (*in the range of a solution*) *where $f(u) = 0$, then equality cannot hold in* (i), *while* (iii) *can be improved to $u'(r) < 0$ for all $r > 0$ such that $0 < u(r) < u(0)$.*

The asymptotic behavior (iv) can be considerably strengthened in case conditions (A2) and (F1) below are satisfied; see Lemmas 3.2.1 and 3.2.2.

By condition (iii), if $u(\bar{r}) = 0$ for some $\bar{r} > 0$ then $u(r) \equiv 0$ for $r > \bar{r}$, so that the solution has compact support. The next result gives a necessary and sufficient condition for this situation to occur (see Section 1.3).

PROPOSITION 2. *Assume that constant $c > 1$ and $\delta > 0$ exist such that*

$$f(u) \leqslant 0 \quad \textit{for} \ 0 < u < \delta, \qquad c \int_0^p \rho A(\rho)\, d\rho \leqslant p^2 A(p) \quad \textit{for} \ 0 < p < \delta.$$

Then radial ground states of (1) *have compact support if and only if*

$$\int_0 \frac{1}{H^{-1}(|F(t)|)}\, dt < \infty.$$

The final section of the paper concerns the *uniqueness* of radially symmetric ground states. The situation here is quite delicate, and accordingly we shall require further restrictions on the function $f(u)$ and the operator $A(p)$:

(A1′) $p^3 \Omega(p)\, \Omega'(p)$ is increasing (see Section 3, condition (A1), for a more general hypothesis),

(A2) $p^{2-m}A(p) \to 1$ as $p \to 0$, for some $m > 1$,

(A3) $\Omega'(p) \geqslant [\Omega(p)]^{\mu}$ for all p near 0 and some $\mu \in [0, 2)$,

(F1) $F(u) < 0$ for $0 < u < \beta$, $F(\beta) = 0$,

(F2) $f(u)$ is positive and non-increasing for $\beta < u < \gamma$,

(F3) $f(u)$ is locally Lipschitz continuous on (β, γ).

THEOREM B. *Assume* $n \geqslant 2$ *and that* $A(p)$ *is of class* C^1 *for* $p > 0$. *Suppose also that the hypotheses* (A1′), (A2), (A3) *and* (F1), (F2), (F3) *are satisfied. Then equation* (1) *cannot have more than one radially symmetric ground state* $u = u(r)$ *such that* $u(0) < \gamma$.

THEOREM C. *Let the hypotheses of Theorem* B *hold, except that*

(i) $\mu = 0$ *in* (A3);

(ii) $\gamma < \infty$ *in* (F2), (F3);

(iii) (β, γ) *is replaced by* $(\beta, \gamma]$ *in* (F3).

Then equation (1) *cannot have more than one radially symmetric ground state such that* $u(0) \leqslant \gamma$.

The requirement (F2) in Theorem B is fairly strong, and under appropriate conditions on the operator $A(p)$ can be significantly weakened. In particular if A satisfies (A1′)–(A3) and also

(A2′) $p^{2-\nu}A$ is non-decreasing on $(0, \infty)$, where $\nu > 1$ is constant,

then it is enough to suppose that f satisfies (F1) and (F3) and, in place of (F2), the weaker condition

(F2′) $f(u)/(u-\beta)^{\nu-1}$ is positive and non-increasing for $\beta < u < \gamma$.

Note that (A2′), (F2′) reduce simply to (F2) when $\nu \to 1$.

For the Laplace operator ($A \equiv 1$) we can take $m = 2$ in (A2) and $\mu = 0$ in (A3), and also $v = 2$ in (A2′). Then (F2′) becomes condition (S) in [PS1]. Conditions (A2′) and (A3) also are obeyed by the degenerate Laplace operator $A \equiv p^{m-2}$, with $v = m$ and with $\mu = 0$ for $1 < m \leqslant 2$ and $\mu = (m-2)/(m-1)$ for $m > 2$. For the mean curvature operator one can take $m = 2$ in (A2) and $\mu = 0$ in (A3) exactly as for the Laplace operator. On the other hand (A2′) is not satisfied for any $v > 1$. Consequently (F2′) cannot be directly used in this case (but see the footnote in the introductory part of Section 3). Finally, condition (A1′) holds for the degenerate Laplace operator for all $m > 1$ and $n \geqslant 2$, and also for the mean curvature operator—see the introduction to Section 3.

The fact that f in Theorem B needs to be Lipschitz continuous *only on* (β, γ) is worth noting, both because the natural expectation is that if Lipschitz continuity is needed at all it should be for the whole range of values $(0, \gamma)$ which a solution might encounter, and also because in previous uniqueness results for equation (2) Lipschitz continuity or even continuous differentiability were required at all values $u \in (0, \gamma)$. Thus Theorem B is, even for equation (2), a generalization of earlier work. Similar comments apply also for Theorem C.

Even after giving up both hypothesis (A3) and the Lipschitz continuity of f, a result of considerable interest still persists, describing the structure of the set of all ground states when uniqueness fails.

Proposition 3. *Assume* $n \geqslant 2$ *and that* $A(p)$ *is of class* C^1 *with* $\{pA(p)\}' > 0$ *for* $p > 0$. *Suppose hypotheses* (A1′), (A2), (F1) *and* (F2) *are satisfied* (*or, in place of* (F2), *the conditions* (A2′) *and* (F2′)). *If u and v are radially symmetric ground states of* (1) *with* $u(0), v(0) \leqslant \gamma$, *then necessarily*

$$u(0) = v(0).$$

Moreover if $u \not\equiv v$ *then* $u(r) \neq v(r)$ *for all* $r > 0$ *such that* $u(r) \in (0, \gamma)$. (*If u and v have compact support, then also* $\inf\{r > 0; u(r) = 0\} \neq \inf\{r > 0; v(r) = 0\}$.)

In other words, even if the radial ground state itself is not unique for a given function $f(u)$, *its central value is*. Note moreover that if either u or v does *not* have compact support then $u(r) \neq v(r)$ for all $r > 0$. Of course, if one adds to the hypotheses of Proposition 3 the assumption that the Cauchy problem for the equation satisfied by radial solutions of (1) has only a unique solution, then the ground state is unique. This happens in particular for Theorems B and C above.

Under the hypotheses of Theorem B, and with the help of Proposition 3, it is clear that the only way there could be *more than one* radial ground state of (1) with central value $u(0)$ in $(0, \gamma]$, $\gamma < \infty$, is that there are *no radial ground states* with $u(0) \in (0, \gamma)$ and *at least* two radial ground states with $u(0) = \gamma$.

This remark is of particular importance for the degenerate Laplace operator when $m > 2$, since Theorem C cannot be applied in this case because assumption (i) fails.

The examples of the Laplace operator, the degenerate Laplace operator and the mean curvature operator are sufficiently important to be considered separately. By combining our previous conclusions of existence and uniqueness we can state the following results for these cases.

COROLLARY 1. *Suppose $n \geqslant 2$. Assume conditions* (a), (b) *of Theorem* A *are satisfied, and that, for some constant $m > 1$,*

(c) $f(u)/(u-\beta)^{m-1}$ *is non-increasing for $\beta < u < \gamma$.*

There exists a radially symmetric ground state $u = u(r)$ of the equation

$$\operatorname{div}\{|Du|^{m-2}\, Du\} + f(u) = 0$$

with the following properties (*see Theorem A*):

$$\text{if} \quad \gamma = \infty, \quad \text{then} \quad u(0) \in (\beta, \infty),$$
$$\text{if} \quad \gamma < \infty, \quad \text{then} \quad u(0) \in (\beta, \gamma].$$

Suppose also that f is Lipschitz continuous on (β, γ). Then if $\gamma = \infty$ the solution is unique, and if $\gamma < \infty$ the solution is unique in the class where $u(0) < \gamma$.

If $f(u) \leqslant 0$ for small u, then the solution is positive for all $r \geqslant 0$ if and only if

$$\int_0 |F(t)|^{-1/m}\, dt = \infty.$$

COROLLARY 2. *Let the preceding hypotheses hold, except that* (c) *is replaced by*

$$(\mathrm{c}') \quad f \text{ is non-increasing for } \beta < u < \gamma$$

$$(\mathrm{c}'') \quad \max_{[0, \beta]} |F(u)| + F(\gamma) < 1.$$

Then there exists a radially symmetric ground state $u = u(r)$ of the equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) + f(u) = 0$$

with the same properties as in Corollary 1.

If $f(u) \leqslant 0$ for small u, then the solution is positive for all $r \geqslant 0$ if and only if

$$\int_0 |F(t)|^{-1/2}\, dt = \infty.$$

In both corollaries the solutions have the general properties listed in Propositions 1 and 2. Also, in the first corollary, when $m=2$, and in the second corollary, *all* ground states are necessarily radially symmetric if $f \in C^{1+\varepsilon}$ near $u=0$ and $f'(0)<0$. This follows at once from the symmetry results of Gidas, Ni & Nirenberg and from work of Franchi & Lanconelli.

Corollary 1 also shows the possibility in certain cases of ground states which are *not* radially symmetric, a point first raised by Kichenassamy & Smoller. That is, should a *compact support* radial ground state exist, then two or more of these, with translated origins and disjoint supports, would still constitute a ground state, but clearly not one which is radially symmetric. All the more, in such a case the family of non-radially symmetric ground states is non-denumerable. To obtain *existence* of a compact support ground state it is enough by Corollary 1 that the function $f(u)$ satisfy the condition

$$\int_0 |F(t)|^{-1/m}\, dt < \infty. \tag{7}$$

In particular, in the natural case when $f'(0)$ exists and is negative, condition (7) becomes simply $m>2$, as noted in [KS], Section 1 C (the argument there does not, however, quite conclusively demonstrate the existence of the requisite compact support ground states, since the final hypothesis of Theorem 1.2 is not easy to verify). Finally, remarkably enough, condition (7) can hold even for the Laplace operator, for example whenever

$$f(u) \leqslant -\text{const}.u^{1-\varepsilon}, \qquad \varepsilon > 0,$$

for small u, see [KK4]. This relation on course is not covered by the Gidas–Ni–Nirenberg theorem, since $f \notin \mathbf{C}^{1+\sigma}([0,\infty))$.

For the existence results of both Corollaries 1 and 2 we emphasize again that there is no need to assume that $f(u)$ is Lipschitz continuous, even though such an assumption is certainly necessary for uniqueness. At the same time, if the hypothesis of Lipschitz continuity is dropped in either of the above corollaries, the partial uniqueness result of Proposition 3 still remains true, and in particular the central value $u(0)$ is still unique. The weakening of the Lipschitz condition to the subinterval (β, γ), and the complete removal of this hypothesis in Proposition 3, are of course improvements of previous results even for the case of the Laplace operator.

A detailed table of contents is given at the beginning of the paper. Chapters II and III, respectively covering existence and uniqueness, can be read independently of each other. Chapter I, on the other hand, is fundamental for all further results.

Since we shall be dealing throughout with radially symmetric solutions of (1) it is possible to consider the dimension n simply as a real parameter greater than 1; we shall consistently follow this point of view in the sequel.

Some particular cases of the above results were announced earlier in two papers [FLS1] and [FLS2].

1. General Theory

1.1. *Preliminary Results*

A radially symmetric ground state $u = u(r)$ of equation (1) can be considered as a solution of the problem

$$\begin{gathered} (Au')' + \frac{n-1}{r} Au' + f(u) = 0, \qquad r > 0 \\ u'(0) = 0,\ u \geqslant 0 \quad \text{for } r > 0; \qquad u \to 0 \quad \text{as } r \to \infty; \qquad u \not\equiv 0, \end{gathered} \tag{$*$}$$

where $A = A(|u'|)$ and n is the dimension, which for our purposes may be considered any real number greater than 1.

As noted in the Introduction, we shall always maintain the hypotheses (H1)–(H3) without further comment. Moreover, to avoid confusion *we shall specifically define the function Au' to be* 0 *whenever $u' = 0$.*

Because of the possible singularity of A when $u' = 0$ (which occurs in particular for the degenerate Laplace operator, and equally because we do not require the operator $A(p)$ itself to be differentiable) it is necessary to be precise concerning the meaning of a solution of the differential equation in ($*$), namely

$$(Au')' + \frac{n-1}{r} Au' + f(u) = 0, \qquad r > 0. \tag{1.1.1}$$

Here we shall treat *classical* solutions, with the precise meaning that $u \in C^1([0, \infty))$ with $u'(0) = 0$, and $w = Au' \in C^1((0, \infty))$.

In this section we shall prove some simple but important identities for non-negative classical solutions of the equation (1.1.1). For the moment we shall not assume that $u \to 0$ as $r \to \infty$. Obvious specializations of these results also hold when the domain $[0, \infty)$ for u is replaced by some subinterval of $[0, \infty)$.

LEMMA 1.1.1. *If u is a classical solution of* (1.1.1), *then* $w = Au' \in C^1([0, \infty))$ *and*

$$w(r) = -\int_0^r \left(\frac{t}{r}\right)^{n-1} f(u(t))\,dt, \qquad w(0) = 0, \qquad w'(0) = -\frac{1}{n} f(u(0)).$$

Proof. Obviously w satisfies the equation

$$w' + \frac{n-1}{r} w + f(u) = 0, \qquad r > 0, \tag{1.1.2}$$

and $w(0) = 0$ because $u'(0) = 0$. Writing (1.1.2) in the form

$$(r^{n-1} w)' = -r^{n-1} f(u), \tag{1.1.3}$$

an easy integration now yields

$$r^{n-1} w(r) = -\int_0^r t^{n-1} f(u(t))\,dt,$$

whence also

$$w'(r) = -f(u(r)) + \frac{n-1}{r^n} \int_0^r t^{n-1} f(u(t))\,dt, \qquad r > 0.$$

By l'Hopital's rule the right hand side approaches $-f(u(0))/n$ as $r \to 0$, completing the proof.

Note further that

$$\lim_{r \to 0} \frac{w(r)}{r} = w'(0) = -\frac{1}{n} f(u(0))$$

so in particular w/r is bounded as $r \to 0$.

LEMMA 1.1.2. *Suppose u is a classical solution of* (1.1.1). *Then for any pair of numbers r_0, $r \geqslant 0$ we have*

$$H(p) + F(u) = H(p_0) + F(u_0) - (n-1) \int_{r_0}^r Au'^2 \frac{dr}{r}, \tag{1.1.4}$$

where $H(p)$ is defined by (4),

$$F(u) = \int_0^u f(t)\,dt$$

is the primitive of f vanishing at $u = 0$, and $u = u(r)$, $p = |u'(r)|$, $u_0 = u(r_0)$, $p_0 = |u'(r_0)|$.

Proof. We note first that $Au'^2/r = (w/r)\,u'$, so this quantity is locally integrable on $[0, \infty)$. Now put

$$I(r) = H(p) + F(u) + (n-1)\int_{r_0}^{r} Au'^2 \frac{dr}{r}.$$

We must show that $dI/dr = 0$ when $r > 0$. But

$$\frac{dF}{dr} = f(u)\,u', \qquad \frac{d}{dr}\int_{r_0}^{r} Au'^2 \frac{dr}{r} = \frac{Au'^2}{r},$$

and by (5)

$$\frac{dH}{dr} = \frac{d}{dr}\int_0^{\Omega(p)} p(\Omega)\,d\Omega = u'w' = u'(Au')'$$

since $\Omega(p) = w$ sign u', and so $d\Omega/dr = (dw/dr)$ sign u'. Hence, because u satisfies (1.1.1) when $r > 0$,

$$\frac{d}{dr}\,I(r) = \left((Au')' + \frac{n-1}{r}\,Au' + f(u)\right)u' = 0.$$

LEMMA 1.1.3. *Suppose $n > 1$ and let u be a classical solution of* (1.1.1). *If $\bar{r}$ $(\geqslant 0)$ is a critical point of u, then either*

$$u(r) \leqslant u(\bar{r}) \text{ for } r > \bar{r}, \qquad \text{and} \qquad f(u(\bar{r})) \geqslant 0$$

or

$$u(r) \geqslant u(\bar{r}) \text{ for } r > \bar{r}, \qquad \text{and} \qquad f(u(\bar{r})) \leqslant 0.$$

Proof. Let $\bar{r}$ be a critical point of u. For contradiction, we suppose that there exist $r_1, r_2 \geqslant \bar{r}$ such that $u(r_1) > u(\bar{r})$ and $u(r_2) < u(\bar{r})$. Then we have $u(\bar{r}) = u(\tilde{r})$ for some $\tilde{r}$ between r_1 and r_2. Obviously u is non-constant on $[\bar{r}, \tilde{r}]$. Now, from (1.1.4) with $r_0 = \bar{r}$ and $r = \tilde{r}$, we obtain

$$H(\tilde{p}) + (n-1)\int_{\bar{r}}^{\tilde{r}} \frac{Au'^2}{r}\,dr = 0.$$

Hence since $n > 1$ and because $H > 0$, $A > 0$ for $p > 0$ we get $u' \equiv 0$ on $[\bar{r}, \tilde{r}]$. This contradiction proves that $u(r) \leqslant u(\bar{r})$ for every $r > \bar{r}$ or $u(r) \geqslant u(\bar{r})$ for every $r > \bar{r}$. In the first case it is easy to see that $u'(\bar{r}) = 0$, $Au' = w(\bar{r}) = 0$, $w'(\bar{r}) = (Au')' \leqslant 0$. Hence by (1.1.2), or by the relation $w'(0) = -f(u(0))/n$ if $\bar{r} = 0$, we have $f(u(\bar{r})) \geqslant 0$. In the same way it follows in the second case that $f(u(\bar{r})) \leqslant 0$.

1.2. *Behavior of Solutions*

In this section we shall obtain some elementary but useful results concerning classical solutions of $(*)$. We maintain always the hypotheses (H1)–(H3).

LEMMA 1.2.1. *If u is a classical solution of $(*)$ then*

$$u' \to 0 \qquad as \quad r \to \infty$$

and

$$H(p) + F(u) = (n-1)\int_r^\infty \frac{Au'^2}{\rho}\,d\rho \geqslant 0 \tag{1.2.1}$$

$$(n-1)\int_0^\infty \frac{Au'^2}{r}\,dr = F(u(0)). \tag{1.2.2}$$

Proof. Let $r \to \infty$ in (1.1.4). Since $F(u)$ then tends to zero (recall $u \to 0$ as $r \to \infty$), and the integral tends to some limit, also $H(p)$ tends to a limit. Hence $p \to$ limit, which must necessarily be zero since $u \to 0$ as $r \to \infty$. This proves the first part of the lemma. For (1.2.1), we again let $r \to \infty$ in (1.1.4), and then for (1.2.2) we set $r = 0$ in (1.2.1).

In view of (1.2.2) it is clear that $F(u(0)) > 0$ and so a fortiori $u(0) > 0$. Consequently if there exists a solution of $(*)$ then necessarily

$$B = \{u > 0 \mid F(u) > 0\} \neq \varnothing$$

In the sequel we shall put $\beta = \inf B$. The lemma shows that $u(0) > \beta$.

PROPOSITION 1.2.2. *A necessary condition for Problem $(*)$ to have a solution is that*

$$\max_{[0,\beta]} |F| < H(\infty). \tag{1.2.3}$$

Proof. Put $r_0 = 0$ in (1.1.4); since $H(p)$ is strictly increasing we get

$$\begin{aligned} F(u(0)) - F(u) &= H(p) + (n-1)\int_0^r \frac{Au'^2}{r}\,dr \\ &< H(\infty) + (n-1)\int_0^\infty \frac{Au'^2}{r}\,dr = H(\infty) + F(u(0)), \end{aligned}$$

by (1.2.2). Since the range of $u(r)$ strictly includes $(0, \beta]$, the result follows at once.

LEMMA 1.2.3. *Let u be a classical solution of* (*). *If $u(a)=0$ for some $a>0$ then $u\equiv 0$ for $r\geqslant a$.*

Proof. Clearly $u'(a)=0$. Consequently (1.2.1) with $r=a$ yields

$$\int_a^\infty \frac{Au'^2}{r}\,dr=0.$$

Hence $u'\equiv 0$ and $u\equiv 0$ for $r\geqslant a$.

LEMMA 1.2.4. *Let u be a classical solution of* (*). *Then $r=0$ is a maximum of u and $u'(r)\leqslant 0$ for $r\geqslant 0$.*

Proof. By Lemma 1.1.3, it is evident that the critical point $r=0$ must be a maximum of u for otherwise the condition $u\to 0$ at infinity would be impossible.

Suppose that $u'(\rho)>0$ for some $\rho>0$. Since $u(\rho)\leqslant u(0)$ by what was just shown, it follows that there must be some minimum point $\bar r$ contained in $(0,\rho)$. But then $u(r)\geqslant u(\bar r)$ for $r>\bar r$. If $u(\bar r)>0$ it would be impossible to have $u\to 0$ as $r\to\infty$. If $u(\bar r)=0$ then $u\equiv 0$ for $r\geqslant \bar r$ and so $u'(\rho)=0$, which is again a contradiction.

Even more, by using Lemma 1.1.3 we get

COROLLARY 1.2.5. *If u is a classical solution of* (*) *then $f(u(0))\geqslant 0$.*

The situation is somewhat simpler when f has the property that $F(u)\leqslant 0$ at all points u (in the range of a solution) for which $f(u)=0$ (this is the case, for example, when the conditions (a), (b) noted in the Introduction are satisfied and $u(0)\in(0,\gamma]$). To begin with, the conclusion of Corollary 1.2.5 can be strengthened to $f(u(0))>0$. Indeed if $f(u(0))=0$, then $F(u(0))\leqslant 0$, contradicting (1.2.2).

The possibility $u'(\rho)=0$, $u(\rho)>0$ for some $\rho>0$ can also be ruled out in Lemma 1.2.4. For if this happens, then by the result of Lemma 1.2.4 we must have $u(r)\geqslant u(\rho)$ for $r<\rho$ and $u(r)\leqslant u(\rho)$ for $r>\rho$. Consequently $w'(\rho)$ can be neither positive nor negative. Hence $w'(\rho)=0$. That is, $w(\rho)=w'(\rho)=0$ and so $f(u(\rho))=0$ by (1.1.2). But in this case again $F(u(\rho))\leqslant 0$. Hence by (1.2.1) we get (since $u'(\rho)=0$)

$$F(u(\rho))=(n-1)\int_\rho^\infty \frac{Au'^2}{r}\,dr=0$$

so that $u'(r)\equiv 0$, $u(r)\equiv u(\rho)$ for $r\geqslant\rho$, which contradicts the condition $u\to 0$ as $r\to\infty$. We state this result as

PROPOSITION 1.2.6. *Let u be a classical solution of* (*). *Suppose that $F(u)\leqslant 0$ at all values $u\leqslant u(0)$ for which $f(u)=0$. Then we have*

(i) $u'(r) < 0$ *for* $r > 0$ *as long as* $u(r) > 0$,

(ii) $f(u(0)) > 0$.

1.3. *Compact Support*

In this section we prove a sufficient condition and a related necessary condition for a solution of $(*)$ to have compact support.

For the Laplace operator the result of Proposition 1.3.1 was obtained by Peletier and Serrin [PS2], as well as a conclusion closely related to Proposition 1.3.2.

PROPOSITION 1.3.1. *Let u be a classical solution of* $(*)$. *If* $\beta > 0$ *and*

$$\int_0 \frac{1}{H^{-1}(|F(s)|)}\, ds < \infty, \tag{1.3.1}$$

then u has compact support.

Proof. Let R be such that $0 \leqslant u(r) < \beta$ for $r \geqslant R$. Then by (1.2.1) we have $p = |u'| > 0$ at all $r \geqslant R$ for which $u(r) > 0$. Hence by Lemma 1.2.3 either $u \equiv 0$ for all sufficiently large r or $u > 0$ and $u' < 0$ for $r \geqslant R$. In the first case we are done. Otherwise, again by (1.2.1),

$$u'(r) < -H^{-1}(|F(u(r))|) \qquad \text{for} \quad r \geqslant R.$$

Hence

$$\int_{u(r)}^{u(R)} \frac{ds}{H^{-1}(|F(s)|)} > r - R \qquad \text{for} \quad r \geqslant R.$$

Since $u(r) \to 0$ as $r \to \infty$, this inequality implies

$$\int_0^{u(R)} \frac{ds}{H^{-1}(|F(s)|)}\, ds = \infty. \tag{1.3.2}$$

The assertion is proved.

The next result is a partial converse.

PROPOSITION 1.3.2. *Assume that there exist constants $c > 1$, $\delta > 0$ and an increasing function Φ: $[0, \delta) \to \mathbb{R}$ with $\Phi(0) = 0$ such that*

(i) $|F(u)| \leqslant \Phi(u)$ *for every* $u \in (0, \delta)$

(ii) $c \int_0^p \rho A(\rho)\, d\rho \leqslant p^2 A(p)$ *for* $p \in (0, \delta)$.

Let u be a classical solution of (*). *Then* $u(r)>0$ *for every* $r>0$ *if*

$$\int_0 \frac{ds}{H^{-1}(\Phi(s))} = \infty. \tag{1.3.3}$$

Remarks. (a) Condition (i) is satisfied with $\Phi = |F|$ if there exists $\delta > 0$ such that $f(u) \leqslant 0$ for $0 \leqslant u \leqslant \delta$.

(b) Condition (ii) holds with $c = m$ for the degenerate Laplacian ($A(p) = p^{m-2}$, $m > 1$) and with any $c \in (1, m)$ for the generalized mean curvature operator ($A(p) = (1+p^2)^{-s/2}\, p^{m-2}$, $m > 1$).

(c) Condition (ii) is equivalent to $G(p) \leqslant \text{const.}\ H(p)$; see relations (3) and (6) in the Introduction.

Proof of Proposition 1.3.2. For contradiction we assume $u(r) = 0$ for some $r > 0$. From Lemmas 1.2.3 and 1.2.4 there exists a constant $a > 0$ such that $u'(r) \leqslant 0$ and $0 < u(r) \leqslant u(0)$ for $0 < r < a$, and $u(r) \equiv 0$ for $r \geqslant a$. Then by (1.2.1)

$$H(p) = -F(u) + (n-1) \int_r^a A u'^2 \frac{d\rho}{\rho}, \qquad r < a.$$

Now by (4) and hypothesis (ii)

$$p^2 A(p) = H(p) + \int_0^p \rho A(\rho)\, d\rho \leqslant H(p) + \frac{1}{c}\, p^2 A(p),$$

so

$$p^2 A(p) \leqslant \frac{c}{c-1}\, H(p), \tag{1.3.4}$$

provided of course that $p \in (0, \delta)$. It follows that if r_0 is sufficiently close to a then

$$H(p) \leqslant |F(u)| + \frac{(n-1)\, c}{c-1} \int_r^a H(p) \frac{d\rho}{\rho} \leqslant \Phi(u) + c_1 \int_r^a H(p)\, d\rho$$

for $r_0 < r < a$, where $c_1 = (n-1)\, c/(c-1)\, r_0$.

Applying Gronwall's inequality we obtain

$$H(p) \leqslant \Phi(u) + c_1 \int_r^a \Phi(u(t))\, e^{c_1(t-r)}\, dt.$$

Now observe that $t \to \Phi(u(t))$ is decreasing, because Φ is increasing and u is decreasing. Thus for $r_0 < r < a$, we have

$$H(p) \leqslant \Phi(u)\left(1 + c_1 \int_r^a e^{c_1(t-r)}\,dt\right) = e^{c_1(a-r)}\Phi(u) \leqslant e^{c_1(a-r_0)}\Phi(u) = c_2\Phi(u).$$

with $c_2 > 1$. This inequality in turn yields

$$-u' \leqslant H^{-1}(c_2\Phi(u)), \qquad r_0 < r < a.$$

By the following Lemma 1.3.3 there exists a constant $c_3 > 0$ such that

$$H^{-1}(c_2 q) \leqslant c_3 H^{-1}(q) \qquad \text{for} \quad 0 < q < \Phi(u_0), \quad u_0 = u(r_0).$$

Therefore

$$\frac{-u'}{H^{-1}(\Phi(u))} \leqslant c_3, \qquad r_0 < r < a.$$

Integrating this inequality on (r_1, r_2), $r_0 < r_1 < r_2 < a$, we obtain

$$\int_{u(r_2)}^{u(r_1)} \frac{ds}{H^{-1}(\Phi(s))} \leqslant c_3(r_2 - r_1).$$

When $r_2 \to a$ this yields

$$\int_0^{u(r_1)} \frac{ds}{H^{-1}(\Phi(s))} \leqslant c_3(a - r_1).$$

Because $u(r_1) > 0$, we have obtained a contradiction with the hypothesis (1.3.3). The proof is complete.

LEMMA 1.3.3. *If the hypothesis* (ii) *of the preceding proposition is satisfied, then for every fixed* $\theta > 1$ *and* $q_0 > 0$, *there exists a positive constant* d *such that*

$$H^{-1}(\theta q) \leqslant d H^{-1}(q) \qquad \textit{for every} \quad q \in (0, q_0).$$

Remark. For the special case $A(p) = p^{m-2}$, $m > 1$, we have $H(p) = (m/(m-1))\,p^m$ so that

$$\frac{H^{-1}(\theta q)}{H^{-1}(q)} = \left(\frac{\theta q}{q}\right)^{1/m} = \theta^{1/m}$$

so we can take $d = \theta^{1/m}$. For a general function A the proof is more complicated.

Proof. Put

$$\varphi(p)=\int_0^p \rho A(\rho)\, d\rho.$$

For every $p\in(0,\delta)$ we have by hypothesis (ii)

$$\varphi(p)\leqslant\frac{1}{c}p^2A(p)=\frac{1}{c}p\varphi'(p),$$

or equivalently, putting $q=\varphi(p)$,

$$\frac{(\varphi^{-1})'(q)}{\varphi^{-1}(q)}\leqslant\frac{1}{cq}\qquad\text{if}\quad 0<q<\varphi(\delta).$$

An integration on $(q, \lambda q)$, $\lambda>1$, yields

$$\log\frac{\varphi^{-1}(\lambda q)}{\varphi^{-1}(q)}\leqslant\frac{1}{c}\log\lambda\qquad\text{if}\quad 0<q<\frac{\varphi(\delta)}{\lambda}. \tag{1.3.5}$$

On the other hand, using (ii) and (1.3.4)

$$\varphi(p)\leqslant\frac{1}{c}p^2A(p)\leqslant\frac{H(p)}{c-1}=c'H(p),\qquad c'=\frac{1}{c-1}.$$

Therefore, putting $q=\varphi(p)/c'$ we get

$$H^{-1}(q)\leqslant\varphi^{-1}(c'q)\qquad\text{if}\quad 0<q<\varphi(\delta)/c'. \tag{1.3.6}$$

Moreover, since $pA(p)$ is positive and increasing,

$$\varphi(p)\geqslant\int_{p/2}^p \rho A(\rho)\, d\rho\geqslant\left(\frac{p}{2}\right)^2 A\left(\frac{p}{2}\right)\geqslant H\left(\frac{p}{2}\right),$$

by (4). Hence, putting $q=\varphi(p)$,

$$H^{-1}(q)\geqslant\tfrac{1}{2}\varphi^{-1}(q). \tag{1.3.7}$$

Then for $0<q<\varphi(\delta)/c'\theta$ we have from (1.3.6) and (1.3.7)

$$\frac{H^{-1}(\theta q)}{H^{-1}(q)}\leqslant 2\,\frac{\varphi^{-1}(c'\theta q)}{\varphi^{-1}(q)}.$$

If $c'\theta\leqslant 1$ then the last ratio is $\leqslant 1$, while if $c'\theta>1$ it is $\leqslant(c'\theta)^{1/c}$ by (1.35). On the other hand $H^{-1}(\theta q)/H^{-1}(q)$ is bounded for $\varphi(\delta)/c'\theta\leqslant q<q_0$. Thus in all cases $H^{-1}(\theta q)/H^{-1}(q)$ is bounded, which completes the proof.

COROLLARY. *Assume $\beta > 0$ and that* (i) *in Proposition* 1.3.2 *holds. Suppose also that $p^{2-m}A(p) \to 1$ as $p \to 0$ for some $m > 1$. Then $u(r) > 0$ for every $r > 0$ if*

$$\int_0 \frac{ds}{(\Phi(s))^{1/m}} = \infty$$

and only if

$$\int_0 \frac{ds}{|F(s)|^{1/m}} = \infty.$$

Proof. From (4) and L'Hopital's rule we get $p^{-m}H(p) \to 1 - 1/m$ as $p \to 0$, and the results follow at once from Propositions 1.3.1 and 1.3.2.

If $F(u)$ is monotone for all sufficiently small u, then by taking $\Phi(s) = |F(s)|$ we get necessary and sufficient conditions for compact support.

2. EXISTENCE

In this chapter, we consider the existence question for Problem $(*)$.

THEOREM I. *Assume that, for suitable constants $0 < \beta < \gamma$, $\gamma \leqslant \infty$,*

(a) $F(u) < 0$ *for* $0 < u < \beta$; $\quad F(\beta) = 0$,

(b) $f(u) > 0$ *for* $\beta \leqslant u < \gamma$; $\quad f(\gamma) = 0$ *if* $\gamma < \infty$,

(c) $F(\gamma) + \max\limits_{[0,\beta]} |F| < H(\infty)$ *if* $H(\infty) < \infty$,

(d) $\liminf\limits_{u \to \infty} \dfrac{H^{-1}(F(u))}{u} = 0$ *if* $H(\infty) = F(\gamma) = \infty$ *and* $\Omega(\infty) < \infty$,

(e) $\liminf\limits_{u \to \infty} \dfrac{H^{-1}(F(u))}{u} < \infty$ *if* $\Omega(\infty) = H(\infty) = F(\gamma) = \infty$.

Then $(*)$ has a solution with $u(0) \in (\beta, \gamma]$ and $u' \leqslant 0$.

Remark. If $\Omega(\infty) = \infty$ then necessarily $H(\infty) = \infty$. We will in fact see in the sequel that

$$\lim_{p \to \infty} \frac{H(p)}{\Omega(p)} = \infty \qquad \text{if} \quad \Omega(\infty) = \infty$$

(see (2.1.15)). We note explicitly that condition (d) is non-empty. Indeed it is possible to have $\Omega(\infty)<\infty$ and $H(\infty)=\infty$, as can be seen from the example

$$A(p)=\frac{1}{1+\sqrt{1+p^2}},$$

since in this case

$$H(p)=\frac{1}{2}\left[\frac{p^2}{1+\sqrt{1+p^2}}-\log\frac{1+\sqrt{1+p^2}}{2}\right].$$

2.1. *Existence for Regular Operators*

In this section we prove Theorem I under the following extra conditions,

$$\begin{aligned}&A \text{ is of class } C^1 \text{ in } [0,\infty) \text{ and } \Omega'=pA'+A>0 \text{ for all } p\geqslant 0;\\ &f \text{ is Lipschitz continuous in } [0,\gamma] \text{ if } \gamma<\infty \ (\text{in } [0,\infty) \text{ if } \gamma=\infty).\end{aligned} \tag{2.11}$$

Even this special case of Theorem I is new and of interest of itself.

Consider the Cauchy problem

$$\begin{gathered}(Av')'+\frac{n-1}{r}Av'+f(v)=0, \qquad r>0\\ v(0)=\xi, \qquad v'(0)=0\end{gathered} \tag{2.1.2}$$

with $\xi\in[\beta,\gamma)$. The existence of a unique solution of this Cauchy problem, at least in some neighborhood of $r=0$, follows from Propositions A.1 and A.2 in the Appendix (here we make strong use of (2.1.1)). For convenience we shall write $w=Av'$, so that $w(0)=0$ and $w'(0)=-f(\xi)/n<0$ (as in Lemma 1.1.1). Thus $v'<0$ and $v<\xi$ in some interval to the right of zero. The solution can be continued either for all $r>0$, with $v(r)>0$ and $v'(r)<0$, or else reaches a first point $r=R$ where $v(R)=0$, $v'(R)\leqslant 0$ or where $v'(R)=0$ and $v(R)>0$. To prove this, note first that, by (1.1.4) and the fact (see (b)) that $F(u)$ is positive and increasing for $u>\beta$,

$$H(|v'|)\leqslant F(\xi)-F(v)\leqslant F(\xi)+\max_{[0,\beta]}|F|=M(\xi) \tag{2.1.3}$$

as long as v exists and $0\leqslant v\leqslant\xi$. If $H(\infty)=\infty$, or by (c) if $H(\infty)<\infty$, we have $M<H(\infty)$. Hence (2.1.3) implies

$$|v'|\leqslant H^{-1}(M) \tag{2.1.4}$$

as long as v exists and $0\leqslant v\leqslant\xi$. This being shown, the result now follows by the standard continuation theory of ordinary differential equations. In what follows we shall assume, except where otherwise explicitly stated, that v is continued *exactly* until a first point is reached where either $v' = 0$, $v > 0$ or $v = 0$. If no such point occurs then we assume v is continued for all $r > 0$ (of course with $v' < 0$).

Now put

$$I^+ = \{\xi \in [\beta, \gamma) \mid \inf v > 0\}$$

$$I^- = \{\xi \in [\beta, \gamma) \mid v(R) = 0 \text{ for some finite } R > 0\}.$$

Clearly I^+ and I^- are disjoint. We shall show that both the sets I^+ and I^- are open in $[\beta, \gamma)$ and non-empty. Once this is done the theorem follows at once, for then $I^+ \cup I^-$ cannot cover $[\beta, \gamma)$ and so there exists some $\bar{\xi} \in [\beta, \gamma)$ which is neither in I^+ nor I^-. The corresponding solution v is decreasing, but neither reaches $v = 0$ nor remains bounded from zero. Consequently it must exist for all $r > 0$ and be a (non-compact support) solution of $(*)$. The proof, then, consists of the following four steps:

Step 1. I^- is open,

Step 2. I^+ is open,

Step 3. I^+ is non-empty,

Step 4. I^- is non-empty.

Step 1. I^- is open. Let $\bar{\xi} \in I^-$. Then the solution $\bar{v}$ vanishes at $r = R$, and by the uniqueness of the Cauchy problem for smooth functions A and f, necessarily $\bar{v}'(R) < 0$. The solution can be extended to values $\bar{v} < 0$ by defining $f(u) \equiv 0$ for $u < 0$. Then for all nearby values of ξ the solution also must reach points where $v < 0$. Clearly for these values of ξ we have $v'(r) < 0$ for $r > 0$. Consequently all such values are in I^-, and I^- is open.

Step 2. I^+ is open. Let $\bar{\xi} \in I^+$. Then for the corresponding solution $\bar{v}$ either there exists $R > 0$ such that $\bar{v}(R) > 0$ and $\bar{v}'(R) = 0$, or $\bar{v}'(r) < 0$ on $0 < r < \infty$ and $\lim_{r \to \infty} v(r) > 0$. If $\bar{v}$ has a vanishing derivative at $r = R$, then necessarily $\bar{w}(R) = 0$, $\bar{w}'(R) > 0$ (if $\bar{w}'(R) = 0$ we would have $f(\bar{v}(R)) = 0$ and $\bar{v} \equiv \bar{v}(R)$ by uniqueness). Hence $\bar{v}$ can be continued to larger values of r with $\bar{v}' > 0$. Then for all nearby values of ξ the solution also can be continued to reach values $v' > 0$ *before* reaching $v = 0$. In turn these solutions must have some (possibly different) value $R > 0$ such that $v(R) > 0$, $v'(R) = 0$, and $v'(r) < 0$ for $0 < r < R$. Consequently all such values are in I^+.

These remains the case that $\bar{v}'(r) < 0$ on $0 < r < \infty$, with $\lim_{r \to \infty} \bar{v}(r) = l > 0$. As in the proof of Lemma 1.2.1, it is clear that $\bar{v}'(r) \to 0$ as $r \to \infty$. Hence $\bar{w}(r) \to 0$ as $r \to \infty$ and so by (1.1.2) also $\bar{w}'(r) \to$ limit, necessarily zero, as $r \to \infty$. Hence $f(l) = 0$ by (1.1.2).

Since $f(u)>0$ for $\beta \leqslant u<\gamma$, and since $0<l<\xi<\gamma$, it follows that $0<l<\beta$ and so $F(l)<0$ by (a).

By (1.1.4) applied to $\bar{v}$, with $r_0=0$ and $r_1 \to \infty$, we get

$$(n-1)\int_0^{\infty} \frac{\bar{A}\bar{v}'^2}{r}\,dr+F(l)-F(\bar{\xi})=0$$

where $\bar{A}=A(|\bar{v}'|)$. Choose $\bar{R}>0$ so large that

$$(n-1)\int_{\bar{R}}^{\infty} \frac{\bar{A}\bar{v}'^2}{r}\,dr<\frac{1}{4}|F(l)|.$$

In turn, choose $\delta>0$ so that when $\xi \in [\beta, \gamma)$ and $|\xi-\bar{\xi}|<\delta$ we have

(1) $|F(\xi)-F(\bar{\xi})|<\frac{1}{4}|F(l)|$,

(2) the corresponding solution v exists at least for $0 \leqslant r \leqslant \bar{R}$ and satisfies $v(r)>0$ when $0 \leqslant r \leqslant \bar{R}$,

(3) $(n-1)\left|\int_0^{\bar{R}} \frac{Av'^2-\bar{A}\bar{v}'^2}{r}\,dr\right|<\frac{1}{4}|F(l)|$.

Conditions (2) and (3) are possible because of the continuous dependence of solutions of the Cauchy problem on the initial conditions, together with the fact that w/r is uniformly bounded (see Proposition A3 in the Appendix) and v' is uniformly bounded (by (2.1.4)).

Now consider any $\xi \in [\beta, \gamma)$ with $|\xi-\bar{\xi}|<\delta$, and let v be the corresponding solution of the Cauchy problem. For any $r \geqslant \bar{R}$ for which v is defined, we have by (1.1.4) with $r_0=0$,

$$\begin{aligned}
F(\xi)-F(v(r)) &\geqslant (n-1)\int_0^{r} \frac{Av'^2}{r}\,dr \geqslant (n-1)\int_0^{\bar{R}} \frac{Av'^2}{r}\,dr \\
&\geqslant (n-1)\int_0^{\bar{R}} \frac{\bar{A}\bar{v}'^2}{r}\,dr-\frac{1}{4}|F(l)| \\
&\geqslant (n-1)\int_0^{\infty} \frac{\bar{A}\bar{v}'2}{r}\,dr-\frac{1}{2}|F(l)| \\
&= F(\bar{\xi})-F(l)-\frac{1}{2}|F(l)| \geqslant F(\xi)+\frac{1}{4}|F(l)|
\end{aligned}$$

Hence $-F(v(r)) \geqslant \frac{1}{4}|(F(l)|$ and therefore of course $\xi \in I^+$, possibly with $v'(R)=0$, $v(R)>0$ at some value $R>\bar{R}$. This completes Step 2.

Step 3. $I^+ \neq \emptyset$. We shall show that $\beta \in I^+$. Let v denote the corresponding solution of the Cauchy problem. Applying (1.1.4) with $r_0 = 0$ there results (since $F(\beta) = 0$)

$$H(|v'(r)|) + (n-1) \int_0^r \frac{Av'^2}{r} \, dr + F(v(r)) = 0. \tag{2.1.5}$$

Thus $F(v(r)) < 0$ and so $v(r) > 0$ for all r for which v is defined.

If $v'(R) = 0$, $v(R) > 0$ for some $R > 0$ then obviously $\beta \in I^+$. Otherwise $v' < 0$ and $v > 0$ for all $r > 0$, and $v \to l$, $v' \to 0$ as $r \to \infty$ (as before). Letting $r \to \infty$ in (2.1.5) yields

$$F(l) = -(n-1) \int_0^\infty \frac{Av'^2}{r} \, dr < 0$$

so that $l > 0$. Thus again $\beta \in I^+$, completing the proof of Step 3.

Step 4. $I^- \neq \emptyset$. Consider any $\xi \in [\beta, \gamma)$. The corresponding solution v is defined on an interval $[0, R)$, with $R = R_\xi$ possibly infinite, and

$$\lim_{r \to R} v = m \geqslant 0, \qquad v'(r) < 0 \qquad \text{for} \quad r \in (0, R).$$

We assert to begin with that $f(m) \leqslant 0$. Indeed, if $m > 0$ and $R < \infty$ then clearly $v(R) = m$, $w(R) = 0$ and $w'(R) > 0$. Hence by (1.1.2) we have $f(m) < 0$. If $m = 0$ or $R = \infty$ then $f(m) = 0$ as in Step 2. It follows next from (b) that $m < \beta$ and $F(m) \leqslant 0$.

Let $\bar{\beta} \in (\beta, \gamma)$ be fixed. Suppose $\xi > \bar{\beta}$, and define $\bar{R} = \bar{R}(\xi) \in (0, R)$ by

$$v(\bar{R}) = \bar{\beta} \qquad (\text{note } \bar{\beta} > \beta > m).$$

Now suppose $\xi \notin I^-$. Then $\lim_{r \to R} v'(r) = 0$ (since the case $m = 0$, $R < \infty$ does not occur). Put $\bar{M} = \bar{M}(\xi) = \sup_{[\bar{R}, R]} |v'| = |v'(R_0)|$ where $R_0 = R_0(\xi) \in [\bar{R}, R)$. The identity (1.1.4) in $[R_0, R]$ (or in $[R_0, \infty)$ if $R = \infty$) gives

$$\begin{aligned} H(\bar{M}) &= F(m) - F(v(R_0)) + (n-1) \int_{R_0}^R Av'^2 \frac{d\rho}{\rho} \\ &\leqslant \sup_{[0, \beta]} |F| + (n-1) \frac{\Omega(\bar{M})}{\bar{R}} \int_{R_0}^R (-v') \, d\rho \\ &\leqslant \sup_{[0, \beta]} |F| + (n-1) \frac{\Omega(\bar{M})}{\bar{R}} \bar{\beta}. \end{aligned} \tag{2.1.6}$$

On the other hand, the same identity in $[\bar{R}, R]$ gives

$$F(\bar{\beta}) = F(m) - H(|v'(\bar{R})|) + (n-1)\int_{\bar{R}}^{R} Av'^2 \frac{d\rho}{\rho} \leqslant (n-1)\frac{\Omega(\bar{M})}{\bar{R}}\bar{\beta}. \quad (2.1.7)$$

Defining

$$C = C(\Omega) = \max\left\{\Omega(2), \frac{\Omega(2)}{\Omega(2)-\Omega(1)}\right\} \quad (2.1.8)$$

we assert that

$$\Omega(p) \leqslant C(1 + H(p)) \quad (2.1.9)$$

for every $p > 0$. The relation (2.1.9) is obvious if $0 < p \leqslant 2$. On the other hand, if $p \geqslant 2$,

$$H(p) \geqslant \int_1^p \rho\, d\Omega(\rho) \geqslant \Omega(p) - \Omega(1) \geqslant \Omega(p)\left(1 - \frac{\Omega(1)}{\Omega(2)}\right).$$

as required. Now by (2.1.6) and (2.1.9) with $p = \bar{M}$,

$$\Omega(\bar{M})\left(1 - C\frac{(n-1)\,\bar{\beta}}{\bar{R}}\right) \leqslant C(1 + \sup_{[0,\beta]} |F|) \quad (2.1.10)$$

We can now prove the following important

LEMMA 2.1.1. *Suppose*

$$\bar{R}(\xi) > 2C\frac{(n-1)\,\bar{\beta}}{\bar{m}}, \quad (2.1.11)$$

where

$$\bar{m} = \min\{1, F(\bar{\beta})/(1 + \sup_{[0,\beta]} |F|)\}. \quad (2.1.12)$$

Then $\xi \in I^-$.

Proof. If $\xi \notin I^-$, then by (2.1.10), (2.1.11) and (2.1.7) we get

$$\Omega(\bar{M}) < 2C(1 + \sup_{[0,\beta]} |F|) < \frac{F(\bar{\beta})\,\bar{R}}{(n-1)\,\bar{\beta}} \leqslant \Omega(\bar{M})$$

which is contradiction.

When γ is finite, the existence of a value ξ satisfying (2.1.11) is an obvious consequence of the continuous dependence of v on ξ. Indeed, in this case $v \to \gamma$ as $\xi \to \gamma$ uniformly on every bounded subset of $[0, \infty)$, and so there exists $\xi_0 \in [0, \gamma)$ such that (2.1.11) holds. Thus $I^- \neq \emptyset$ and Theorem I for the regular case and for $\gamma < \infty$ is proved.

Next suppose that $\gamma = \infty$ and either (c) or (d) holds. From (1.1.4) with $r_0 = 0$, we obtain for any $\xi \in [\beta, \infty)$

$$H(|v'(r)|) < F(\xi) \tag{2.1.13}$$

for every r in $[0, \bar{R}]$, because of course $v(r) \geqslant \bar{\beta}$ when $r \in [0, \bar{R}]$ and so $F(v(r)) > 0$. Therefore, since

$$\frac{\xi - \bar{\beta}}{\bar{R}} = \frac{v(0) - v(\bar{R})}{\bar{R}} \leqslant \sup_{[0, \bar{R}]} |v'|$$

and because $F(\xi) < F(\infty) < H(\infty)$ in case (c), and $H(\infty) = \infty$ in case (d), we have

$$\bar{R} \geqslant \frac{\xi - \bar{\beta}}{H^{-1}(F(\xi))}. \tag{2.1.14}$$

Hence, by conditions (c) or (d), we can choose $\xi_0 \in [\beta, \infty)$ such that $\bar{R} = \bar{R}(\xi_0)$ is so large that (2.1.11) holds. Thus $\xi_0 \in I^-$ by the lemma.

The case (*e*).

Assume finally that $\gamma = \infty$ and (e) holds. In this case, the proof that $I^- \neq \emptyset$ is quite long and will be divided into several steps. First, we need the following result:

If $\Omega(\infty) = \infty$, *then*

$$\lim_{p \to \infty} \frac{H(p)}{\Omega(p)} = \infty. \tag{2.1.15}$$

In fact, let $k > 0$ be fixed. For every fixed $p > k$ we obtain from (5)

$$H(p) \geqslant \int_k^p \rho \, d\Omega(\rho) \geqslant k(\Omega(p) - \Omega(k)).$$

Then

$$\liminf_{p \to \infty} \frac{H(p)}{\Omega(p)} \geqslant k \liminf_{p \to \infty} \left(1 - \frac{\Omega(k)}{\Omega(p)}\right) \geqslant k$$

for every $k > 0$. This proves (2.1.15).

In addition, a crucial role will be played by the following generalized Hadamard inequality; the standard Hadamard inequality is in fact the special case $A \equiv 1$, $\Omega \equiv p$, $H = \frac{1}{2}p^2$ (our attention was drawn to Hadamard's inequality by the work of Kaper and Kwong [KK4]).

LEMMA 2.1.2. *Let u be a continuously differentiable function defined on an interval $I = [r_0, R)$, $0 \leqslant r_0 < R \leqslant \infty$. Suppose $\Omega(\infty) = \infty$, $w = u'A(|u'|) \in \mathbf{C}^1(I)$, $u' < 0$ and $u'(r) \to 0$ as $r \to R$. Then*

$$H(\Omega^{-1}(M)) \leqslant DS. \tag{2.1.16}$$

where

$$M = \sup_I |w|, \qquad S = \sup_I |w'|, \qquad D = u(r_0) - u(R).$$

Proof. Let η be a point on I such that $|w(\eta)| = M$. For every $r \in I$, $r > \eta$, we have

$$S(r-\eta) \geqslant \int_\eta^r w'(s)\, ds = w(r) - w(\eta) = M - |w(r)|,$$

and hence

$$|w(r)| \geqslant M - S(r-\eta).$$

This inequality implies that $R \geqslant \eta + M/S$ and

$$|u'| \geqslant \Omega^{-1}(M - S(r-\eta)) \qquad \text{if} \quad \eta < r < \eta + M/S.$$

Thus we obtain

$$\begin{aligned} D &\geqslant \int_\eta^{\eta+M/S} |u'(s)|\, ds \geqslant \int_\eta^{\eta+M/S} \Omega^{-1}(M - S(r-\eta))\, dr \\ &\geqslant \int_0^M \Omega^{-1}(t)\frac{dt}{S} = \frac{1}{S}\int_0^{\Omega^{-1}(M)} p\, d\Omega(p) = \frac{H(\Omega^{-1}(M))}{S}, \end{aligned}$$

and the assertion is proved.

We can now go back to the solution v of the Cauchy problem (2.1.2). First we prove the following assertion.

Let $c^* > 0$ be such that $H^{-1}(F(\xi))/\xi < c^*$ and put $r^* = (2c^*)^{-1}$. Then

$$v(r^*) \geqslant \tfrac{1}{2}\xi. \tag{2.1.17}$$

In fact, as long as $v(r) > \beta$, from (1.1.4) with $r_0 = 0$ we obtain $H(p) \leqslant F(\xi)$, and hence $p \leqslant H^{-1}(F(\xi)) \leqslant c^*\xi$. Then

$$v(r) = \xi + \int_0^r v'(s)\, ds \geqslant \xi - c^*\xi r = \xi(1 - c^* r).$$

Hence (2.1.17) follows.

We are now able to prove an a priori estimate for v' on the interval $[\bar{R}, R)$, where $\bar{R}$ and R have been defined at the beginning of this step. More precisely, we have the following result.

Suppose $\xi \notin I^-$ and let $d > 0$ be such that

$$H(p) > 2(n-1)\, \bar{\beta} c^* \Omega(p) + \bar{\beta} \sup_{[0,\, \beta]} |f| \qquad \text{for} \quad p > d \tag{2.1.18}$$

(*note that d exists, by* (2.1.15)). *Then if $\xi \geqslant 2\bar{\beta}$ and $H^{-1}(F(\xi))/\xi < c^*$ we have*

$$\sup_{[\bar{R},\, R)} |v'| \leqslant d. \tag{2.1.19}$$

Indeed, from the equation (1.1.2)

$$w' + \frac{n-1}{r} w + f(u) = 0$$

we have

$$S \leqslant \frac{n-1}{\bar{R}} M + \sup_{[0,\, \beta]} |f|, \tag{2.1.20}$$

where we have used the notation of Lemma 2.2.2 (with $r_0 = \bar{R}$). Note that

$$D = v(\bar{R}) - v(r) \leqslant v(\bar{R}) = \bar{\beta}.$$

Hence, by Lemma 2.2.1, which clearly applies since $\xi \notin I^-$, we have

$$\bar{\beta} S \geqslant DS \geqslant H(\Omega^{-1}(M)).$$

Moreover $\bar{\beta} \leqslant \xi/2 \leqslant v(r^*)$, so that $\bar{R} \geqslant r^* = (2c^*)^{-1}$; thus by (2.1.20),

$$\bar{\beta} S \leqslant \bar{\beta}\{2(n-1)\, Mc^* + \sup_{[0,\, \beta]} |f|\}.$$

Then, by (2.1.18) (with $p = \Omega^{-1}(M)$), we must have

$$\Omega^{-1}(M) \leqslant d,$$

and hence (2.1.19) follows since $M = \Omega(\sup |v'|)$.

The last step is to show that if $\xi \notin I^-$ then

$$\frac{1}{2}\xi \leqslant \bar{\beta} + \Omega^{-1}\left(\left(\Omega(2d) + \frac{1}{(n-1)\,c^*}\sup_{[0,\beta]}|f|\right)\exp(2(n-1)\,c^*\bar{R})\right)\bar{R}. \tag{2.1.21}$$

To prove (2.1.21), put $N = 2\sup_{[0,\beta]}|f|$ and consider the Cauchy problem

$$\begin{gathered}(Au')' + \frac{n-1}{r^*}Au' - N = 0, \qquad r^* \leqslant r \leqslant \bar{R}\\ u(\bar{R}) = v(\bar{R}), \qquad u'(\bar{R}) = 2v'(\bar{R})\end{gathered} \tag{2.1.22}$$

An easy comparison argument proves that

$$v \leqslant u \qquad \text{on} \quad r^* < r < \bar{R}. \tag{2.1.23}$$

In fact $v = u$ at $r = r_1$ and $v < u$ in a left neighborhood of $\bar{R}$ (since $u'(r_1) = 2v'(r_1) < v'(r_1) < 0$). For contradiction suppose (2.1.23) does not hold; then there exists a point $\bar{r} \in (r^*, \bar{R})$ such that

$$u(\bar{r}) > v(\bar{r}), \qquad u'(\bar{r}) = v'(\bar{r}) < 0, \qquad u''(\bar{r}) \leqslant v''(\bar{r}).$$

Consequently at $r = \bar{r}$ we have

$$(Au')' \leqslant (Av')', \qquad \frac{n-1}{r^*}Au' \leqslant \frac{n-1}{\bar{r}}Av', \qquad -N < f(v).$$

From the equation in (2.1.22) these inequalities imply

$$(Av')' + \frac{n-1}{\bar{r}}Av' + f(v) > 0,$$

an obvious contradiction.

Now define

$$z = A(|u'|)\,u',$$

so that

$$z' + \frac{n-1}{r^*}z - N = 0, \qquad z(\bar{R}) = -\Omega(2\,|v'(\bar{R})|).$$

Then, if we put

$$U=\left[\Omega(2\,|v'(\bar{R})|)+\frac{r^*}{n-1}\,N\right]\exp((n-1)\,\bar{R}/r^*), \tag{2.1.24}$$

we have $|z|\leqslant U$ and hence

$$|u'|\leqslant \Omega^{-1}(U). \tag{2.1.25}$$

If we integrate (2.1.25) on the interval $[r,\bar{R}]$ there results

$$|u(r)-u(\bar{R})|\leqslant \Omega^{-1}(U)\,\bar{R},$$

which implies

$$u(r)\leqslant u(\bar{R})+\Omega^{-1}(U)\,\bar{R}=v(\bar{R})+\Omega^{-1}(U)\,\bar{R},\qquad r\in[r^*,\bar{R}]. \tag{2.1.26}$$

Combining (2.1.26) and (2.1.23) we get

$$v(r^*)\leqslant v(\bar{R})+\Omega^{-1}(U)\,\bar{R}=\bar{\beta}+\Omega^{-1}(U)\,\bar{R}. \tag{2.1.27}$$

Then from (2.1.17), (2.1.27), (2.1.24) and (2.1.29) we obtain (2.1.21).

LEMMA 2.1.3. *Suppose*

$$\frac{H^{-1}(F(\xi))}{\xi}<c^* \tag{2.1.28}$$

and

$$\frac{\xi}{2}>\bar{\beta}+\frac{2(n-1)\,C\bar{\beta}}{\bar{m}}\,\Omega^{-1}\left(U_1\exp\frac{4(n-1)^2\,Cc^*\bar{\beta}}{\bar{m}}\right), \tag{2.1.29}$$

where $U_1=\Omega(2d)+(1/((n-1)\,c^*))\sup_{[0,\beta]}|F|$. *Then* $\xi\in I^-$.

Proof. If $\xi\notin I^-$ then by Lemma (2.1.1) we get

$$R(\xi)\leqslant\frac{2(n-1)\,C\bar{\beta}}{\bar{m}}$$

(here $\bar{\beta}$ is any fixed number greater than β, while the numbers C, $\bar{m}$ and d are defined respectively by (2.1.8), (2.1.12) and (2.1.18)). Then (2.1.21) contradicts (2.1.29).

To complete the proof of Theorem I in the case when A and f are regular it is now enough to take

$$c^*=2\liminf_{u\to\infty}\frac{H^{-1}(F(u))}{u}$$

and then to choose ξ_0 (sufficiently large) so that (2.1.28) and (2.1.29) are satisfied. Then $\xi_0 \in I^-$ and I^- is not empty.

Remark. The solution v we have just obtained has the property

$$\beta < v(0) < \gamma. \tag{2.1.30}$$

Moreover when $\gamma = \infty$ and either (c) or (d) holds, we can choose $\Gamma > \bar{\beta}$ such that

$$\frac{H^{-1}(F(\Gamma))}{\Gamma - \bar{\beta}} < \frac{\bar{m}}{2(n-1)\,\bar{\beta}C} \tag{2.1.31}$$

where $\bar{m}$ is defined by (2.1.12). Then by (2.1.14) the inequality (2.1.11) is satisfied with $\bar{R} = \bar{R}(\Gamma)$. Hence $\Gamma \in I^-$. Analogously, in the case (e) if we choose Γ such that

$$\begin{aligned} &\frac{H^{-1}(F(\Gamma))}{\Gamma} < c^* \\ &\frac{\Gamma}{2} > \bar{\beta} + \frac{2(n-1)\,C\bar{\beta}}{\bar{m}}\,\Omega^{-1}(U_1 \exp(4(n-1)^2\, Cc^*\bar{\beta}/\bar{m})). \end{aligned} \tag{2.1.32}$$

Then by Lemma 2.1.3 we have $\Gamma \in I^-$. Thus in all cases, recalling that $\beta \in I^+$, this implies that there exists $\xi \in (\beta, \Gamma)$ such that $\xi \notin I^+ \cup I^-$. The corresponding solution of the Cauchy problem (2.1.2) is then defined for all $r > 0$ and is a solution of $(*)$ having the property

$$\beta < v(0) < \Gamma. \tag{2.1.33}$$

Obviously, we also obtain a solution of $(*)$ verifying (2.1.33) if in Theorem I we assume either (2.1.31) and (2.1.32) holds for some values $\beta < \bar{\beta} < \Gamma$ instead of either hypothesis (d) or (e), respectively.

2.2. *Existence for General Operators*

In this section we complete the proof of Theorem I without the extra conditions on A and f assumed in Section 2.1. The importance of this step lies not only in the minimal continuity behavior required of A and f, but also in the fact that degenerate operators can be considered. For example, the operator $A(p) \equiv |p|^{m-2}$, $m > 2$, fails to have $\Omega' > 0$ for $p = 0$ and thus does not satisfy (2.1.1) even though it is covered by Theorem I. Our proof is not entirely simple, but this is perhaps to be expected in view of the generality of the conclusion.

In what follows we may suppose without loss of generality that f is extended to have values for all $u \in \mathbb{R}$, with $f(u) \equiv 0$ for $u \leqslant 0$. We may also suppose without loss of generality that $f(u) \equiv 0$ for $u \geqslant \gamma$ if $\gamma < \infty$. Con-equently $F(u) = 0$ for $u \leqslant 0$ and $F(u) = F(\gamma)$ for $u \geqslant \gamma$ if $\gamma < \infty$.

For every $\varepsilon > 0$ we define

$$\Omega_\varepsilon(p) = \int_{\mathbb{R}} \Omega(p + \varepsilon t)\, J(t)\, dt + \varepsilon p, \qquad p \in \mathbb{R}$$

and

$$F_\varepsilon(u) = \int_{\mathbb{R}} F(u + \varepsilon(t-1))\, J(t)\, dt, \qquad u \in \mathbb{R}$$

where $\Omega(t) = tA(|t|)$, $t \in \mathbb{R}$, and J is a C^∞ function on $\mathbb{R}$ such that $0 \leqslant J \leqslant 1$, $J(t) = 0$ if $|t| \geqslant 1$, $J(t) = J(|t|) > 0$ if $|t| < 1$, and $\int_{\mathbb{R}} J(t)\, dt = 1$.

Clearly $\Omega_\varepsilon(p)$ is an odd C^∞ function on $\mathbb{R}$, with $\Omega'_\varepsilon(p) > 0$ and $\Omega_\varepsilon(p) \nearrow \infty$ as $p \to \infty$. If we put

$$A_\varepsilon(p) = \frac{\Omega_\varepsilon(p)}{p} \quad \text{if } p \neq 0, \qquad A_\varepsilon(0) = \lim_{p \to 0} \frac{\Omega_\varepsilon(p)}{p},$$

then A_ε is a C^∞ function which verifies the hypotheses (H2) and (H3) (see Introduction) and satisfies (2.1.1). In addition we have

$$\frac{d}{du} F_\varepsilon(u) = \int_{\mathbb{R}} f(u + \varepsilon(t-1))\, J(t)\, dt = \int_{\mathbb{R}} f(u - \varepsilon + \varepsilon t)\, J(t)\, dt \equiv f_\varepsilon(u);$$

clearly f_ε is a C^∞ function and $f_\varepsilon(0) = 0$. Moreover, since $F_\varepsilon(0) = 0$,

$$F_\varepsilon(u) = \int_0^u f_\varepsilon(t)\, dt.$$

It is easy to show that, for every $\varepsilon > 0$ sufficiently small, there exist $\gamma_\varepsilon \leqslant \infty$ and $\beta_\varepsilon \in (0, \gamma_\varepsilon)$, such that $F_\varepsilon(u) < 0$ for $0 < u < \beta_\varepsilon$, $F_\varepsilon(\beta_\varepsilon) = 0$, $f_\varepsilon(u) > 0$ for $\beta_\varepsilon \leqslant u < \gamma_\varepsilon$. Moreover $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0$, while

if $\gamma < \infty$, then $\gamma_\varepsilon < \infty$ and $f_\varepsilon(u) \equiv 0$ for $u > \gamma_\varepsilon$;

if $\gamma = \infty$, then $\gamma_\varepsilon = \infty$.

Finally, if we put

$$H_\varepsilon(p) = \int_0^p \rho\, d\Omega_\varepsilon(\rho) \equiv p\Omega_\varepsilon(p) - \int_0^p \Omega_\varepsilon(\rho)\, d\rho,$$

then since $\Omega_\varepsilon \to \Omega$ as $\varepsilon \to 0$ uniformly on every compact subset of $\mathbb{R}$, we have $H_\varepsilon \to H$ as $\varepsilon \to 0$ uniformly on every compact subset of $\mathbb{R}$. Moreover when $H(\infty)=\infty$ it is clear that also $H_\varepsilon(p) \nearrow \infty$ as $p \to \infty$, that is $H_\varepsilon(\infty)=\infty$.

Now consider the problem

$$(A_\varepsilon u'_\varepsilon) + \frac{(n-1)}{r} A_\varepsilon u'_\varepsilon + f_\varepsilon(u_\varepsilon) = 0, \qquad r>0$$
$$u'_\varepsilon(0)=0,\ u_\varepsilon(r) \geqslant 0 \text{ for } r>0; \qquad u_\varepsilon(r) \to 0 \text{ as } r \to \infty; \qquad u_\varepsilon \neq 0. \tag{$*$}_\varepsilon$$

If $\gamma_\varepsilon < \infty$, i.e. the case $\gamma < \infty$, then by the results proved in Section 2.1, this problem has a positive solution u_ε, such that

$$\beta_\varepsilon < u_\varepsilon(0) < \gamma_\varepsilon \tag{2.2.1}$$

for every $\varepsilon > 0$ sufficiently small.

Next suppose $\gamma_\varepsilon = \infty$, and so of course $\gamma = \infty$. In this case, by conditions (c), (d) or (e), there exist constants $\Gamma > \bar\beta > \beta$ such that either (2.1.31) or (2.1.32) holds. Of course also $\bar\beta > \beta_\varepsilon$ for every $\varepsilon > 0$ small enough.

Now let $C_\varepsilon = C(\Omega_\varepsilon)$ as in (2.1.8) and let $\bar m_\varepsilon = \bar m(F_\varepsilon)$ as in (2.1.12). If either condition (c) or (d) holds, then clearly there exists $\varepsilon_0 > 0$ such that

$$\frac{H_\varepsilon^{-1}(F_\varepsilon(\Gamma))}{\Gamma - \bar\beta} < \frac{\bar m_\varepsilon}{2(n-1)\,\bar\beta C_\varepsilon}$$

for every ε in $(0, \varepsilon_0)$, where Γ has been defined in (2.1.31). Therefore, from the principal remark at the end of Section 2.1, the solution u_ε clearly has the property

$$\beta_\varepsilon < u_\varepsilon(0) < \Gamma, \qquad \varepsilon \in (0, \varepsilon_0). \tag{2.2.2}$$

If condition (e) holds, then the proof of the analogous a priori estimate requires some further argument since in (2.1.32) the further quantities c^* and d also appear. Choose now $\bar d > 0$ such that the following slightly stronger version of (2.1.18) holds:

$$H(p) > 4(n-1)\,\bar\beta c^* \Omega(p) + \bar\beta \sup_{[0,\beta]} |f| \qquad \text{for} \quad p > \bar d.$$

Then, if $p > 2\bar d$, so that $p + \varepsilon t > \bar d$ for $|t| < 1$ and ε small, we have

$$\begin{aligned}
H_\varepsilon(p) &= p\Omega_\varepsilon(p) - \int_0^p \Omega_\varepsilon(\rho)\,d\rho \\
&= p\left\{\int_{\mathbb{R}} \Omega(p+\varepsilon t)\,J(t)\,dt + \varepsilon p\right\} - \int_0^p \left(\int_{\mathbb{R}} \Omega(\rho+\varepsilon t)\,J(t)\,dt + \varepsilon\rho\right)d\rho \\
&= \int_{\mathbb{R}} \left\{p\Omega(p+\varepsilon t) - \int_0^p \Omega(\rho+\varepsilon t)\,d\rho\right\} J(t)\,dt + \frac{1}{2}\varepsilon p^2
\end{aligned}$$

$$= \int_{\mathbb{R}} \left\{ (p+\varepsilon t)\, \Omega(p+\varepsilon t) - \int_{\varepsilon t}^{p+\varepsilon t} \Omega(\rho)\, d\rho \right.$$

$$\left. - \varepsilon t \Omega(p+\varepsilon t) \right\} J(t)\, dt + \frac{1}{2}\, \varepsilon p^2$$

$$= \int_{\mathbb{R}} \left\{ H(p+\varepsilon t) - \varepsilon t \Omega(p+\varepsilon t) + \int_0^{\varepsilon t} \Omega(\rho)\, d\rho \right\} J(t)\, dt + \frac{1}{2}\, \varepsilon p^2$$

$$\geqslant \int_{\mathbb{R}} \left\{ (4(n-1)\, \bar{\beta} c^* - \varepsilon)\, \Omega(p+\varepsilon t) + \bar{\beta} \sup_{[0, \beta_\varepsilon]} |f_\varepsilon| \right\} J(t)\, dt + \frac{1}{2}\, \varepsilon p^2$$

$$\geqslant 3(n-1)\, \bar{\beta} c^* \Omega_\varepsilon(p) + \bar{\beta} \sup_{[0, \beta_\varepsilon]} |f_\varepsilon| \frac{1}{2}\, \varepsilon p (p - 6(n-1)\, \bar{\beta} c^*)$$

$$\geqslant 2(n-1)\, \bar{\beta} c^* \Omega_\varepsilon(p) + \bar{\beta} \sup_{[0, \beta_\varepsilon]} |f_\varepsilon|$$

if ε is small. Hence we can repeat our previous arguments to show that (2.2.2) holds for ε small and with an obviously modified value for Γ (i.e. with d replaced by $2\bar{d}$ in (2.1.32)).

We claim that there exists a positive constant M such that

$$-M \leqslant u_\varepsilon'(r) \leqslant 0, \qquad r \geqslant 0, \tag{2.2.3}$$

for every ε in $(0, \varepsilon_0)$. The inequality $u_\varepsilon' \leqslant 0$ follows from Lemma 1.2.4. On the other hand, if $\gamma = \infty$ then by (1.1.4) and (2.2.2) we have

$$H_\varepsilon(|u_\varepsilon'(r)|) \leqslant F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(r)) \leqslant F_\varepsilon(\Gamma) + \sup_{[0, \beta_\varepsilon]} |F_\varepsilon|.$$

Hence, using condition (c) in case $H(\infty) < \infty$, we have $H_\varepsilon(|u_\varepsilon'|) < H(\infty)$ for every sufficiently small ε in $(0, \varepsilon_0)$. Choose $\bar{M}$ so that

$$H_\varepsilon(|u_\varepsilon'|) \leqslant \bar{M} < H(\infty).$$

Thus $u_\varepsilon' \geqslant -H_\varepsilon^{-1}(\bar{M})$. Clearly $H_\varepsilon^{-1}(\bar{M}) \to H^{-1}(\bar{M})$ as $\varepsilon \to 0$, and inequality (2.2.3) therefore follows with $M = H^{-1}(\bar{M}) + 1$, changing, if necessary, the choice of ε_0. Next suppose $\gamma < \infty$. In this case from (1.1.14) and (2.2.1) we get

$$H_\varepsilon(|u'(r)|) \leqslant F_\varepsilon(\gamma_\varepsilon) + \sup_{[0, \beta_\varepsilon]} |F|$$

and (2.2.3) is proved as before.

It now follows from the Arzelá-Ascoli theorem that, for a suitable sequence $\varepsilon_j \downarrow 0$ we have $u_{\varepsilon_j} \to u$ uniformly on every compact subset of $[0, \infty)$. Therefore, because (see Lemma 1.1.1)

$$u'_\varepsilon A_\varepsilon(|u'_\varepsilon|) = -\int_0^r f_\varepsilon(u_\varepsilon(\rho)) \left(\frac{\rho}{r}\right)^{n-1} d\rho, \tag{2.2.4}$$

we see that

$$\Omega_{\varepsilon_j}(|u'_{\varepsilon_j}|) = |u'_{\varepsilon_j}|\, A_{\varepsilon_j}(|u'_{\varepsilon_j}|) \to g(r) \qquad \text{for all} \quad r \geqslant 0, \tag{2.2.5}$$

where

$$g(r) = \int_0^r f(u(\rho)) \left(\frac{\rho}{r}\right)^{n-1} d\rho.$$

Moreover, since $\Omega_\varepsilon(p) \to \Omega(p)$ uniformly on compact subsets of $p \geqslant 0$, it follows that

$$u'_{\varepsilon_j} \to -\Omega^{-1}(g(r)) \qquad \text{for all} \quad r \geqslant 0.$$

Hence from the relation

$$u_\varepsilon(r) = u_\varepsilon(0) + \int_0^r u'_\varepsilon(\rho)\, d\rho$$

together with (2.2.3), (2.2.5) and Lebesgue's dominated convergence theorem we get

$$u(r) = u(0) - \int_0^r \Omega^{-1}(g(\rho))\, d\rho;$$

thus u is continuously differentiable and a classical solution of (1.1.1). In turn of course (2.2.5) and (2.2.4) supply the relations

$$u'_{\varepsilon_j} \to u' \qquad \text{and} \qquad \Omega_{\varepsilon_j}(|u'_{\varepsilon_j}|) \to \Omega(|u'|) \qquad \text{for all} \quad r \geqslant 0. \tag{2.2.6}$$

Obviously $u \geqslant 0$, $u' \leqslant 0$ and

$$\beta < u(0) \leqslant \gamma \qquad (\leqslant \Gamma < \infty \text{ if } \gamma = \infty);$$

note that the possibility $u(0) = \beta$ is ruled out by Lemma 1.2.1.

It remains only to be shown that $u \to 0$ as $r \to \infty$. Put $l = \lim_{r \to \infty} u(r)$. As in Step 2 of the proof in Section 2.1 we find easily that $u' \to 0$ as $r \to \infty$ and $f(l) = 0$. Of course also $0 \leqslant l \leqslant \gamma$ so that necessarily either

$$0 \leqslant l < \beta \qquad \text{or} \qquad l = \gamma,$$

the second possibility occurring only in the case $\gamma<\infty$. Now

$$0 \leqslant \int_R^\infty A_\varepsilon u_\varepsilon'^2 \frac{d\rho}{\rho} \leqslant \frac{A_\varepsilon(M)\,M}{R} \int_R^\infty -u_\varepsilon'(\rho)\,d\rho = \frac{A_\varepsilon(M)\,M}{R}\,u_\varepsilon(R)$$
$$\leqslant \frac{2A(M)\,M\gamma}{R} \qquad \left(\text{or } \frac{2A(M)\,M\Gamma}{R} \quad \text{if } \gamma=\infty\right) \tag{2.2.7}$$

for every $R>0$.

On the other hand, by (1.2.2) applied to the solution u_ε,

$$F_\varepsilon(u_\varepsilon(0)) = (n-1) \int_0^\infty A_\varepsilon u_\varepsilon'^2 \frac{dr}{r}.$$

Letting $\varepsilon=\varepsilon_j \to 0$ and using (2.2.3), (2.2.4) to show that the integrand is uniformly bounded, together with (2.2.6) and (2.2.7) and the dominated convergence theorem, then gives

$$F(u(0)) = (n-1) \int_0^\infty A u'^2 \frac{dr}{r}. \tag{2.2.8}$$

In the case $\gamma<\infty$, if $l=\gamma$ then $u \equiv \gamma$, because u is decreasing function with $u(0) \leqslant \gamma$. In turn (2.2.8) gives

$$F(\gamma)=0,$$

which however is impossible since (a), (b) imply $F(\gamma)>0$. Thus in all cases $0 \leqslant l < \beta$. Finally, because $u(r) \to l$ and $u'(r) \to 0$ as $r \to \infty$, we obtain from (1.1.4)

$$(n-1) \int_0^\infty A u'^2 \frac{d\rho}{\rho} = F(u(0)) - F(l).$$

From this result and (2.2.8) it now follows that $F(l)=0$, and so $l=0$ by hypothesis (a). This completes the proof.

3. Uniqueness

In this section, we prove our main uniqueness results. We shall require throughout that the operator $A(p)$ be *continuously differentiable* on $(0, \infty)$ rather than simply continuous as in the hypothesis (H2). We recall from the Introduction the definition

$$\Omega(p) = pA(p), \quad p>0; \qquad \Omega(0)=0.$$

By hypothesis (H3) it is clear that $\Omega(p)>0$ for $p>0$ and $\Omega'(p)\geqslant 0$. We shall strengthen this in the sequel, requiring henceforth that

(H3′) $\Omega(p)>0,\quad E(p)=\Omega'(p)>0$ for $p>0$.

In terms of the variational integrand $G(p)$ in the Introduction, these are exactly the conditions $G'(p)$, $G''(p)>0$ for $p>0$.

THEOREM I. *Suppose $n>1$, and assume also the following hypotheses:*

(A1) *$S(l,p)>0$ for $0<l<p$, where*

$$S(l,p)=M(l)-M(p)+\frac{n-2}{n-1}N(l)(N(l)-N(p))$$

and

$$M(p)=1/p^3\Omega\Omega',\qquad N(p)=1/p\Omega.$$

(A2) *$p^{2-m}A(p)\to 1$ as $p\to 0$, for some constant $m>1$,*

(A3) *$\Omega'(p)\geqslant\{\Omega(p)\}^{\mu}$ for all $p>0$ near zero, where μ is a constant in* $[0,2)$,

(F1) *$F(u)\leqslant 0$ for $0<u<\beta$, $F(\beta)=0$,*

(F2) *f is positive and non-increasing on (β,γ), where $\gamma\in(\beta,\infty]$,*

(F3) *f is locally Lipschitz continuous on (β,γ).*

Then (*) *has at most one solution u such that $u(0)<\gamma$.*

THEOREM II. *Let the hypotheses of Theorem* I *hold, with the following exceptions:*

(i) *$\mu=0$ in condition* (A3);

(ii) *$\gamma<\infty$ in conditions* (F2) *and* (F3);

(iii) *(β,γ) is replaced by $(\beta,\gamma]$ in condition* (F3).

Then (*) *has at most one solution u such that $u(0)\leqslant\gamma$.*

THEOREM III. *Suppose the hypotheses of Theorem* I (*or Theorem* II) *hold, with the exception that condition* (F2) *is weakened to*

(F2′) *The function $u\to f(u)/(u-\beta)^{\nu-1}$ is positive and non-increasing for $\beta<u<\gamma$, where ν is a constant $\geqslant 1$.*

Assume also

$$(\mathrm{A}2') \quad p^{2-\nu}A \textit{ is non-decreasing on } (0, \infty).^1$$

Then $(*)$ *has at most one solution such that* $u(0) < \gamma$ (*or* $u(0) \leqslant \gamma$).

If $\gamma = \infty$ the condition $u(0) < \gamma$ is satisfied for any solution, and so can be omitted from the statement of Theorem I. Similarly the condition $u(0) \leqslant \gamma$ can be dropped in Theorem II if f satisfies the condition $f(u) \leqslant 0$ for $u > \gamma$. Indeed in this case, if $u(0) > \gamma$ then (1.1.4) gives a contradiction when applied to the interval $(0, r_0)$ where $u(r) \in (\gamma, u(0))$.

When conditions (A3) and (F3) are dropped from Theorems I–III, we still obtain results of interest, as stated in Proposition 3 in the Introduction.

Condition (A2′) together with (A2) forces $\nu \leqslant m$. In the particular case $A(p) = p^{m-2}$ condition (A2) is trivially satisfied, while (A2′) holds with $\nu = m$; similarly (A3) is valid for $\mu = 1$.

Conditions (A2) and (A3) concern the behavior of $A(p)$ and $A'(p)$ near $p = 0$, whereas (A1) has a global character. Some simpler sufficient conditions can be formulated in order that (A1) holds; in particular we have the following result.

LEMMA 3.0. *The hypothesis* (A1) *is satisfied if either of the following conditions holds*:

$$\text{(i)} \quad n > 2 \quad \textit{and} \quad M + \frac{1}{2}\frac{n-2}{n-1} N^2 \textit{ is non-increasing};$$

$$\text{(ii)} \quad n < 2 \quad \textit{and} \quad M + \frac{n-2}{n-1} N^2 \textit{ is non-increasing};$$

$$\text{(iii)} \quad n = 2 \quad \textit{and} \quad M \textit{ is decreasing}.$$

Proof. Write $\Psi(p) = M(p) + ((n-2)/(n-1))\, N^2(p)$. Then

$$S(l, p) = \Psi(l) - \Psi(p) + \frac{n-2}{n-1} N(p)(N(l) - N(p))$$

and the conclusion for case (ii) follows at once since N is decreasing (by (H3)). Condition (i) is proved in essentially the same way; that is, putting

$$\Psi^*(p) = M(p) + \frac{1}{2}\frac{n-2}{n-1} N^2(p)$$

[1] When $F(\gamma) < \infty$ the interval $(0, \infty)$ can be replaced by a smaller set $(0, p_0)$, where p_0 is defined by $H(p_0) = F(\gamma)$. Indeed by (1.2.1) and (1.2.2), when $u(r) > \beta$,

$$H(p) < H(p) + F(u) \leqslant F(u(0)) < F(\gamma),$$

so $p < p_0$.

we get for $0 < l < p$,

$$S(l, p) = \Psi^*(l) - \Psi^*(p) + \frac{1}{2}\frac{n-2}{n-1}\{N(l) - N(p)\}^2$$

$$> \Psi^*(l) - \Psi^*(p) \geqslant 0$$

since $n > 2$. Condition (iii) is obvious.

It is clear that (i) and (iii) are most simply satisfied when M is decreasing. This in turn is implied by condition (A1′) in the Introduction, as is the condition $\Omega'(p) > 0$ for $p > 0$.

Remark. For the operator

$$A(p) = (1+p^2)^{-s/2}\, p^{m-2}, \qquad s \geqslant 0,$$

we have

$$\begin{aligned}
\Omega(p) &= (1+p^2)^{-s/2}\, p^{m-1},\\
E(p) &= \Omega'(p) = (1+p^2)^{-s/2-1}\, p^{m-2}\{(m-1) + (m-1-s)\, p^2\},\\
M(p) &= (1+p^2)^{s+1}\, p^{-2m}\{(m-1) + (m-1-s)\, p^2\}^{-1},\\
N^2(p) &= (1+p^2)^s\, p^{-2m}.
\end{aligned}$$

Clearly (H3′) holds if and only if $m \geqslant 1+s$, or $m > 1$ if $s = 0$. In this case, moreover, it is easy to check that M is decreasing. Hence (A1) and (A1′) are satisfied for $n \geqslant 2$.

When $1 < n < 2$ we write

$$\Psi(p) = M(p) - \frac{1}{m-1} N^2(p) + \left(\frac{n-2}{n-1} + \frac{1}{m-1}\right) N^2(p).$$

Now

$$M(p) - \frac{1}{m-1} N^2(p) = \frac{s}{m-1}\, \frac{(1+p^2)^s}{p^{2(m-1)}}\, \frac{1}{(m+1) + (m-1-s)p^2}.$$

This quantity is non-increasing provided

$$s \geqslant 0, \qquad m \geqslant s+1 \text{ (or } m > 1 \text{ if } s = 0).$$

Hence $\Psi(p)$ itself is non-increasing and (A1) holds if we suppose also that

$$\frac{n-2}{n-1} + \frac{1}{m-1} \geqslant 0, \qquad \text{that is} \quad n \geqslant 2 - \frac{1}{m}.$$

(It is not hard to see that this condition is best possible).

The following three sections contain introductory material, including a discussion of the asymptotic behavior of solutions in Section 3.2 and a monotone separation theorem in Section 3.3.

Theorems I–III are almost immediate corollaries of Proposition 3 in the Introduction (with assumption (A1′) being replaced by (A1) and also with $n > 1$ rather than $n \geqslant 2$). Consequently our principal effort can be devoted to proving Proposition 3. The proofs of Theorems I and II are then given at the end of Section 3.4, and the proof of Theorem III in Section 3.5. Theorems A, B, C of the Introduction are the special cases of Theorems I–III when $n \geqslant 2$ and (A1) is replaced by the stronger condition (A1′).

The main Theorems I–III require the satisfaction of conditions (A1) and (A2). These conditions can be avoided provided that other hypotheses are suitably strengthened. This result is given in Theorem IV of Section 6. In the final section of the paper we add several remarks concerning the exterior Dirichlet problem.

3.1. *Preliminary Results.*

Denote by $l = l(u, p)$ the positive function determined implicitly by the equation

$$H(l) = H(p) + F(u)$$

on the domain

$$P = \{(u, p) \colon u, p > 0,\ 0 < H(p) + F(u) < H(\infty)\}.$$

Obviously l is well-defined and strictly positive since H is increasing. Moreover (recall that $H'(p) = pE(p)$ by (4) and that $E(p) > 0$ for $p > 0$).

$$\frac{\partial l}{\partial u} = \frac{f(u)}{lE(l)}, \qquad \frac{\partial l}{\partial p} = \frac{pE(p)}{lE(l)}. \tag{3.1.1}$$

We also define $K(u, p)$ on P by the formula

$$K(u, p) = \frac{(lA(l))^{\alpha-1}}{l}\{l^2 A(l) - p^2 A(p)\}, \qquad \alpha = 1/(n-1).$$

Then the following result holds.

LEMMA 3.1.1. *Suppose* $(u, p) \in P$. *Then*

(i) $\dfrac{\partial}{\partial p}\{lA(l)\} > 0;$

(ii) $\dfrac{\partial}{\partial p}\left\{\dfrac{K(u, p)}{p}\right\} > 0$ *if and only if* $S(l, p) > 0$.

Moreover, if $F(u) \leqslant 0$, *then*

$$\text{(iii)} \quad K(u, p) \leqslant 0.$$

Proof. The first relation is obvious since

$$\frac{\partial}{\partial p}\{lA(l)\} = \frac{\partial}{\partial p}\Omega(l) = E(l)\frac{pE(p)}{lE(l)} = \frac{pE(p)}{l}. \tag{3.1.2}$$

The third result also follows at once since $p^2 A(p)$ is increasing and $l \leqslant p$ when $F(u) \leqslant 0$. Finally $l = p$ and $K = 0$ if and only if $F(u) = 0$.

We turn to the second conclusion. By direct calculation, keeping in mind (3.1.1) and (3.1.2), we have

$$\begin{aligned}
\frac{\partial}{\partial p}\left\{\frac{K(u,p)}{p}\right\} &= \frac{\partial}{\partial p}\left\{(\Omega(l))^{\alpha-1}\left(\frac{\Omega(l)}{p} - \frac{\Omega(p)}{l}\right)\right\} \\
&= (\alpha-1)(\Omega(l))^{\alpha-2}\frac{pE(p)}{l}\left(\frac{\Omega(l)}{p} - \frac{\Omega(p)}{l}\right) \\
&\quad + (\Omega(l))^{\alpha-1}\left(-\frac{\Omega(l)}{p^2} + \frac{\Omega(p)}{l^2}\frac{pE(p)}{lE(l)}\right) \\
&= (\Omega(l))^{\alpha}\, p\Omega(p)\, E(p)\, S(l,p),
\end{aligned}$$

and (ii) is proved.

We next introduce an important identity (Lemma 3.1.3) for solutions of (*). We begin by showing that the solution u is twice differentiable whenever $u' \neq 0$.

LEMMA 3.1.2. *Suppose* $u' \neq 0$ *at some point* $r > 0$. *Then at this point* u'' *exists and satisfies* (1.1.1) *in the form*

$$E(p)\, u'' + \frac{n-1}{r} A(p)\, u' + f(u) = 0 \qquad \text{where} \quad p = |u'|. \tag{3.1.3}$$

Proof. Suppose $u'(r) < 0$. By virtue of Lemma 1.1.1

$$u'(r) = -\Omega^{-1}\left(\int_0^r \left(\frac{t}{r}\right)^{n-1} f(u(t))\, dt\right) \tag{3.1.4}$$

Since A is continuously differentiable and $\Omega' = E$, it follows that Ω^{-1} is differentiable and $(\Omega^{-1})' = 1/E$, when $p > 0$. Hence from (3.1.4) we see that u'' exists and

$$u'' = -\frac{1}{E(p)}\left\{f(u(r)) - \int_0^r \frac{n-1}{r}\left(\frac{t}{r}\right)^{n-1} f(u(t))\,dt\right\}$$

$$= \frac{1}{E(p)}\left\{-f(u) + \frac{n-1}{r}\,\Omega(p)\right\},$$

as required. The proof when $u' > 0$ is similar.

LEMMA 3.1.3. *If u is a solution of $(*)$ such that $u(r) > 0$ and $F(u(r)) \leqslant 0$ on (r_0, r_1) we have*

$$\text{(i)}\quad (u(r), |u'(t)|) \in P \qquad \textit{for any}\quad r \in (r_0, r_1).$$

Moreover, if we write $p = p(r) = |u'(r)|$ and $l = l(r) = l(u(r), p(r))$, then

$$\text{(ii)}\quad r(lA(l))^{\alpha}\big|_{r_0}^{r_1} = \int_{r_0}^{r_1} K(u, p)\,dr, \qquad \alpha = \frac{1}{n-1}. \tag{3.1.5}$$

Proof. (i) It is obvious from (1.2.1) that $H(p) + F(u) > 0$ on (r_0, r_1). Since $F(u) \leqslant 0$ we get in turn that $p > 0$ and $H(p) + F(u) \leqslant H(p) < H(\infty)$. Thus (i) follows from the definition of P.

(ii) If $r \in (r_0, r_1)$ we have

$$\frac{d}{dr}\{r(lA(l))^{\alpha}\} = (lA(l))^{\alpha} + \alpha r(lA(l))^{\alpha-1}\,E(l)\left\{\frac{\partial l}{\partial u}u' + \frac{\partial l}{\partial p}p'\right\}$$

$$= (lA(l))^{\alpha} + \alpha r(lA(l))^{\alpha-1}\,u'\{f(u) + E(p)\,u''\}\,l^{-1}$$

$$= \frac{(lA(l))^{\alpha-1}}{l}\{l^2A(l) - p^2A(p)\} = K(u, p),$$

where (3.1.1) was used at the second step and equation (3.1.3) at the third (recall that $p > 0$ in (r_0, r_1) by (i)). Integrating over (r_0, r_1) completes the proof.

3.2. *Asymptotic Behavior.*

We can now obtain the following important results concerning the asymptotic behavior of a solution of $(*)$.

LEMMA 3.2.1. *Suppose* (F1) *holds and let* u *be a solution of* (*). *Then*

(i) *if* $u>0$ *for all* $r>0$ *there exists* $\lambda \geqslant 0$ *such that*

$$r^{n-1} l A(l) \to \lambda \qquad as \quad r \to \infty, \tag{3.2.1}$$

(ii) $\lim_{r\to\infty} r^{n-1} \int_r^\infty A u'^2 \frac{d\rho}{\rho} = 0.$

Proof. (i) By Lemma 3.1.3 we get

$$(r^{n-1} l(r) A(l(r)))^\alpha = (R^{n-1} l(R) A(l(R)))^\alpha + \int_R^r K(u, p)\, d\rho \tag{3.2.2}$$

for $r>R$, where R is so large that $u(r)<\beta$ for $r>R$; see (F1).

The right hand side of (3.2.2) goes to a limit $\bar{\lambda}$ as $r\to\infty$, since $K(u,p)\leqslant 0$ by Lemma 3.1.2 (ii). Obviously $\bar{\lambda} \in [-\infty, \infty)$. On the other hand, the left hand side is $\geqslant 0$. Hence $\bar{\lambda}\geqslant 0$ and there exists $\lambda \geqslant 0$ such that (3.2.1) holds. Note that this proof shows also that $K(u,p)\in L^1((r,\infty))$ for any $r>0$.

(ii) It is enough to consider the case when $u>0$ for all $r>0$. By Lemma 1.2.1 and the definition of l we have

$$(n-1)\int_r^\infty A u'^2 \frac{d\rho}{\rho} = H(p) + F(u) = H(l).$$

On the other hand, by equation (4) in the introduction,

$$r^{n-1} H(l) = r^{n-1}\left(l^2 A(l) - \int_0^l \rho A(\rho)\, d\rho\right) \leqslant r^{n-1} l^2 A(l) \to 0$$

since $l\to 0$ $(0<l<p)$ and $r^{n-1} l A(l) \to \lambda$ as $r\to\infty$.

In the proof of the following lemma we use an ingenious idea of Kaper & Kwong [KK2].

LEMMA 3.2.2. *Suppose hypotheses* (A2) *and* (F1) *are satisfied and let* u *be a positive solution of* (*). *Then, with* λ *defined as in* (3.2.1), *we have*

(i) $\lambda = 0$ *when* $n \leqslant m$;

(ii) *when* $n>m$

$$r^{(n-m)/(m-1)} u(r) \to \frac{m-1}{n-m} \lambda^{1/(m-1)} \qquad as \quad r\to\infty.$$

Proof. (i) Since $l \to 0$ and $r^{n-1} l A(l) \to \lambda$ as $r \to \infty$, from (A2) it follows that

$$r^{(n-1)/(m-1)} l = (r^{n-1} l A(l))^{1/(m-1)} (l^{2-m} A(l))^{-1/(m-1)} \to \lambda^{1/(m-1)} \qquad (3.2.3)$$

as $r \to \infty$. Therefore since $0 < l < p$ for large r,

$$\liminf_{r \to \infty} r^{(n-1)/(m-1)} p(r) \geqslant \lambda^{1/(m-1)}.$$

Because $p = p(r) = |u'(r)| = -u'(r)$ is integrable on $[0, \infty)$, the last inequality implies $\lambda = 0$ if $n \leqslant m$.

(ii) The proof for the case $n > m$ lies much deeper. By (3.2.3), l is integrable on (R, ∞), where $R > 0$ is such that $u(r) < \beta$ for every $r > R$. Moreover $F(u(r)) < 0$ for $r > R$ by (F1), so that also

$$H(p(r)) = H(l(r)) + |F(u(r))| \qquad \text{for} \quad r > R.$$

In particular, because H is strictly increasing we obtain

$$l < p = H^{-1}(H(l) + |F(u)|) \qquad \text{for} \quad r > R. \qquad (3.2.4)$$

Now, from (4) and (A2) it follows easily that $p^{-m} H(p) \to (m-1)/m$ as $p \to 0$. Thus since $m > 1$ we see that for every $\varepsilon > 0$ there exists $r_\varepsilon \geqslant R$ such that

$$\begin{aligned} p &< \left(1 + \frac{\varepsilon}{2}\right) \left\{ \frac{m}{m-1} H(p) \right\}^{1/m} \\ &= \left(1 + \frac{\varepsilon}{2}\right) \left\{ \frac{m}{m-1} (H(l) + |F(u)|) \right\}^{1/m} \\ &\leqslant (1 + \varepsilon) \left\{ l^m + \frac{m}{m-1} |F(u)| \right\}^{1/m} \\ &\leqslant (1 + \varepsilon) \left(l + \left(\frac{m}{m-1} |F(u)| \right)^{1/m} \right) \end{aligned} \qquad (3.2.5)$$

on (r_ε, ∞). Therefore we obtain from (3.2.4) and (3.2.5)

$$\begin{aligned} \int_r^\infty l(\rho)\, d\rho &< \int_r^\infty p(\rho)\, d\rho \\ &< (1 + \varepsilon) \left(\int_r^\infty l(\rho)\, d\rho + \text{const.} \int_r^\infty |F(u(\rho))|^{1/m}\, d\rho \right) \end{aligned} \qquad (3.2.6)$$

for every $r > r_\varepsilon$.

We estimate the second integral on the right side by using Hölder's inequality, namely

$$\int_r^\infty |F(u)|^{1/m}\,d\rho$$
$$\leqslant\left(\int_r^\infty B(l)\,|F(u)|\,d\rho\right)^{1/m}\cdot\left(\int_r^\infty (B(l))^{-1/(m-1)}\,d\rho\right)^{1-1/m}, \quad (3.2.7)$$

where $B(l)=((lA(l))^{\alpha-1}/l$. Since $l<p$ and $r>r_\varepsilon$,

$$|F(u)|=H(p)-H(l)=p^2A(p)-l^2A(l)-\int_0^p \rho A(\rho)\,d\rho+\int_0^l \rho A(\rho)\,d\rho$$
$$\leqslant p^2A(p)-l^2A(l)=\frac{|K(u,p)|}{B(l)}.$$

Then, from the fact that $K(u,p)\in L^1$ we get

$$B(l)\,F(u)\in L^1. \quad (3.2.8)$$

On the other hand, by (3.2.1), (3.2.3), and a direct computation using L'Hopital's rule, the quantity

$$r^{(n-m)/(m-1)}\left(\int_r^\infty (B(l))^{-1/(m-1)}\,d\rho\right)^{1-1/m}$$

tends to a finite limit as $r\to\infty$. In fact it is enough to show that

$$\frac{B(l)^{-1/(m-1)}}{r^{-m(n-m)/(m-1)^2-1}}$$

tends to a limit, or equivalently that

$$\frac{B(l)}{r^{m(n-m)/(m-1)+(m-1)}}$$

tends to a limit. But

$$B(l)=[lA(l)]^{(2-n)/(n-1)}/l$$

while by (3.2.1) and (3.2.3)

$$r^{n-1}lA(l)\to\text{limit},\qquad lr^{(n-1)/(m-1)}\to\text{limit}.$$

The conclusion now follows immediately, since

$$\frac{m(n-m)}{m-1}+(m-1)=(n-2)+\frac{n-1}{m-1}.$$

From (3.2.7) and (3.2.8) we then get

$$r^{(n-m)/(m-1)}\int_r^\infty |F(u)|^{1/m}\,d\rho\to 0 \qquad \text{as} \quad r\to\infty. \tag{3.2.9}$$

Finally, again by (3.2.3) and L'Hospital's rule,

$$r^{(n-m)/(m-1)}\int_r^\infty l\,d\rho\to\frac{m-1}{n-m}\,\lambda^{1/(m-1)}. \tag{3.2.10}$$

From (3.2.6), (3.2.9) and (3.2.10) we obtain the assertion since $\int_r^\infty p\,d\rho=u(r)$.

3.3. *Monotone Separation Theorems*

Our purpose is to study the separation properties of pairs of solutions of Problem (*).

Let u and v be two solutions of (*), which we assume throughout to satisfy the conditions

$$u(0)\leqslant\gamma, \qquad v(0)\leqslant\gamma.$$

By Lemma 1.2.6, in view of (F1) and (F2), it is clear that

$$u'(r)<0 \textit{ whenever } r>0 \textit{ and } u(r)>0;$$

$$v'(r)<0 \textit{ whenever } r>0 \textit{ and } v(r)>0.$$ [2]

Thus both u and v possess inverses on the domain where they are positive. We denote by r and s the inverses of u and v, defined respectively on the intervals $(0, u(0))]$ and $(0, v(0)]$. Note also the principal fact that

$$u(0) \textit{ and } v(0) \textit{ must exceed } \beta,$$

by virtue of Lemma 1.2.1 and (F1).

LEMMA 3.3.1. *Assume that* (F1), (F2) *hold. If* $r(u)-s(u)>0$ *on some open interval* I *of the domain of* r *and* s, *then* $r(u)-s(u)$ *can have at most*

[2] To be precise in the application of Lemma 1.2.6, the reader may note that condition (F1) is used to obtain $u'(r)<0$ when $u\leqslant\beta$ while condition (F2) is used when $u\geqslant\beta$.

one critical point on I. Moreover if such a point exists it must be a strict maximum.

Proof. By Lemma 3.1.2, equation (1.1.1) can be written in the form (3.1.3). Then it is immediately verified that the function $r = r(u)$ satisfies the equation

$$E\left(\left|\frac{1}{r_u}\right|\right) r_{uu} - \frac{n-1}{r} A\left(\left|\frac{1}{r_u}\right|\right) r_u^2 - r_u^3 f(u) = 0$$

for $u \in I$ (clearly $r > 0$ and $r_u < 0$). A similar equation also holds for $s = s(u)$.

Now suppose $r - s$ has a critical point in I. Then $r_u = s_u < 0$ and $r > s > 0$ at this point. Consequently by subtraction we find

$$(r-s)_{uu} = (n-1)\frac{A}{E}\left(\frac{1}{r} - \frac{1}{s}\right) r_u^2 < 0$$

which proves the assertion.

Lemma 3.3.2. *Suppose* (A1) *and* (F1) *are satisfied. If* $r - s$ *has two zeros in* $(0, \beta]$, *say* ξ_0 *and* ξ_1, *then* $r - s = 0$ *for all* u *between* ξ_0 *and* ξ_1.

Proof. Suppose the conclusion is false. Then without loss of generality we can assume that $\xi_0 < \xi_1$ and $r - s > 0$ for all $u \in [\xi_0, \xi_1]$.

In the open interval (ξ_0, ξ_1) the difference $r - s$ can have at most one critical point by Lemma 3.3.1, and at the same time at least one, since it vanishes at the endpoints. If follows that there exists a point $\xi_2 \in (\xi_0, \xi_1)$ such that $r_u = s_u$ at $u = \xi_2$, while

$$r_u < s_u \qquad \text{for} \quad u \in (\xi_2, \xi_1).$$

Now put $r_1 = r(\xi_1)$, $r_2 = r(\xi_2)$, $l_1 = l(r_1)$, $l_2 = l(r_2)$. By (3.1.5) we have

$$r_2(l_2 A(l_2))^\alpha - r_2(l_2 A(l_2))^\alpha = \int_{r_1}^{r_2} K(u, p)\, dr = \int_{\xi_2}^{\xi_1} K(u, p) \frac{du}{p},$$

where p denotes the function $p(r(u)) = |u'(r(u))|$.

A similar identity naturally holds for the solution v. In analogy with the preceding notation, we denote by s_1, s_2, m_1, m_2 the quantities relative to v corresponding to the quantities r_1, r_2, l_1, l_2 for u.

Subtracting the two identities, and using the fact that $r_1 = s_1$ and $l_2 = m_2$ (because $r_u(\xi_2) = s_u(\xi_2)$), we obtain

$$\begin{aligned}(r_2 - s_2)\{l_2 A(l_2)\}^\alpha - r_1\{(l_1 A(l_1))^\alpha - (m_1 A(m_1))^\alpha\} \\ = \int_{\xi_2}^{\xi_1} \left(\frac{K(u,p)}{p} - \frac{K(u,q)}{q}\right) du, \qquad (3.3.1)\end{aligned}$$

where, by analogy with the above notation, $q = q(s(u)) = |v'(s(u))|$. Note particularity that $p > q$ for $u \in (\xi_2, \xi_1)$ since $r_u < s_u$.

Now let

$$\frac{\partial}{\partial p}\left(\frac{K(u, p)}{p}\right) = D(u, p).$$

Then by the mean value theorem, for any fixed $u \in (\xi_2, \xi_1)$,

$$\frac{K(u, p)}{p} - \frac{K(u, q)}{q} = D(u, \tilde{p})(p - q)$$

where $\tilde{p} \in (p, q)$. Let $\tilde{l} = l(u, \tilde{p})$, see Section 3.1. By the definition of l and the fact that $H(\cdot)$ is strictly increasing it is clear that $\tilde{l} \leqslant \tilde{p}$, with $\tilde{l} = \tilde{p}$ if and only if $F(u) = 0$ (recall $F(u) \leqslant 0$ for $u \in (\xi_2, \xi_1)) \subset (0, \beta]$.

By Lemma 3.1.2 (ii) we have $D(u, \tilde{p}) > 0$ if $S(\tilde{l}, \tilde{p}) > 0$, so by (A1) also $D(u, \tilde{p}) > 0$ if $\tilde{l} < \tilde{p}$. Thus

$$\frac{K(u, p)}{p} - \frac{K(u, q)}{q} < 0 \qquad \text{if} \quad \tilde{l} < \tilde{p}.$$

On the other hand, if $\tilde{l} = \tilde{p}$ then $D(u, \tilde{p}) = S(\tilde{l}, \tilde{p}) = 0$ and $K(u, p)/p = K(u, q)/q$. Hence the right hand side of (3.3.1) is non-positive.

But also $r_2 > s_2$ and $l_1 \leqslant m_1$ because $r_u \leqslant s_u$ at $u = \xi_1$. Thus the left hand side of (3.3.1) is strictly positive, a contradiction.

LEMMA 3.3.3. *Suppose hypotheses* (F1) *and* (F2) *are satisfied. If* $r(u) - s(u) > 0$ *on an interval* $I = (0, a)$, *then* $(r - s)' < 0$ *on* I.

Proof. By Lemma 3.3.1 either $r(u) - s(u)$ is everywhere decreasing on I, or $r(u) - s(u)$ is increasing for u near zero. In the first case we are done, so let us assume for contradiction that $r_u - s_u > 0$ in some interval $0 < u < \delta$. As in the previous lemma, we put $r = r(u)$, $p = p(u) = |u'(r(u))|$. Then equation (1.2.1) becomes

$$H(p) + F(u) = (n-1) \int_0^u \frac{Ap}{r}\, dt, \qquad 0 < u < \delta.$$

A similar identity holds for the solution v, replacing p by $q = q(u) = |v'(s(u))|$.

Now writing $A = A(p)$, $B = A(q)$ and subtracting the two identities we get

$$H(p) - H(q) = (n-1) \int_0^u \left(\frac{Ap}{r} - \frac{Bq}{s}\right) dt. \tag{3.3.2}$$

Define

$$\varphi = \varphi(u) = \int_0^u \left(\frac{Ap}{r} - \frac{Bq}{s} \right) dt.$$

Since $s_u < r_u < 0$ on $(0, \delta)$ we have $p > q$ for $0 < u < \delta$. Hence, because H is strictly increasing, (3.3.2) gives

$$(n-1)\,\varphi = H(p) - H(q) > 0, \qquad 0 < u < \delta. \tag{3.3.3}$$

Also, since $r > s$,

$$qs\varphi' = qs\left(\frac{Ap}{r} - \frac{Bq}{s} \right) < q(Ap - Bq) = q \int_q^p d(\rho A(\rho))$$

$$< \int_q^p \rho d(\rho A(\rho)) = H(p) - H(q) = (n-1)\,\varphi,$$

by (5).

Now we integrate the inequality

$$\frac{\varphi'}{\varphi} < \frac{(n-1)}{qs},$$

obtaining for $0 < u < \bar{u} < \delta$,

$$\log \frac{\varphi(\bar{u})}{\varphi(u)} < (n-1) \int_u^{\bar{u}} \frac{1}{qs}\, dt = (n-1) \int_{s(\bar{u})}^{s(u)} \frac{ds}{s} = (n-1) \log \frac{s(u)}{s(\bar{u})}.$$

Hence

$$(s(\bar{u}))^{n-1}\,\varphi(\bar{u}) < (s(u))^{n-1}\,\varphi(u). \tag{3.3.4}$$

On the other hand, since $r > s$ and $Bq > 0$, we have

$$0 < (s(u))^{n-1}\,\varphi(u) < (r(u))^{n-1}\,\varphi(u) < (r(u))^{n-1} \int_0^u \frac{Ap}{r}\, dt = r^{n-1} \int_r^\infty Au'^2 \frac{d\rho}{\rho}.$$

Thus by Lemma 3.2.1 (ii), it follows that $(s(u))^{n-1}\,\varphi(u) \to 0$ as $u \to 0$. From (3.3.4) we then deduce that $\varphi(\bar{u}) \leqslant 0$, which contradicts (3.3.3).

3.4. *Uniqueness Theorem* I

In this section we shall prove Theorem I. We first obtain several preliminary results, of interest in themselves.

THEOREM 3.4.1. *Suppose hypotheses* (F1) *and* (F2) *are satisfied and let* u *and* v *be two solutions of* $(*)$ *satisfying* $u(0), v(0) \leqslant \gamma$. *Then there exists a value* $R \geqslant 0$ *such that* $u(R) = v(R) > 0$.

THEOREM 3.4.2. *Suppose hypotheses* (A1), (A2) *and* (F1) *are satisfied. Let* u *and* v *be two solutions of* $(*)$ *such that* $u(R) = v(R) \in (0, \beta]$ *for some* $R > 0$. *Then* $u(r) \equiv v(r)$ *whenever* $u(r) \leqslant \beta$.

THEOREM 3.4.3 *Suppose hypothesis* (F2) *is satisfied. Let* u *and* v *be two solutions of* $(*)$ *such that* $u(R) = v(R) \in [\beta, \gamma]$ *for some* $R \geqslant 0$. *Then either* $u(r) \equiv v(r)$ *whenever* $u(r) \geqslant \beta$, *or* $u(0) = v(0)$ *and* $u(r) \neq v(r)$ *whenever* $r > 0$ *and* $u(r) \geqslant \beta$.

In the following proofs, we shall retain the notation of Section 3.3, with $r(u)$ and $s(u)$ respectively denoting the inverses of u and v.

Proof of Theorem 3.4.1. Suppose for contradiction that $r(u) > s(u)$ for each $u \in (0, v(0)]$. By Lemma 3.3.3, we get $r'(u) < s'(u)$ on $(0, v(0))$, which is impossible since $s'(u) \to -\infty$ as $u \to v(0)$, while $r'(v(0))$ is finite since $r(v(0)) > s(v(0))$.

Proof of Theorem 3.4.2. If the conclusion of the theorem does not hold, then by Lemma 3.3.2 it is easy to verify that *either* there exists $u_1 \in (0, \beta)$ such that

$$u(r) \equiv v(r) \qquad \text{when} \quad u(r) \leqslant u_1,$$

or there exists $U \in (0, \beta]$ such that

$$\begin{gathered} r(u) \neq s(u) \qquad \text{for} \quad u \in (0, U) \\ r(U) = s(U) = R \ \text{(say)}. \end{gathered} \tag{3.4.1}$$

If the first alternative holds, then we consider the inverse functions $r(u)$ and $s(u)$, which satisfy

$$r(u_1) = s(u_1), \qquad r'(u_1) = s'(u_1).$$

We shall show that in fact $r(u) \equiv s(u)$ whenever $u \in (0, \beta]$. To do this, first write equation (1.1.2) as a first order system

$$\begin{aligned} w' &= \frac{n-1}{r} w + f(u) \\ u' &= -\Omega^{-1}(|w|), \end{aligned}$$

where we use the fact that $u' < 0$, $w < 0$ whenever $u(r) \leqslant \beta$; see footnote 2 in Section 3.3.

We now consider the inverse function $r(u)$ on the interval $u \in [u_1, \beta]$, and similarly define

$$w(u) = w(r(u)).$$

Then one finds at once that

$$r'(u) = -\frac{1}{\Omega^{-1}(w)}$$

$$w'(u) = \left(\frac{n-1}{r} w + f(u)\right) \Big/ \Omega^{-1}(w),$$

and of course the same first order system is satisfied by the inverse functions $s(u)$ and $z(u) = w(s(u))$.

Clearly we have as well that

$$r(u_1) = s(u_1), \qquad w(u_1) = z(u_1).$$

We can now apply the standard uniqueness theorem for the Cauchy initial value problem to this system, noting that $\Omega^{-1}(w)$ is of class C^1 (since $w < 0$ and $\Omega^{-1} \in C^1$ from (H3$'$)) and that $f(u)$ is continuous in the *independent* variable u. Hence $r(u) = s(u)$ and $w(u) = z(u)$ for $u \in (0, \beta]$, which in turn implies that $u(r) \equiv v(r)$ whenever $u(r) \leqslant \beta$.

Thus to prove the theorem it is enough to show that the second possibility, namely (3.4.1), cannot happen. Therefore suppose for contradiction that (3.4.1) holds, say with $r(u) > s(u)$ for $u \in (0, U)$. Hence $r' - s' < 0$ on $(0, U)$ by Lemma 3.3.3. Keeping in mind (3.2.1), from (3.1.5) we get[3]

$$R(LA(L))^\alpha - \lambda^\alpha = -\int_0^U \frac{K(u, p)}{p}\, du \qquad (\alpha = 1/(n-1)),$$

where $L = l(R)$ and $p = p(u) = |u'(r(u))|$. An analogous formula holds for the function v. Then, with an obvious meaning for the symbols, we obtain by subtraction

$$R\{(LA(L))^\alpha - (MA(M))^\alpha\} - (\lambda^\alpha - \mu^\alpha)$$
$$= \int_0^U \left(\frac{K(u, q)}{q} - \frac{K(u, p)}{p}\right) du. \tag{3.4.2}$$

[3] The result holds when $u > 0$ for all $r > 0$. Otherwise we get the same formula with $\lambda = 0$.

Evidently $L \leqslant M$ by (3.1.1) since $p < q$ for $u \in (0, U)$. Moreover, by Lemma 3.2.2 we have $\lambda \geqslant \mu$ when $u > v > 0$ on (R, ∞). On the other hand, if v has compact support then correspondingly we have $\mu = 0$, and one sees again that $\lambda \geqslant \mu$.

Hence the left hand side of (3.4.2) is non-positive. Moreover, exactly as in the proof of Lemma 3.3.2, the right hand side is non-negative (note that $K(u, p)$ and $K(u, q)$ are interchanged in (3.4.2) from (3.3.1)). It follows now that both sides of (3.4.2) vanish, which implies that

$$L = M, \qquad \lambda = \mu$$

and $\tilde{l} = \tilde{p}$ for each $u \in (0, U)$ (recall here that $\tilde{p} \in (p, q)$ is obtained as in the proof of Lemma 3.3.2 by use of the mean value theorem). The relation $\tilde{l} = \tilde{p}$ in turn shows that $F(u) \equiv 0$ for all $u \in (0, U)$.

This being shown, also $f(u) \equiv 0$ for $u \in (0, U)$. However the condition $L = M$ implies $r'(U) = s'(U)$. Hence, using the uniqueness argument of the first alternative once again, we get $r \equiv s$ for $u \in (0, \beta)$, an obvious contradiction.

Proof of Theorem 3.4.3. If $u(0) \neq v(0)$, then we can suppose $u(0) > v(0)$ (say); hence, without loss of generality,

$$\begin{gathered} u(r) > v(r) \qquad \text{for} \quad 0 \leqslant r < R \\ u(R) = v(R) \geqslant \beta. \end{gathered} \tag{3.4.3}$$

Now put $w_1 = A(|u'|)\, u'$, $w_2 = A(|v'|)\, v'$ and $\omega = w_1 - w_2$. Then, since

$$\beta < v < u < u(0) \leqslant \gamma \qquad \text{on } (0, R),$$

from (1.1.3) and the hypothesis (F2) follows

$$\omega' + \frac{n-1}{r}\omega = f(v) - f(u) \geqslant 0, \qquad 0 < r < R.$$

Consequently $r^{n-1}\omega$ is non-decreasing on $(0, R)$, so that $\omega(r) \geqslant 0$ for $0 \leqslant r \leqslant R$ because $\omega(0) = 0$. But then $u'(r) \geqslant v'(r)$ because $pA(p)$ is increasing. By integration, $u(R) - u(0) \geqslant v(R) - v(0)$, so from (3.4.3) in turn $u(0) \leqslant v(0)$, a contradiction.

In the remaining case $u(0) = v(0)$. If the theorem fails, then there will exist $u_0 \in (\beta, u(0))$ such that

$$u(r) = v(r) \qquad \text{whenever} \quad u(r) \in [u_0, u(0)]$$

or there will exist points $0 \leqslant r_1 < r_2$ such that (say)

$$\begin{aligned} u(r) > v(r), \qquad & r_1 < r < r_2, \\ u(r_1) = v(r_1), & \\ u(r_2) = v(r_2) \geqslant \beta. & \end{aligned} \tag{3.4.4}$$

If the first alternative holds, then again we consider the inverse functions $r(u)$ and $s(u)$, which satisfy

$$r(u_0) = s(u_0), \qquad r'(u_0) = s'(u_0).$$

As in the proof of Theorem 3.4.2, the uniqueness of the Cauchy problem for the first order system in the variable u applies. Consequently $r(u) \equiv s(u)$ for $u \in [\beta, u(0)]$, as required.

In the second case, namely (3.4.2), it is clear that that $\omega(r_1) \geqslant 0$, while $r^{n-1}\omega(r)$ is non-decreasing on (r_1, r_2). Hence $\omega(r) \geqslant 0$ and $u'(r) \geqslant v'(r)$ on (r_1, r_2). But $u(r_2) = v(r_2)$, so in fact $u'(r) \equiv v'(r)$ on (r_1, r_2), which contradicts the fact that $u > v$ on this interval.

Proposition 3 (First part). *Suppose all the hypotheses of Theorem* I *hold, except* (A3) *and* (F3). *Then if u and v are two solutions of* (*) *with $u(0), v(0) \leqslant \gamma$, we have necessarily*

$$u(0) = v(0).$$

Moreover if $u \not\equiv v$ then $u(r) \neq v(r)$ for all $r > 0$ where $u(r) > 0$. (If u and v have compact support, then also $\inf\{r > 0;\ u(r) = 0\} \neq \inf\{r > 0;\ v(r) = 0\}$.)

Remark. In particular, if $u \not\equiv v$ then necessarily either $u \geqslant v$ for all $r > 0$ or $u \leqslant v$ for all $r > 0$.

Proof. By Theorems 3.4.2 and 3.4.3, it is evident that either $u(r) \equiv v(r)$ or else $u(r) \neq v(r)$ for all $r > 0$ such that $u(r) > 0$. In the latter case, Theorem 3.4.1 implies that $u(0) = v(0)$. The final part of the theorem is a consequence of Lemma 3.3.3 (the monotone separation theorem).

We can now prove Theorem I. Let u and v be two solutions of (*) such that $u(0), v(0) < \gamma$. By Proposition 3 if the solutions are not identical, then $u(0) = v(0)$ but $u(r) \neq v(r)$ for small $r > 0$.

On the other hand, by (A3), (F3) and the uniqueness conclusions of the Appendix (Proposition A.4) we have $u \equiv v$ for all suitably small r (where $u(r) \geqslant \beta$). This contradiction completes the proof.

Theorem II is proved in the same way as Theorem I except that we use Proposition A.2 instead of Proposition A.4 of the Appendix.

3.5. *Uniqueness Theorem* II

In the previous section we considered the case when condition (F2) holds. Here we shall study the case in which the hypothesis (F2) is replaced by (A2′), (F2′).

We begin with the following analogue of Theorem 3.4.3.

THEOREM 3.5.1. *Suppose* (A2′) *and* (F2′) *are satisfied. Let u and v be two solutions of* (*) *such that* $u(0) \neq v(0)$ *but* $u(R) = v(R) \geqslant \beta$ *for some* $R > 0$. *Then* $u(R) = v(R) = \beta$ *and*

(i) $u(r) \neq v(r)$ *when* $r \in [0, R)$,

(ii) $u'(R) \neq v'(R)$.

Proof. As in the demonstration of Theorem 3.4.3 the problem reduces to the case where (3.4.3) holds. Define

$$\bar{u} = u - \beta, \qquad \bar{v} = v - \beta, \qquad \theta = \sup_{0 \leqslant r < R} \frac{\bar{u}(r)}{\bar{v}(r)},$$

so that obviously $\theta > 1$. Clearly

$$\bar{v}(r) \leqslant \bar{u}(r) \leqslant \theta \bar{v}(r) \qquad \text{for} \quad 0 \leqslant r \leqslant R,$$

and moreover there will necessarily be a first point $\eta \in [0, R]$ where the second equality holds:

$$\bar{u}(\eta) = \theta \bar{v}(\eta).$$

As before, we put

$$w_1 = u' A(|u'|), \qquad w_2 = v' A(|v'|),$$

and also introduce the function

$$\omega = w_1 - \theta^{\nu - 1} w_2$$

(no confusion should result from this slightly different definition for ω). Then by (1.1.3) we get

$$(r^{n-1} \omega)' = r^{n-1} (\theta^{\nu - 1} f(v) - f(u)).$$

Now, for $0 \leqslant r \leqslant R$,

$$\theta^{\nu-1} f(v) - f(u) = (\theta\bar{v})^{\nu-1} \frac{f(v)}{(v-\beta)^{\nu-1}} - \bar{u}^{\nu-1} \frac{f(u)}{(u-\beta)^{\nu-1}}$$

$$\geqslant \frac{f(u)}{(u-\beta)^{\nu-1}} \{(\theta\bar{v})^{\nu-1} - \bar{u}^{\nu-1}\} \geqslant 0,$$

by (F2′) and the fact that $v < u$ and $\bar{u} \leqslant \theta\bar{v}$. Hence, since $\omega(0) = 0$ we get $\omega(r) \geqslant 0$ for $0 \leqslant r \leqslant R$. This implies that

$$u'(r) \geqslant \theta v'(r). \tag{3.5.1}$$

Indeed, let $p = |u'(r)|$, $q = |v'(r)|$ and suppose for contradiction that $p > \theta q$ (recall $u' < 0$). Then

$$0 \leqslant \omega = \theta^{\nu-1} q A(q) - p A(p) = (\theta q)^{\nu-1} \frac{A(q)}{q^{\nu-2}} - p^{\nu-1} \frac{A(p)}{p^{\nu-2}}$$

$$< p^{\nu-1} \left\{ \frac{A(q)}{q^{\nu-2}} - \frac{A(p)}{p^{\nu-2}} \right\} \leqslant 0$$

since, by (A2′) the quantity $A(p)/p^{\nu-2}$ is non-decreasing, while $p > \theta q > q$. This is a contradiction, proving (3.5.1).

Now integrate the relation (3.5.1) from η to R, assuming that $\eta < R$. We get

$$u(R) - u(\eta) \geqslant \theta(v(R) - v(\eta))$$

and since $\bar{u}(\eta) = \theta\bar{v}(\eta)$ it follows that

$$\bar{u}(R) \geqslant \theta\bar{v}(R), \tag{3.5.2}$$

a result which also trivially holds if $\eta = R$.

Since $u(R) = v(R)$, the inequality (3.5.2) yields

$$u(R) - \beta \geqslant \theta(u(R) - \beta). \tag{3.5.3}$$

This is clearly impossible unless $u(R) = v(R) = \beta$.

Thus suppose finally that this last condition holds. We must show that $u'(R) \neq v'(R)$. First if $\eta < R$, then the proof of (3.5.2) equally shows that $\bar{u}(r) \geqslant \theta\bar{v}(r)$ for $\eta < r \leqslant R$. But by definition $\bar{u} \leqslant \theta\bar{v}$, so that $\bar{u} \equiv \theta\bar{v}$ for $\eta < r \leqslant R$. Hence $u'(R) = \theta v'(R)$, as required.

On the other hand, if $\eta = R$ then we use the fact that $\bar{u}(r) < \theta\bar{v}(r)$ for $0 \leqslant r < R$. In this case, surely $\bar{u}'(R) \geqslant \theta\bar{v}'(R)$, while at the same time the inequality $\bar{u}'(R) > \theta\bar{v}'(R)$ contradicts the definition of θ. Hence again $\bar{u}'(r) = \theta\bar{v}'(R) \neq \bar{v}'(R)$. This completes the proof.

THEOREM 3.5.2. *Suppose* (A2$'$) *and* (F2$'$) *are satisfied. Let u and v be two solutions of* (*) *such that $u(0) = v(0)$ and $u(R) = v(R) \geqslant \beta$ for some $R > 0$. Then $u(r) \equiv v(r)$ whenever $u(r) \geqslant \beta$.*

Proof. As in the proof of Theorem 3.4.3, if the theorem fails, there will be points $0 \leqslant r_1 < r_2$ such that

$$u(r) > v(r), \qquad r \in (r_1, r_2),$$
$$u(r_1) = v(r_1),$$
$$u(r_2) = v(r_2) \geqslant \beta.$$

Modifying slightly the proof of Theorem 3.5.1, we let

$$\theta = \sup_{r_1 \leqslant r < r_2} \frac{\bar{u}(r)}{\bar{v}(r)}.$$

The previous proof then leads to the same result as before, e.g.

$$u(r_2) - \beta \geqslant \theta(v(r_2) - \beta). \tag{3.5.3$'$}$$

Note however that equality *cannot hold in* (3.5.3$'$) as it did in (3.5.3). This is because we have

$$\theta^{v-1} f(v) - f(u) = (\theta^{v-1} - 1)\, f(u) > 0 \qquad \text{at } r_1,$$

so that the integration leading to (3.5.1) yields instead the modified result

$$u'(r) > \theta v'(r), \quad r_1 < r < r_2. \tag{3.5.1$'$}$$

This being the case, (3.5.3$'$) now contradicts the earlier condition $u(r_2) = v(r_2) \geqslant \beta$, completing the proof.

Remark. A result similar to Theorems 3.5.1 and 3.5.2, but restricted to the degenerate Laplace operator and to the function $f(u)/(u - \beta)^{m-1}$ being decreasing (rather than non-increasing), was recently obtained by Diaz and Saa. They also required Lipschitz continuity for the function $f(u)$.

PROPOSITION 3 (Second part). *Suppose all the hypotheses of Theorem* III *hold, except* (A3) *and* (F3). *Then the conclusion is the same as for the first part of the Proposition* 3 *in Section* 3.4.

Proof. By Theorems 3.5.1. and 3.5.2 there can be only the following three cases:

(i) $u(r) \neq v(r)$ for all $r > 0$ such that $u(r) > 0$

(ii) $u(0) = v(0)$ and $u(r) \equiv v(r)$ for $r > 0$

(iii) $u(0) \neq v(0)$ and $u(r) \equiv v(r)$ when $u(r) \leqslant \beta$; $u(R) = v(R) = \beta$, $u'(R) \neq v'(R)$.

In case (i), by Theorem 3.4.1[4] we get $u(0) = v(0)$. On the other hand, case (iii) clearly can't happen because we would then have both $u'(R) = v'(R)$ and $u'(R) \neq v'(R)$.

To obtain Theorem III, we argue exactly as at the end of Section 4 for the case of Theorem I.

3.6. *Uniqueness Theorem* IV

The main Theorems I–III require the strong assumptions (A1) and (A2), though at the same time conditions (F1) and (F2′) are quite general. The purpose of this Section is to show that (A1), (A2) can be avoided as principal hypotheses, provided on the other hand the conditions (F1), (F2′) and (F3) are considerably strengthened.

The alternative result is of particular importance for the degenerate operator $A(p) = p^{2-m}$ in the case when $1 < n < 2$. Indeed we have earlier noted in the remarks following Proposition 3.0 that (A1) fails for this operator precisely when

$$1 < n < 2 - \frac{1}{m}.$$

The appropriately strengthened versions of (F1), (F2′), (F3) are the following:

(G1) $f(u) \leqslant 0$ for $0 < u < \alpha$, for some $\alpha > 0$.

(G2) The function $u \to f(u)/(u-\alpha)_v$ is positive and non-increasing for $\alpha < u < \gamma$, where v is the constant in (A2′).

(G3) f is locally Lipschitz continuous on (α, γ).

Conditions (G1), (G2) imply that $f(\alpha) = 0$ and $f(u) > 0$ for $\alpha < u < \gamma$. Even more, by (G2), the indeterminate form

$$\lim_{u \downarrow \alpha} \frac{f(u)}{(u-\alpha)^v}$$

must exist, either positive or infinite.

[4] Note that this theorem continues to apply for the hypothesis (F2′) as well as for (F2), since the only use which was made of the latter assumption was to guarantee that $f(u) > 0$ when $u \in (\beta, \gamma)$.

THEOREM IV. *Suppose* $n > 1$ *and assumes hypotheses* (A2′), (A3) *and* (G1), (G2), (G3). *Then* (*) *has at most one solution* u *such that* $u(0) < \gamma$.

As we have already noted in the Introduction, conditions (A3) and (G3) can be omitted and an interesting conclusion still retained. *That is, if* u *and* v *are two solutions of* (*) *and* (A2), (G1), (G2) *are satisfied, then the conclusions of Proposition* 3 *hold, with* β *replaced by* α. We shall in fact prove only this result, since, as in Sections 4 and 5, the result of Theorem IV is then an immediate corollary.

A result corresponding to Theorem II can also be obtained, but can be left to the reader.

The proof will follow the outline of those already given, and accordingly we can omit many of the details. We begin with an important lemma.

LEMMA 3.6.1. *Let* (G1) *hold and let* u *be a solution of* (*) *with* $u(0) < \gamma$. *Then there exists a number* $\tilde{\lambda} \geqslant 0$ (*finite*) *such that*

$$r^{n-1} pA(p) \to \tilde{\lambda} \qquad \text{as} \quad r \to \infty, \tag{3.6.1}$$

where $p = |u'(r)|$.

The proof follows immediately from the identity (see (1.1.3))

$$r^{n-1} pA(p) - r_1^{n-1} p_1 A(p_1) = \int_{r_1}^{r} \rho^{n-1} f(u)\, d\rho \tag{3.6.2}$$

and the fact that $f(u) \leqslant 0$ for $0 < u < \alpha$. For a similar argument, see also [PS2], Lemma 5 (i).

LEMMA 3.6.2. *Suppose* u *and* v *are two solutions of* (*) *with* $u(0), v(0) < \gamma$. *If* $u \leqslant v$ *for all sufficiently large* r, *then* $\tilde{\lambda} \leqslant \tilde{\mu}$ (*here* $\tilde{\mu}$ *is the limit value in* (3.6.1) *for the function* v).

Proof. Let $\Omega(p) = pA(p)$ for $p > 0$. Suppose for contradiction that $\tilde{\lambda} < \tilde{\mu}$. Then for all sufficiently large r we have $q > 0$ (since $\tilde{\mu} > 0$), and moreover by Lemma 3.6.1,

$$\lim_{r \to \infty} \frac{\Omega(p)}{\Omega(q)} = \frac{\tilde{\lambda}}{\tilde{\mu}} < 1.$$

Hence $p < q$ for all large r. But then

$$u(r) = \int_r^\infty p(\rho)\, d\rho < \int_r^\infty q(\rho)\, d\rho = v(r),$$

contradicting the hypothesis of the lemma. (A similar but less general result is given in [PS2], Lemma 8).

The following stages of the proof follow those in Sections 3–5. It is convenient here to introduce the condition

(G2′) $f(u) > 0$ for $\alpha < u < \gamma$.

This is an obvious consequence of (G2). Moreover, we note that if (G1), (G2′) hold, then the hypothesis of Lemma 1.2.6 is satisfied, so that for any solution of (*) we have $u'(r) < 0$ for $r > 0$ as long as $u(r) > 0$.

LEMMA 3.6.3. *Let* (G1), (G2′) *hold. Then the conclusion of Lemma* 3.3.1 *remains true.*

The proof is exactly the same as for Lemma 3.3.1. The next result is the analogue of Lemma 3.3.2.

LEMMA 3.6.4. *Suppose* (G1) *is satisfied. If $r - s$ has two zeros in* $(0, \alpha]$, *say ξ_0 and ξ_1, then $r - s \equiv 0$ for all u between ξ_0 and ξ_1.*

Proof. We argue by contradiction as in the proof of Lemma 3.3.2, but replacing β by α throughout. Moreover, instead of the identity (3.3.2), we have from (3.6.2)

$$r_2^{n-1}\Omega(p_2) - r_1^{n-1}\Omega(p_1) = \int_{\xi_2}^{\xi_1} r^{n-1}\frac{f(u)}{p}\,du.$$

In turn, in place of (3.3.3) we get, in an obvious notation,

$$\begin{aligned}(r_2^{n-1} - r_1^{n-1})\,\Omega(p_2) - r_1^{n-1}(\Omega(p_1) - \Omega(q_1))\\ = \int_{\xi_2}^{\xi_1}\left\{\frac{r(u)^{n-1}}{p} - \frac{s(u)^{n-1}}{q}\right\} f(u)\,du \leqslant 0,\end{aligned} \tag{3.6.3}$$

since $p < q$ and $r > s$ (recall that $f(u) \leqslant 0$ for $0 < u \leqslant \alpha$). On the other hand, as in the proof of Lemma 3.3.2, the left side of (3.6.3) is strictly positive, a contradiction.

LEMMA 3.6.5. *Suppose* (G1), (G2′) *hold. Then the conclusion of Lemma* 3.3.3 *remains true.*

The proof is exactly the same as before, as is also the case for the following

THEOREM 3.6.6. *Suppose* (G1), (G2′) *are satisfied. Then the conclusion of Theorem* 3.4.1 *remains true.*

THEOREM 3.6.7. *Suppose hypothesis* (G1) *is satisfied. Let* u *and* v *be two solutions of* (*) *such that* $u(R)=v(R)\in(0,\alpha]$ *for some* $R>0$. *Then* $u(r)\equiv v(r)$ *whenever* $u(r)\leqslant\alpha$.

Proof. We proceed as in the proof of Theorem 3.4.2, but replacing β by α throughout and basing our identities on (3.6.2) rather than (3.1.5). There results (see (3.6.3))

$$R^{n-1}(\Omega(\tilde{p})-\Omega(\tilde{q}))-(\tilde{\lambda}-\tilde{\mu})=-\int_0^{\tilde{u}}\left\{\frac{r(u)^{n-1}}{p}-\frac{s(u)^{n-1}}{q}\right\}f(u)\,du. \qquad (3.6.4)$$

Continuing as in the proof of Theorem 3.4.2, we have $\tilde{p}\leqslant\tilde{q}$, $p<q$ for $u\in(0,\tilde{u})$ and $\tilde{\lambda}\geqslant\tilde{\mu}$ by Lemma 3.6.2. Thus the only possibility for maintaining (3.6.4) is

$$\tilde{p}=\tilde{q}, \qquad \tilde{\lambda}=\tilde{\mu}, \qquad f(u)\equiv 0 \qquad \text{for} \quad u\in(0,\tilde{u}).$$

Integration of (1.1.3) then gives $u\equiv v$ for $r\geqslant R$, a contradiction.

THEOREM 3.6.8. *Suppose* (A2′) *and* (G2) *are satisfied. Let* u *and* v *be two solutions of* (*) *such that* $u(0)\neq v(0)$, *but* $u(R)=v(R)\geqslant\alpha$ *for some* $R>0$. *Then* $u(R)=\alpha$ *and*

(i) $u(r)\neq v(r)$ *when* $0\leqslant r<R$,

(ii) $u'(R)\neq v'(R)$.

The proof is exactly the same as that of Theorem 3.5.1, except that β is replaced by α. The same is the case for the following Theorem 3.6.9, the analogue of Theorem 3.5.2.

THEOREM 3.6.9. *Suppose* (A2′) *and* (G2) *are satisfied. Then Theorem* 3.5.2 *holds with* β *replaced by* α.

We can now prove

PROPOSITION 3 (Third part). *Suppose that* (G1), (G2) *and* (A2′) *hold. Then if* u *and* v *are two solutions of* (*) *with* $u(0), v(0)\leqslant\gamma$, *we have necessarily*

$$u(0)=v(0).$$

Moreover if $u\not\equiv v$ *then* $u(r)\neq v(r)$ *for all* $r>0$ *where* $u(r)>0$. (*If* u *and* v *have compact support, then also* $\inf\{r>0;\ u(r)=0\}\neq\inf\{r>0;\ v(r)=0\}$.)

This is obtained in exactly the same way as the second part of Proposition 3, with only the following differences:

Theorem 3.6.6 replaces 3.4.1

Theorem 3.6.7 replaces 3.4.2

Theorem 3.6.8 replaces 3.5.1

Theorem 3.6.9 replaces 3.5.2.

3.7. *Remarks on the Exterior Problem*

The work of this paper carries over without essential change to the study of the following exterior Neumann problem for the equation (1.1):

$$\operatorname{div}(A(\,|Du|)\,Du) + f(u) = 0, \qquad x \in B$$
$$Du = 0 \qquad \text{on } \partial B, \qquad u(x) \to 0 \qquad \text{as} \quad |x| \to \infty,$$

where B is the *exterior* of a ball of radius $b > 0$ in $\mathbf{R}^n$, $n > 1$, and where we are concerned with non-negative, non-trivial solutions. Just as for the ground state problem for equation (1.1), solutions of the above problem can be expected to be radially symmetric with respect to the center of the ball, and according we restrict discussion to that case.

Thus we consider the following direct analogue of the problem (*) introduced in Section 1.1,

$$\begin{aligned} &(Au')' + \frac{n-1}{r} Au' + f(u) = 0, \qquad r \geqslant a \\ &u'(b) = 0, \qquad u \geqslant 0 \qquad \text{for} \quad r \geqslant b; \qquad u \to 0 \qquad \text{as} \quad r \to \infty; \qquad u \not\equiv 0. \end{aligned} \tag{*}$$

We treat classical solutions of (**), with the precise meaning that $u \in \mathbf{C}^1([b, \infty))$ and also $w = Au' \in \mathbf{C}^1([b, \infty))$.

The discussion exactly parallels that already given, the only exception being that the role of $r = 0$ in the ground state problem is here played by the point $r = b$. This in fact simplifies the argument in several places, since the problem is no longer singular at the initial point b. The main results are then the same as before, namely: the principal properties of solutions given in Section 1.2; the compact support theorems of Section 1.3; the Existence Theorem of Chapter 2; and the Uniqueness Theorems I–IV of Chapter 3.

Appendix: The Cauchy Problem

The initial value problem at $r = 0$ for equation (1.1.1) is singular due to the term $(n-1)/r$ as well as the possible singularity of $A(p)$ when $p = 0$. We state here the main results which are required in the paper.

Consider the initial value problem

$$\begin{aligned} &(Au')' + \frac{n-1}{r} Au' + f(u) = 0, \qquad r > 0 \\ &u(0) = \xi > 0, \qquad u'(0) = 0 \end{aligned} \tag{P}$$

for a classical solution u (see Section 1.1).

Proposition A1. *Suppose the hypotheses* (H1)–(H3) *are satisfied. Then problem* (P) *has a classical solution in a neighborhood of the origin.*

Proof. Put $\Omega(p) = pA(|p|)$ for $p \neq 0$, $\Omega(0) = 0$.[5] Existence of a solution is proved by applying the Schauder fixed point theorem to the operator

$$(Tu)(r) = \xi - \int_0^r \Omega^{-1}\left(\int_0^\rho f(u(t)) \left(\frac{t}{\rho}\right)^{n-1} dt\right) d\rho;$$

see [NS].

Proposition A2. *Suppose that A is of class C^1 on* $(0, \infty)$ *and that $f(u)$ is Lipschitz continuous for $u \in J$, where J is a subinterval of* $(0, \infty)$ *containing ξ. Assume also that the derivative of the* (*increasing*) *function Ω is bounded from zero on every bounded subset of* $(0, \infty)$. *Then the solution of* (P) *is unique as long as it exists and remains in J.*

Proof. Suppose that u and v are two different solutions of (P) whose values lie in J. Then the function

$$\omega = \Omega(u') - \Omega(v')$$

is a solution of the Cauchy problem

$$\omega' + \frac{n-1}{r}\omega = \psi(r), \qquad \omega(0) = 0,$$

where $\psi(r) = f(v(r)) - f(u(r))$. It follows that

$$|\omega(r)| = \int_0^r \psi(t) \left(\frac{t}{r}\right)^{n-1} dt \leqslant \frac{r}{n} \sup_{[0,\, r]} |\psi(t)|. \tag{1}$$

On the other hand, since f is Lipschitz continuous on J, we obtain, for appropriate values r_0, $M > 0$,

$$|\psi(r)| \leqslant M\, |u(r) - v(r)| \qquad \text{for} \quad r \in [0, r_0].$$

[5] This extends the domain of $\Omega(p)$ from $p \geqslant 0$ to all $p \in \mathbf{R}$.

Hence recalling that $u(0)=v(0)$, the principal hypothesis of the proposition yields

$$|\psi(r)| \leqslant M \int_0^r |u'(s)-v'(s)|\, ds \leqslant \frac{M}{k} \int_0^r |\omega(s)|\, ds, \tag{2}$$

where $k>0$ is the infimum of the derivative of $\Omega(p)$ on the set $(0, p_0)$, and p_0 is a bound for $|u'(r)|$ and $|v'(r)|$ on the interval $[0, r_0]$; here we recall that both u' and v' are zero when $r=0$, and are also continuous.

Combining (1) and (2) yields

$$|\omega(r)| \leqslant \frac{1}{n}\frac{M}{k} r \int_0^r |\omega(s)|\, ds$$

for $r \leqslant r_1$. It now follows from Gronwall's inequality that $\omega \equiv 0$ for $r \leqslant r_1$. Consequently $\Omega(u') \equiv \Omega(v')$, and $u' \equiv v'$ for $r \leqslant r_1$. With the initial point $r=r_0$ replaced by $r=\rho>0$, for an appropriate value ρ, the same proof can be reapplied as often as necessary to give uniqueness of any continuation of the solution whose values lie in J.

PROPOSITION A3. *Suppose that the hypotheses of Proposition* A2 *are satisfied. Then solutions of problem* (P) *depend continuously on the initial data* ξ.

Proof. We proceed in essentially the same way as in the demonstration of Proposition A2. Let $u(0)=\xi$, $v(0)=\xi+h$. Then on any compact subset of values r for which the solutions are defined, we have

$$|u(r)-v(r)| \leqslant h + \int_0^r |u'(s)-v'(s)|\, ds$$

(it can be assumed that $h>0$). Consequently (2) can be replaced by

$$|\psi(r)| \leqslant M\left(h + \frac{1}{k}\int_0^r |\omega(s)|\, ds\right),$$

with the constants M, k as before. In turn

$$|\omega(r)| \leqslant \frac{1}{n} Mr\left(h + \frac{1}{k}\int_0^r |\omega(s)|\, ds\right),$$

so by Gronwall's inequality

$$|\omega(r)| \leqslant \frac{M}{n} rh \exp\left(\frac{M}{2nk} r^2\right).$$

This gives

$$|u' - v'| \leqslant \frac{M}{nk} rh \exp\left(\frac{M}{2nk} r^2\right),$$

and

$$|u - v| \leqslant h \exp\left(\frac{M}{2nk} r^2\right),$$

completing the proof.

Proposition A2 does not apply to the important case when $A(p) = p^{m-2}$ and $m > 2$, since then $\Omega'(p) = (m-1) p^{m-2}$ is not bounded from zero on bounded subsets of $(0, \infty)$. Therefore it is important to have the following extension of its validity.

PROPOSITION A4. *Suppose A is of class C^1 on $(0, \infty)$ and that $f(u)$ is non-vanishing and Lipschitz continuous for $u \in J$, where J is a subinterval of $(0, \infty)$ containing ξ. Assume also that $\Omega'(p) > 0$ for $p \neq 0$ and*

$$\Omega'(p) \geqslant |\Omega(p)|^{\mu} \tag{3}$$

for all sufficiently small $p \neq 0$, where μ is a fixed exponent in $[0, 2)$. Then the solution of (P) is unique as long as it exists and remains in J.

Condition (3) clearly holds when $A(p) = p^{m-2}$, with $\mu = 0$ if $m \leqslant 2$ and $\mu = 1$ (and $p \leqslant m-1$) if $m > 2$. It is also satisfied when $A(p) = (1 + p^2)^{-s/2} p^{m-2}$ for $s \geqslant 0$, $m > 1$, as one easily checks. We are indebted to L. Veron for the idea of the proof.

Proof of Proposition A4. It can be assumed that $f(u) > 0$ for $u \in J$, the opposite case being treated similarly. We proceed now as in the proof of Proposition A.2, until relation (2). Here the estimate

$$|u'(s) - v'(s)| \leqslant \frac{1}{k} |\omega(s)|, \qquad s \in (0, r_0)$$

must be replaced by

$$|u'(s) - v'(s)| \leqslant \frac{1}{\inf \Omega'(\tilde{p})} |\omega(s)|, \qquad s \in (0, r_0), \tag{4}$$

where the infimum is taken over all intermediate values $\tilde{p}$ between $|u'(s)|$ and $|v'(s)|$. Note, for this estimate, that necessarily $u'(r)$, $v'(r)$ are negative for $r > 0$ because $f(u) > 0$ when $u \in J$, see Lemma 1.1.1 or the proof of Proposition A1.

Obviously, for any solution of (P)

$$\Omega(u') = -\int_0^r f(u(t))(t/r)^{n-1}\,dt.$$

Therefore supposing without loss of generality that $|u'(s)| \leqslant |v'(s)|$ in (4), we find that

$$\Omega(\tilde{p}) > \Omega(|u'|) = -\int_0^s f(u(t))(t/s)^{n-1}\,dt \geqslant \frac{s}{2n} f(u(0)) = \frac{s}{2n} f(\xi),$$

provided that r_0 is chosen even smaller, if necessary, so that $f(u(r)) \geqslant \frac{1}{2} f(u(0))$ for $0 \leqslant r \leqslant r_0$. Using (3) and the fact that $\mu \geqslant 0$, we get

$$\Omega'(\tilde{p}) \geqslant \left(\frac{s}{2n} f(\xi)\right)^{\mu}, \qquad s \in (0, r_0),$$

and of course the same holds when $|v'(s)| \leqslant |u'(s)|$. Hence from (4) and the first inequality of (2) follows

$$|\psi(r)| \leqslant M \int_0^r \left(\frac{2n}{sf(\xi)}\right)^{\mu} |\omega(s)|\,ds.$$

In turn, using (1) for $r \in (0, r_0)$ there holds

$$|\omega(r)| \leqslant Cr \int_0^r \frac{|\omega(s)|}{s^{\mu}}\,ds,$$

where $C = Mn^{\mu-1}(2/f(\xi))^{\mu}$.

One can now apply Gronwall's lemma (for this purpose, it is convenient to replace ω by ω/r), with the conclusion

$$\left|\frac{\omega(r)}{r}\right| \leqslant C \int_0^t \frac{|\omega(s)|}{s^{\mu}}\,ds \cdot \exp\left(C \int_t^r \frac{ds}{s^{\mu-1}}\right)$$

for any $t \in (0, r)$. Of course $|\omega(s)/s|$ is bounded on $(0, r_0)$ by (1). Hence, recalling that $\mu < 2$ and letting $t \to 0$, we get $\omega(r) \equiv 0$ for $r \in (0, r_0)$. The procedure can of course be repeated as often as necessary, proving the theorem.

Remark. The condition that $f(u(0)) > 0$ is essential for the validity of Proposition A.4. Indeed, consider the equation

$$((u')^3)' + \frac{n-1}{r}(u')^3 + 8(1-u) = 0,$$

that is, the case $A(p)=p^2$ and $f(u)=8(1-u)$. If we take $\xi=1$, then $f(\xi)=0$, and the solution of the Cauchy problem (P) is in fact not unique, there being at least the three solutions,

$$u\equiv 1, \qquad u=1-at^2, \qquad u=1+at^2,$$

where $a=\sqrt{1/(n+2)}$.

One can restore uniqueness in case $f(\xi)=0$ by appropriately strengthening the Lipschitz condition at $u=\xi$, but (for once) we shall not pursue this further.

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