Cosets, genericity, and the Weyl group

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A B S T R A C T

In a connected group of finite Morley rank in which, generically, elements belong to connected nilpotent subgroups, proper normalizing cosets of definable subgroups are not generous. We explain why this is true and what consequences this has for the abstract theory of Weyl groups in groups of finite Morley rank.

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The only known infinite simple groups of finite Morley rank are the simple algebraic groups over algebraically closed fields and this is a motivation, among many others, for a classification project of these groups. This project borrows ideas and techniques from the Classification of the Finite Simple Groups but at the same time it may provide, sometimes, a kind of simplified version of the finite case. This is mostly due to the existence of well-behaved notions of genericity and connectivity in the infinite case, which unfortunately have no direct finite analogs.

The present note deals with a very specific and technical topic concerning such arguments based on genericity in the case of infinite groups of finite Morley rank, which serve here to bypass allegro potential complications of various kinds, including finite combinatorics. As a result, we show similarities with algebraic groups at least as far as a theory of Weyl groups is concerned, and naturally this applies also to non-algebraic configurations which are encountered throughout much of the current work in the area.

In a connected reductive algebraic group, maximal (algebraic) tori are conjugate and cover the group generically, with the Weyl group governing essentially the structure of the entire group. In the abstract context, we use the term "generous" to speak of a subset "whose union of conjugates is generic in the group", the typical property of tori in the classical algebraic case. There are at least two abstract versions of tori in groups of finite Morley rank, which coincide at least in the case of a reductive algebraic group, *decent tori* on the one hand and *Carter subgroups* on the other. The main caveat with these two more abstract notions, for a seemingly complete analogy with algebraic groups

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in general, is in both cases an unknown existence, more precisely the existence of a *nontrivial* decent torus on the one hand and the existence of a *generous* Carter subgroup on the other. Still, both notions provide fragments, frequently sufficient, of the theory of algebraic groups and their Cartan subgroups. Here we provide results typically adapted to groups with a generous Carter subgroup, but which also apply in many more situations.

For both notions there are conjugacy theorems, the conjugacy of maximal decent tori [Che05] and of generous Carter subgroups [Jal06]. This gives a natural notion of Weyl group in each case, $N(T)/C^\circ(T)$ for some maximal decent torus T or $N(Q)/Q$ for some generous Carter subgroup Q. In either case the Weyl group is finite and, as with classical Weyl groups and algebraic tori in algebraic groups, its determination and its action on the underlying subgroup is fundamental in the abstract context.

As an element of the Weyl group is a coset in the ambient group, it is then useful to get a description of such cosets, even though recovering from such a description the structure and the action of the Weyl group is in general a particularly delicate task. This is mostly due to the fact that, in practice, one can only get a generic, and thus weak, description of the coset. In [CJ04] such arguments were however developed intensively, and this was highly influenced by one of the most critical aspects of the early work, notably by Nesin, on the so-called "bad" groups of finite Morley rank [BN94, Theorem 13.3]. In this paper, a pathological coset, whose representative is typically a Weyl element which should not exist, is usually shown to be both generous *and* non-generous, and then the coset does not exist. This is the main protocol, sometimes referred to as "coset arguments", for the limitation of the size of the Weyl group. Generosity is usually obtained by unexpected commutativity between the Weyl elements and the underlying subgroup, and in general this may depend on the specific configuration considered. It is certainly the pathological property in any case, and we shall prove here at a reasonable level of generality that the existing cosets should be non-generous.

In particular, we rearrange as follows the protocol of [CJ04] in the light of further developments of [Jal06] concerning generosity.

Theorem 1 *(Generix and the cosets). Let G be a connected group of finite Morley rank in which, generically, elements belong to connected nilpotent subgroups. Then the coset wH is not generous for any definable subgroup H and any element w normalizing H but not in H.*

The assumption on the generic elements of G in Theorem 1 can take several forms, and we will explain this shortly. The most typical case where Theorem 1 applies is however the case in which $H = Q$ is a generous Carter subgroup of G. In particular, the present paper is also an appendix to [Jal06] on the structure of groups of finite Morley rank with such a generous Carter subgroup, and more precisely a follow-up to Section 3.3 in that paper.

The general idea of the protocol of [CJ04] has been used repeatedly in various contexts, most notably to get a fine description of p-torsion in terms of connected nilpotent subgroups of bounded exponent and of decent tori [BC07]. Applied to the most natural kind of Weyl groups, the protocol shows that centralizers of decent tori are connected in any connected group, implying in particular that the Weyl group $N(T)/C^\circ(T)$ attached to a decent torus T acts faithfully on T. This corresponds to the most typical and smooth applications of the protocol in [CJ04], generally a lemma expedited at the early stage of the analysis of each configuration considered there. With [Che05] and [Jal06], and eventually the finiteness of conjugacy classes of uniformly definable families of decent tori of [FJ08, Theorem 6.4], it became clear that, for that specific lemma, the protocol had implementations autonomous from these specific configurations. Proofs without any reference to [CJ04] appear in [AB08, Fré07b], with a conceptually better and more general implementation in the second case.

A much more delicate use of the protocol can be found in [CJ04, Proposition 6.17]. It is proved there, still in a specific configuration, that the centralizer of a certain finite subgroup of a decent torus is connected, with then a much more restrictive faithful action of the Weyl group. As this special application of the protocol addresses the main intrinsic difficulty of the subject, we mostly refer to this example. As we will see below, the key point is that generosity is in general related to a finiteness property, as opposed to a uniqueness property, a delicate aspect treated "by hand" in [CJ04, Proposition 6.17] and much more conceptually here.

Theorem 1 has general consequences on the action of the Weyl group on the underlying subgroup, under either interpretation of the notion of Weyl group. Back to the concrete example of a reductive algebraic group, the maximal algebraic torus is a divisible abelian subgroup, and the Weyl group acts faithfully on it. The main corollary of Theorem 1 is a general form of this in the abstract context of groups of finite Morley rank.

Corollary 2. *Let G be a connected group of finite Morley rank in which, generically, elements belong to connected nilpotent subgroups. Suppose that H is a definable connected generous subgroup, that w is an element normalizing H but not in H, of finite order n modulo H, and that $\{h^n \mid h \in H\}$ is generic in H. Then $C_H(w) < H$.*

In the case of a connected reductive algebraic group, the subgroup H in Corollary 2 is typically the maximal torus T and w a representative of a nontrivial element of order n of the Weyl group. In the finite Morley rank case, H may typically be a generous n-divisible Carter subgroup Q, and w a representative of a nontrivial element of order n of the Weyl group $N(Q)/Q$. One gets then, for instance if Q is a divisible abelian generous Carter subgroup as in Corollary 14 below, consequences qualitatively similar in the finite Morley rank case.

As for Theorem 1, the statement adopted in Corollary 2 is far more general than what it says about this typical case. Less typical applications can be found along the lines of [DJ07] in the context of connected *locally*° *solvable*° groups, the smallest natural class of groups of finite Morley rank containing connected solvable groups and Chevalley groups of type PSL_2 and SL_2 over algebraically closed fields.

1. Technicalities and environment

Before passing to the proofs, we review briefly the background needed in our arguments, as well as facts related to our results.

Groups of finite Morley rank are equipped with a rudimentary notion of finite dimension on their definable sets, satisfying as axioms a few basic properties of the natural dimension of varieties in algebraic geometry over algebraically closed fields. By definable we mean definable by a first-order logic formula, possibly with parameters and possibly in quotients by definable equivalence relations. The dimension, or "rank", of a definable set A is denoted by $\mathrm{rk}(A)$.

The finiteness of the rank implies the descending chain condition on definable subgroups, and this naturally gives abstract versions of classical notions of the theory of algebraic groups:

- The *definable hull* of an arbitrary subset of the ambient group is the smallest definable subgroup containing that set. It is contained in the Zariski closure in the case of an algebraic group.
- The *connected component* G° of a group G of finite Morley rank is the smallest (normal) definable subgroup of finite index of G, and G is *connected* when $G = G^\circ$.

A fundamental property of a connected group of finite Morley rank is that it cannot be partitioned into two definable *generic* subsets, that is two subsets of maximal rank [Che79]. Our arguments make full use of the following simpler properties.

Fact 3.

(1) *A connected group of finite Morley rank acting definably on a finite set must fix it pointwise.*
(2) *A connected group of finite Morley rank acting definably on a group H of finite Morley rank induces a trivial action on H/H°.*

Proof. The first item is a well-known application of connectedness: as elements of the base set have finite orbits, their (definable) stabilizers are of finite index, and hence cannot be proper. The second item is a special case of the first which does not seem to be specifically mentioned in the literature: as H° is definably characteristic in H, the acting group induces an action on H/H°, and we are then in presence of the action of a connected group on a finite set. □

Following [Jal06], we say that a definable subset of a group G of finite Morley rank is *generous* in G when the union of its G-conjugates is generic in G. In our proof of Theorem 1, we are essentially going to reuse lines of arguments of [Jal06] for dealing with generosity, both for characterizing it and for applying it. When working with generosity in very general contexts, one has to inspect closely each conjugacy class of each individual element of the set considered. The reader can find in [Jal06, §2.2] such an analysis, done there for definable connected subgroups. Another approach for this analysis was mentioned to the author by Cherlin, with a more conceptual geometric proof, dual in some sense, giving also a few more rank equalities. We take here the opportunity to recast these computations in terms of permutation groups, not only because it generalizes naturally, but also as it certainly might be useful in this more general context.

Given a permutation group (G, Ω) and a subset H of Ω, we denote by $N(H)$ and by $C(H)$ the *setwise* and the *pointwise* stabilizer of H respectively, that is $G_{\{H\}}$ and $G_{(H)}$ in a usual permutation group theory notation. We also denote by H^G the set $\{h^g \mid (h, g) \in H \times G\}$, where h^g denotes the image of h under the action of g, as in the case of an action by conjugation. Subsets of the form H^g for some g in G are also called *G-conjugates* of H. Notice that the set H^G can be seen, alternatively, as the union of G-orbits of elements of H, or also as the union of G-conjugates of H. When considering the action of a group on itself by conjugation, as we will do below, all these terminologies and notations are the usual ones, with $N(H)$ and $C(H)$ the *normalizer* and the *centralizer* of H respectively.

We note that in this paper we work only with "exact" normalizers $N(H) = \{g \in G \mid H^g = H\}$, or "stabilizers", as opposed to "generic stabilizers", where the equality $H^g = H$ is understood up to a symmetric difference of lower rank.

Fact 4. *(See [Jal06, Proposition 2.9].) Let (G, Ω) be a ranked permutation group, H a definable subset of Ω, and assume that for r between 0 and $\mathrm{rk}(G/N(H))$ the definable subset H_r of H, consisting of those elements of H belonging to a set of G-conjugates of H of rank exactly r, is nonempty. Then*

$$\mathrm{rk}\left(H_r{}^G\right) = \mathrm{rk}(G) + \mathrm{rk}(H_r) - \mathrm{rk}\left(N(H)\right) - r.$$

Proof. One may proceed exactly as in the geometric proof of [Jal06, Proposition 2.9]. First, notice that all sets H_r considered are definable, by the *definability* of the rank in the Borovik–Poizat axioms for ranked structures [BN94, §4].

In the natural geometry associated to the computation, points are the elements of Ω which are G-conjugate to those of H and lines are the G-conjugates of H. The set of flags is the set of couples (point, line) where the point belongs to the line, and one considers the subflag naturally associated to H_r. Projecting on the set of points one gets, by the *additivity* of the rank, $\mathrm{rk}(H_r{}^G) + r$ for the rank of this subflag, and similarly $\mathrm{rk}(G/N(H)) + \mathrm{rk}(H_r)$ by projecting on the set of lines. The equality follows. $\square$

In the context of a permutation group as in Fact 4, we may naturally say that the definable subset H of Ω is *generous* when the subset H^G of Ω is generic in Ω. Of course, this matches with the usual definition in the case of the action of a group on itself by conjugation. Continuing in the general context of permutation groups, Fact 4 has the following corollary characterizing generosity.

Corollary 5. *Assume furthermore $\mathrm{rk}(G) = \mathrm{rk}(\Omega)$ and $\mathrm{rk}(H) \leqslant \mathrm{rk}(N(H))$ in Fact 4. Then H^G is generic in Ω if and only if $\mathrm{rk}(H_0) = \mathrm{rk}(N(H))$. In this case $\mathrm{rk}(H_0) = \mathrm{rk}(H) = \mathrm{rk}(N(H))$, a generic element of Ω lies in only finitely many conjugates of H, and the same applies to a generic element of H.*

Proof. If H^G is generic in Ω, then one has for some r as in Fact 4 that $H_r{}^G$ is generic in Ω, and then

$$0 \leqslant r = \mathrm{rk}(H_r) - \mathrm{rk}\left(N(H)\right) \leqslant \mathrm{rk}(H) - \mathrm{rk}\left(N(H)\right) \leqslant 0,$$

showing that all these quantities are equal to 0. In particular $r = 0$, and $\mathrm{rk}(H_0) = \mathrm{rk}(N(H))$. Conversely, if $\mathrm{rk}(H_0) = \mathrm{rk}(N(H))$, then $\mathrm{rk}(H_0{}^G) = \mathrm{rk}(G) = \mathrm{rk}(\Omega)$ by Fact 4.

For our last statement, we also see with the above inequalities that $\mathrm{rk}(H) = \mathrm{rk}(N(H))$, and as H_0 and $N(H)$ have the same rank it follows that $\mathrm{rk}(H_0) = \mathrm{rk}(H) = \mathrm{rk}(N(H))$. In particular the definable subset H_0 of H is generic in H, and together with the genericity of $H_0{}^G$ in Ω this is exactly the meaning of our two last claims. $\quad\square$

As emphasized in the proofs, Fact 4, and consequently Corollary 5, remain valid in any context where there is a mere definable and additive rank function. We stress the fact that, under the circumstances of Corollary 5, the generosity of H is equivalent to the genericity of the *definable* sets H_0 and $H_0{}^G$ in H and Ω respectively, a clear characterization only in terms of definable subsets that we will fully exploit.

We also mention, parenthetically, that there are uniform bounds on finite sets throughout. This is one of the Borovik–Poizat axioms, usually called *elimination of infinite quantifiers*, which gives uniform bounds on the cardinals of finite sets in uniformly definable families of sets. This implies, in Fact 4 and Corollary 5, that H_0 is exactly the set of elements of H contained in at most m distinct conjugates of H, for some *fixed* finite m. We will not use such uniform bounds, but of course they can be kept in mind.

A typical case in which Fact 4 and Corollary 5 apply is the case in which the permutation group (G, Ω) is interpretable in a group G of finite Morley rank. In the rest of this paper we are only going to consider the action of a group of finite Morley rank on itself by conjugation, so Fact 4 and Corollary 5 will be applied freely.

As G and Ω are the same is this case, the extra assumption $\mathrm{rk}(G) = \mathrm{rk}(\Omega)$ is then automatically satisfied in the characterization of generosity of Corollary 5. The second assumption $\mathrm{rk}(H) \leqslant \mathrm{rk}(N(H))$ is not satisfied in general, but an interesting case in which it holds is the case in which H has the form $x\Gamma$, where Γ is a definable subgroup of G and x is an element of G normalizing Γ: in this case $\Gamma \leqslant N(x\Gamma)$, and thus $\mathrm{rk}(x\Gamma) = \mathrm{rk}(\Gamma) \leqslant \mathrm{rk}(N(x\Gamma))$. In fact, one sees in this case that $N(x\Gamma)$ is exactly the preimage in $N(\Gamma)$ of $C_{N(\Gamma)/\Gamma}(x \bmod \Gamma)$. All cosets considered in this paper are of this type, and we will make full use of Corollary 5 when considering the generosity of such cosets in the rest of the paper.

We insist again on the fact that the characterization of Corollary 5 is in this case essentially the genericity of H_0 in H (in addition to $\mathrm{rk}(H) = \mathrm{rk}(N(H))$), and thus the fact that only finitely many conjugates of H pass through a generic element of H. In general, and we would like to say with probability almost one, there is not uniqueness. It may be seen by considering the generic element g of a connected reductive algebraic group. It lies in a maximal torus T, which lies in a generous Borel subgroup B; T is the unique of its conjugates containing g [Jal06, Corollary 3.8], but there are several conjugates of B containing g (and permuted by the Weyl group $N(T)/T$).

That is all about the background we will use. We do not use decent tori and Carter subgroups in the present work, Theorem 1 and Corollary 2, but, as they correspond so closely to its most typical applications, it may be useful to recall their definitions and to place more precisely our results in context. A decent torus T of a group of finite Morley rank is a definable (connected) divisible abelian subgroup which coincides with the definable hull of its (divisible abelian) torsion subgroup, and a Carter subgroup Q is a definable connected nilpotent subgroup of finite index in its normalizer (and in particular it satisfies $Q = N^\circ(Q)$). Both types of subgroups exist in any group of finite Morley rank, which is trivial in the first case and follows in the second case from a graduated notion of unipotence on certain connected nilpotent subgroups, for which decent tori are precisely the first stones [FJ08, §3.1]. By [Che05], maximal decent tori are conjugate in any group of finite Morley rank, which indeed follows from the fact that $C^\circ(T)$ is generous for any such decent torus T. By [Jal06], generous Carter subgroups are conjugate in any group of finite Morley rank.

We take this opportunity to mention the following correlation between decent tori and generous Carter subgroups.

Fact 6. *If Q is a generous Carter subgroup of a group of finite Morley rank, then $T \leqslant Q \leqslant C^\circ(T)$ for some maximal decent torus T, and $N(T) = C^\circ(T) \cdot N(Q)$.*

Actually, we will prove something slightly more general than Fact 6, expanding a bit the existing theory of generous subgroups in passing.

Recall first that the existence of a generous Carter subgroup is, maybe, the main open question at the moment concerning groups of finite Morley rank. It is equivalent to the question to know whether any connected group of finite Morley rank containing no proper definable connected generous subgroup is nilpotent (see [Jal06, Genericity Conjecture 4.1b–β]). As in [Jal06, §4.2], a minimal counterexample to the question of existence of a generous Carter subgroup in connected groups has tendency to be *semisimple*, i.e., with all its normal solvable subgroups trivial, and has no proper definable connected generous subgroups.

Fact 7. *Let G be a group of finite Morley rank.*

(1) *If Q is a definable nilpotent subgroup of G, then any definable subgroup of Q generous in G is of finite index in Q.*
(2) *If Q and H are definable subgroups of G° generous in G, with Q nilpotent, then $Q^\circ \leqslant H^\circ$ up to conjugacy.*
(3) *If Q is a generous Carter subgroup of G, then Q is, up to conjugacy, the unique minimal definable subgroup of G° generous in G.*

Proof. (1) Assume H is a definable subgroup of Q, generous in G. Then H must be of finite index in its normalizer, by [Jal06, Lemma 2.2] or more generally Corollary 5. Now by the normalizer condition in infinite nilpotent groups of finite Morley rank, H is of finite index in Q.

(2) By Corollary 5 and connectedness of G°, a generic element of G°, say g, is in conjugates of Q and H, say Q and H, and in only finitely many such conjugates. Now by [Jal06, Fundamental Lemma 3.3], $N^\circ(Q \cap H) \leqslant N^\circ(Q) \cap N^\circ(H)$, and as $N^\circ(Q) = Q^\circ$ and $N^\circ(H) = H^\circ$ by generosity of Q and H (using again [Jal06, Lemma 2.2] or Corollary 5), we get $N^\circ(Q \cap H) \leqslant (Q \cap H)^\circ$. In particular $Q \cap H$ has finite index in its normalizer in Q, and is thus of finite index in Q by the normalizer condition in infinite nilpotent groups of finite Morley rank. In particular, $Q^\circ \leqslant H^\circ$.

(3) By (1) and connectedness of Q, Q is minimal for the generosity of definable subgroups of G°. By (2), any definable generous subgroup H of G° contains a conjugate of Q, i.e., $Q \leqslant H^\circ$ up to conjugacy. Hence item (3) follows from the conjugacy of generous Carter subgroups of [Jal06]. □

The core of the proof of Fact 7(2) may seem to be somehow hidden in the use of [Jal06, Fundamental Lemma 3.3], which essentially relies on Fact 3(1). Fortunately, our proof of Theorem 1 below will reproduce the content of that lemma, with cosets instead of subgroups.

Fact 7(3) provides a way to see generous Carter subgroups in the ostensibly wider class of *minimal* definable generous subgroups, where the problem of existence somehow shifts to the problem of conjugacy.

We now add decent tori into the picture.

Fact 8. *Let G be a group of finite Morley rank.*

(1) *If H is a definable generous subgroup of G°, then H° contains a maximal decent torus T of G.*
(2) *If H is a definable connected generous subgroup of G, minimal with respect to this property, and T is a maximal decent torus of G in H, then $T \leqslant Z(H)$.*
(3) *If T is a maximal decent torus and $C^\circ(T)$ contains a unique minimal definable generous subgroup up to conjugacy, say H, then $N(T) = C^\circ(T) \cdot N(H)$.*

Proof. (1) By [Che05], $C^\circ(T)$ is generous for any decent torus T of G. Arguing as in the proof of Fact 7(2), one finds a generic element in $C^\circ(T) \cap H$ and one deduces similarly that $N^\circ(C^\circ(T) \cap H) \leqslant N^\circ(C^\circ(T)) \cap N^\circ(H) = C^\circ(T) \cap H^\circ$. As T is central in $C^\circ(T)$, this implies in particular that $T \leqslant H^\circ$.

(2) By [Che05], $C_H^\circ(T)$ is generous in H. By transitivity of the generosity of definable subgroups [Jal06, Lemma 3.9a], one deduces that $C_H^\circ(T)$ is generous in G, and the minimality of H forces $C_H^\circ(T) = H$, i.e., $T \leqslant Z(H)$.

(3) We have $T \leqslant N^\circ_{C^\circ(T)}(H) = H^\circ$ by generosity of H in $C^\circ(T)$, and thus $T \leqslant Z(H)$. In particular, $N(H) \leqslant N(T)$. Now a Frattini argument gives the desired decomposition: if $w \in N(T)$, then H and H^w are two minimal definable generous subgroups of $C^\circ(T)$, $H^w = H^\alpha$ for some α in $C^\circ(T)$, and $w = w\alpha^{-1}\alpha \in N(H) \cdot C^\circ(T)$. Notice that $C^\circ(T)$ is normal in $N(T)$. $\square$

Fact 6 follows from Facts 7 and 8, together with the remark that the generous Carter subgroup Q of G, containing the maximal decent torus T, must also be generous in $C^\circ(T)$ (by [Jal06, Lemma 2.3] or Corollary 5).

In the presence of a *nontrivial* maximal decent torus T, the *Weyl group* of an arbitrary group of finite Morley rank is naturally defined as in [CJ04, Theorem 1.8] as $N(T)/C^\circ(T)$, and in the presence of a *generous* Carter subgroup Q, it is defined as in [Jal06, §3.3] as $N(Q)/Q$. In the first case the original definition relied on a particular decent subtorus related to the prime $p = 2$, but since the full proof of conjugacy of maximal decent tori of [Che05] it naturally takes this form. We also mention that the term "Weyl group" made its first appearance, beyond the classical algebraic case, in [Nes89] in the context of "bad" groups of Morley rank 3, with all possible definitions equivalent in this case.

In Fact 6, we see that both notions of Weyl group essentially match, with however

$$N(T)/C^\circ(T) \simeq \big(N(Q)/Q\big)/\big(N_{C^\circ(T)}(Q)/Q\big)$$

isomorphic to a possibly proper quotient of $N(Q)/Q$, and thus a sharper notion with the second definition. Hence when both definitions are possible we prefer the second one, though the question of equality in general is an interesting issue.

We note that everything said here with a decent torus T can be stated similarly with a pseudo-torus T, a slightly more general notion of torus with practically the same properties [Fré07b].

Besides, we note that [Fré07a] provides an analysis of non-generous Carter subgroups in inductive contexts for groups of finite Morley rank, the case of "K^*-groups" and a bit more. This yields the conjugacy of such non-generous Carter subgroups, and eventually gives in these specific cases the full conjugacy of Carter subgroups, in the non-generous case as well as in the generous case. In particular, this then gives a notion of Weyl group in the most pathological situation in which all Carter subgroups would be non-generous, the line antipodal to the one pursed in [Jal06] and, seemingly, here.

In Theorem 1 we assume that, *generically*, elements of the ambient group have a prescribed property: to be in a connected nilpotent subgroup. As this property has no first-order character, this can be interpreted in two possible ways. It means either that the group is saturated and that realizations of the generic type have that property, or, more strongly but with no saturation assumption, that the ambient group has a definable generic subset, all of whose elements have the property. This "generic property" is known to be true, in this second form, in the specific case of connected *locally° solvable* of finite Morley rank, the smallest natural class containing connected solvable groups of finite Morley rank and Chevalley groups of type PSL_2 over algebraically closed fields (see [BBC07, Proposition 8.1], and [DJ07, §5.3] for an account on this and related topics). In any case, the assumption in Theorem 1 is much weaker than that of the existence of a generous Carter subgroup, and as the former is known in contexts where the latter is not known, it seems relevant at present to state Theorem 1, and its consequences, under this weak assumption.

2. Cosets and generosity

In the present section we pass to the proof of the technical Theorem 1 on generous cosets, and in the next we will see its main corollary on Weyl groups.

In most applications of the general protocol for computing Weyl groups in groups of finite Morley rank, there is a uniqueness property, and then rank computations for generosity, or non-generosity, follow more or less immediately from the presence of *disjoint* unions. We refer for example to [CJ04, 3.3–3.4], which was essentially extracted from the original works on bad groups [BN94, Theorem 13.3, Claim (d)]. In general, one can use only finiteness instead of uniqueness for generosity, as explained and illustrated abundantly after Corollary 5. The reader can find in [CJ04, Proposition 6.17] a concrete application of the protocol for Weyl groups which uses finiteness only (see actually the preparatory

sequence 6.13–6.16, and more specifically 3.16, in that paper), and we give here a much more conceptual treatment of this aspect via Corollary 5.

Recall that G is a connected group of finite Morley rank in which, generically, elements belong to connected nilpotent subgroups, that H is a definable subgroup of G and w is an element in $N(H) \setminus H$, and we want to show that wH is not generous in G.

Proof of Theorem 1. Assume towards a contradiction wH generous in G.

We may freely apply Corollary 5 to the coset wH, as remarked after that corollary. It follows that $\mathrm{rk}(wH) = \mathrm{rk}(N(wH))$ on the one hand, and, on the other hand, that wH has a definable generic subset, generous in G, all of whose elements can lie in only finitely many conjugates of wH. In this sense, a generic element g of G is, up to conjugacy, a generic element of wH, and contained in only finitely many conjugates of wH. Of course, $N(wH) \leqslant N(H)$, and in fact $N(wH)$ is the preimage in $N(H)$ of $C_{N(H)/H}(w \bmod H)$. As H, wH, and $N(wH)$ have the same rank,

$$N^\circ(wH) = H^\circ.$$

In particular one sees also that w has finite order modulo H.

By assumption, a generic element g of G also belongs to a connected nilpotent subgroup Q and, as taking definable hulls does not affect connectedness and nilpotence of subgroups in groups of finite Morley rank, we may assume Q definable. (We note here that the generic property in G holds either for the realizations of the generic type in case of saturation of G, or on all elements of a definable generic subset of G, if such a subset exists.)

Using the connectedness of G, one concludes from the two preceding paragraphs that a generic element g of G is, on the one hand, in wH (up to conjugacy) and in only finitely many of its conjugates, and, on the other hand, in a definable connected nilpotent subgroup Q. We will get a contradiction from the position of the tightrope walker g.

As $g \in wH \cap Q$, we may also assume w in Q, replacing the original representative w of the coset wH by a representative in Q in necessary. This is possible as we may take g. Then

$$wH \cap Q = w(H \cap Q).$$

Notice that w still has finite order modulo $H \cap Q$, as the original w had that property modulo H. The group $\langle w \rangle (H \cap Q)$ is in particular definable, and $(H \cap Q)^\circ$ is exactly its connected component. From now on we concentrate on the definable subgroup $\langle w \rangle (H \cap Q)$ of Q, and on its normalizer in Q.

$N_Q^\circ(\langle w \rangle (H \cap Q))$ acts by conjugation on the definable subgroup $\langle w \rangle (H \cap Q)$. By Fact 3(2), it induces a trivial action on this group modulo its connected component, that is $(H \cap Q)^\circ$. This means that it normalizes each coset of $(H \cap Q)^\circ$ in $\langle w \rangle (H \cap Q)$. In particular, $N_Q^\circ(\langle w \rangle (H \cap Q))$ normalizes the (possibly larger) coset $w(H \cap Q)$.

At this point we use an argument similar to the one used in [Jal06, Fundamental Lemma 3.3]. We denote by X the set of elements of $w(H \cap Q)$ contained in only finitely many conjugates of wH. We note that the set X is not empty, as it contains the generic element g. We also note that the subset X of wH can be contained in only finitely many conjugates of wH, as it contains the element g which has this property. As $N_Q^\circ(\langle w \rangle (H \cap Q))$ normalizes $w(H \cap Q)$, it also normalizes X, and thus it permutes by conjugation the conjugates of wH containing X. We are now in presence of the definable action of a connected group on a finite set, and it follows from Fact 3(1) that it has a trivial action, or in other words that $N_Q^\circ(\langle w \rangle (H \cap Q))$ normalizes each of these finitely many conjugates of wH containing X. In particular, it normalizes wH.

Hence

$$N_Q^\circ\big(\langle w \rangle (H \cap Q)\big) \leqslant N^\circ(wH) = H^\circ,$$

as noticed earlier, and the definable connected subgroup $N_Q^\circ(\langle w \rangle (H \cap Q))$ of Q then satisfies

$$N_Q^\circ\big(\langle w \rangle (H \cap Q)\big) \leqslant \big(H^\circ \cap Q\big)^\circ \leqslant (H \cap Q)^\circ.$$

But as $(H \cap Q)^\circ$ is exactly the connected component of $\langle w \rangle (H \cap Q)$, this inclusion shows that $\langle w \rangle (H \cap Q)$ has finite index in its normalizer in Q. Now definable subgroups of infinite index of nilpotent groups of finite Morley rank are of infinite index in their normalizers, by the classical finite Morley rank version of the normalizer condition in finite nilpotent groups. One finds thus that $\langle w \rangle (H \cap Q)$ has finite index in Q, and by connectedness of the latter one gets

$$Q = \langle w \rangle (H \cap Q).$$

As $(H \cap Q)$ now has finite index in Q, one gets similarly

$$Q = (H \cap Q).$$

At this point one gets a contradiction, either by noticing that w has been pushed inside H, or that g has been pushed outside Q. □

Theorem 1 has the following slightly more general form, where the connectedness of the ambient group is dropped and the possibly insinuated saturation assumption is slightly weakened. We note that in this corollary we do not require the elementary extension to be saturated itself, but simply that it is satisfies the same assumption as in Theorem 1.

Corollary 9. *Let G be a group of finite Morley rank having an elementary extension G^* in which, generically, elements belong to connected nilpotent subgroups. Then the coset wH is not generous in G for any definable subgroup H of G° and any element w of G° in $N(H) \setminus H$.*

Proof. Assume towards a contradiction wH generous in G. As G is a finite union of translates of G°, wH is generous in G°. As the Morley rank is invariant under elementary equivalence, it follows that the canonical extension $[wH]^*$ of wH, in $[G^*]^\circ = [G^\circ]^*$, is generous in $[G^*]^\circ$. Now one can apply Theorem 1 in $[G^*]^\circ$. □

Theorem 1 also has the following desirable application.

Corollary 10. *Let G be a group of finite Morley rank as in Corollary 9 and let H be a definable subgroup of G°. Then $H \setminus H^\circ$ is not generous in G and, if H is generous in G, then H° is generous in G, and in fact in any definable subgroup containing it.*

Proof. As $H \setminus H^\circ$ is a finite union of cosets of H° normalizing H°, the first claim follows from Corollary 9. Now H° must be generous in G whenever H is, and our last claim is [Jal06, Lemma 3.9] or Corollary 5. □

In particular, when Corollary 10 applies in a connected group of finite Morley rank, then the notion of minimal definable generous subgroup, as in Facts 7 or 8, is the same as the notion of minimal definable *connected* generous subgroup.

3. Cosets and action

As stressed in the introduction, recovering the action of a Weyl group on its underlying subgroup from weak information on the elements of the corresponding cosets is a particularly delicate task. Corollary 2 is however a general result of faithfulness following merely from the non-generosity provided by Theorem 1. The rest of this paper is devoted to the proof of Corollary 2, or rather of what we see as the most interesting intermediary steps.

The most general situation is that of a definable connected generous subgroup H, and we want to examine the action of $N(H)/H$ on H, and much more generally the action on H of elements w in $N(H)$. Typically, H may be a generous Carter subgroup, with then $N(H)/H$ the natural Weyl group, and w a representative of any coset of H in $N(H)$.

We note that a definable generous subgroup H always satisfies

$$N^\circ(H) = H^\circ$$

by Corollary 5, and is in particular of finite index in its normalizer. We note also that there is a basic result of lifting of torsion in groups of finite Morley rank, implying in particular that any element of finite order of $N(H)/H$ lifts to an element of $N(H)$ of finite order (and where the primes involved in both primary decompositions are the same). In particular, choosing an element w of finite order, for example as in Corollary 2, is always a low cost possibility.

The following lemma is the natural continuation of [CJ04, Lemma 3.4] with the present much better understanding of generosity as a finiteness property as opposed to a uniqueness property. It is the finest correlation one can get between generic elements of the coset wH and generic elements of H in the typical situation where the conclusion of Theorem 1 holds. It shows, we think, the real power of the method.

Lemma 11. *Let G be a group of finite Morley rank, H a definable generous subgroup of G, and w an element in $N(H) \setminus H$ such that $\langle w \rangle H \setminus H$ is not generous. Then:*

(1) *The coset wH has a definable subset $[wH]_{\mathrm{gen}}$, whose complement is non-generic in wH, and all of whose elements are in infinitely many conjugates of wH.*
(2) *The subgroup H has a definable generic subset H_{gen} such that, for any x in $[wH]_{\mathrm{gen}}$, the subgroup of $\langle w \rangle H$ containing x and defined as*

$$\bigcap_{g \in G,\, x \in [wH]^g} \big[\langle w \rangle H \big]^g$$

has an empty intersection with $(H_{\mathrm{gen}})^G$.

Proof. As $N^\circ(H) = H^\circ$ by generosity of H and Corollary 5, $\mathrm{rk}(wH) = \mathrm{rk}(N(wH))$, and the first claim follows from the non-generosity of wH by Corollary 5. Again we remark that the sets provided by Corollary 5 are definable.

Now one can apply Corollary 5 to $\langle w \rangle H$ also. The generosity of $\langle w \rangle H$ (following from that of H) then gives a definable subset $[\langle w \rangle H]_0$, generic in $\langle w \rangle H$, and all of whose elements can lie in only finitely many conjugates of $\langle w \rangle H$. If that set had a non-generic intersection with H, then it would have a generic intersection with one of the proper cosets of H in $\langle w \rangle H$, say $w'H$. As all elements lying in this intersection would be contained in only finitely many conjugates of $w'H$, as contained in only finitely many conjugates of $\langle w \rangle H$ and all normalizers are finite modulo H°, Corollary 5 would give the generosity of $w'H$, a contradiction to the assumption that $\langle w \rangle H \setminus H$ is not generous. One may thus consider a generic element of H as an element of $H_{\mathrm{gen}} := H \cap [\langle w \rangle H]_0$, and thus with the property that it is in only finitely many conjugates of $\langle w \rangle H$.

Consider now x generic in wH in the sense of the first claim, i.e., such that x is in infinitely many conjugates of wH. The intersection of subgroups considered in our second claim is a subgroup of $\langle w \rangle H$. It is contained in infinitely many conjugates of the subgroup $\langle w \rangle H$, as x is in infinitely many conjugates of the coset wH and $N(wH)$ has finite index in $N(\langle w \rangle H)$. Hence it contains no conjugates of an element in H_{gen}, as such an element is contained in only finitely many conjugates of $\langle w \rangle H$. $\square$

We mention, parenthetically, that the subgroup as in Lemma 11(2) containing the element x of wH is normalized by $C(x)$. It is definable by the descending chain condition on definable subgroups, and in particular it contains the definable hull of x as a (possibly smaller) subgroup.

In general, an element x of a coset wH has the form $x = wh$ for some h in H and taking powers one gets

$$(wh)^n = w^n h^{w^{n-1}} h^{w^{n-2}} \cdots h$$

for any natural number n (some useful formulas when considering torsion [CJ04, §3.3]). Assuming additionally that the element w of $N(H)$ has finite order n modulo H, which can be done in a general way as explained above, one has

$$(wh)^n = w^n h^n$$

in the easiest case in which w and h commute, with w^n in H. This correlation between the element wh of the coset wH and the nth power of the element h of H will be combined with the full force of the pure genericity argument of Lemma 11 in our proof of Corollary 2.

To this end, our next main step is as follows.

Lemma 12. *Let G be a group of finite Morley rank, H a definable connected generous subgroup, and w an element in $N(H)$ such that $\langle w \rangle H \setminus H$ is not generous in G. Then*

$$\left\{ h^{w^{n-1}} h^{w^{n-2}} \cdots h \mid h \in H \right\}$$

is not generic in H for any multiple n of the (necessarily finite) order of w modulo H.

Proof. Assume towards a contradiction $\{ h^{w^{n-1}} h^{w^{n-2}} \cdots h \mid h \in H \}$ generic in H. Let $\phi : wh \mapsto (wh)^n$ denote the definable map, from wH to H, consisting of taking n-powers. As

$$\phi(wH) = w^n \cdot \left\{ h^{w^{n-1}} h^{w^{n-2}} \cdots h \mid h \in H \right\},$$

our contradictory assumption forces that $\phi(wH)$ must be generic in H.

Let H_{gen} denote the definable generic subset of H provided by Lemma 11(2). By connectedness of H, one gets that $H_{\text{gen}} \cap \phi(wH)$ must be generic in H as well. In particular, $\phi^{-1}(H_{\text{gen}} \cap \phi(wH))$ must be generic in the coset wH, and one finds an element x in this preimage and in the subset $[wH]_{\text{gen}}$ provided by Lemma 11(1).

Now $\phi(x) \in H_{\text{gen}}$, but as $\phi(x) = x^n$, one gets

$$x^n \in H_{\text{gen}} \cap \langle x \rangle,$$

a contradiction to Lemma 11(2), as $\langle x \rangle$ is obviously a subgroup of the subgroup considered in Lemma 11(2). $\square$

Combined with Theorem 1, one gets the following.

Corollary 13. *Let G be a group of finite Morley rank as in Corollary 9, H a definable connected generous subgroup of G, and w an element of G° in $N(H) \setminus H$. Then*

$$\left\{ h^{w^{n-1}} h^{w^{n-2}} \cdots h \mid h \in H \right\}$$

is not generic in H for any multiple n of the (necessarily finite) order of w modulo H.

Proof. As usual, H is of finite index in its normalizer by Corollary 5. By Theorem 1, or rather its slightly more general form, Corollary 9, Lemma 12 applies. □

If w turned out to centralize H in Lemma 12, then one would get

$$\left\{h^{w^{n-1}}h^{w^{n-2}}\cdots h \mid h \in H\right\} = \left\{h^n \mid h \in H\right\}$$

and thus Corollary 2 follows similarly from Theorem 1 and Lemma 12. Again, Corollary 2 could be stated identically in the slightly more general context of groups as in Corollary 9, taking just care to pick up the element w in G° as in Corollary 13.

Not to come to an abrupt end, we mention the following special case of Corollary 2, much typical of a connected reductive algebraic group, where the maximal torus corresponds to our abelian generous Carter subgroup. In this mere application, we do not conclude much more than the faithfulness of the action of the Weyl group, but state it in a form emphasizing various subgroups reminiscent of the *BN*-pair structure of a reductive algebraic group.

Corollary 14. *Let G be a connected group of finite Morley rank with an abelian generous Carter subgroup Q, and assume Q p-divisible for any prime p dividing the order of $N(Q)/Q$. Then Q has (finitely many) proper definable subgroups, corresponding to all subgroups of the form $C_Q(w)$ for w varying in $N(Q) \setminus Q$, and with a canonical definition as the centers of proper cyclic extensions of Q in $N(Q)$. In particular, $N(Q)/Q$ acts faithfully on Q.*

Proof. Let w in $N(Q) \setminus Q$, of finite order n modulo Q. As Q is p-divisible for all primes p dividing the order of $N(Q)/Q$, it is n-divisible, and in particular $Q^n = Q$. Now $C_Q(w) < Q$ by Corollary 2. We have shown that $C_Q(w) < Q$ for any element w in $N(Q) \setminus Q$.

The fact that there are finitely many possibilities for such subgroups $C_Q(w)$ follows from their alternative definitions as

$$C_Q(w) = Z\big(\langle w \rangle Q\big)$$

and from the fact that $N(Q)/Q$ is finite. For a canonical definition of such subgroups, one may then take $Z(\langle w \rangle Q)$, with w varying in $N(Q) \setminus Q$. □

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