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Unifying practical uncertainty representations: II. Clouds

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ABSTRACT

There exist many simple tools for jointly capturing variability and incomplete information by means of uncertainty representations. Among them are random sets, possibility distributions, probability intervals, and the more recent Ferson's p-boxes and Neumaier's clouds, both defined by pairs of possibility distributions. In the companion paper, we have extensively studied a generalized form of p-box and situated it with respect to other models. This paper focuses on the links between clouds and other representations. Generalized p-boxes are shown to be clouds with comonotonic distributions. In general, clouds cannot always be represented by random sets, in fact not even by two-monotone (convex) capacities.

1. Introduction

There exist many different tools for representing imprecise probabilities. Usually, the more general, the more difficult they are to handle. Simpler representations, although less expressive, usually have the advantage of being more tractable. Over the years, several such representations have been proposed. Among them are possibility distributions [20], probability intervals [5], and more recently p-boxes [12] and clouds [15,16]. Comparing their respective expressive power is a natural task. Finding formal relations between such representations also facilitates a unified handling of uncertainty.

In the first part of this paper [8], a generalized notion of p-boxes is studied and related to representations mentioned above. It is shown that any generalized p-box is representable by a pair of possibility distributions, and that generalized p-boxes are special cases of random sets. Fig. 1 recalls the connections established in the companion paper between the studied representations, going from the most (top) to the least (bottom) general.

The present paper completes Fig. 1 by adding clouds to it, making one step further towards the unification of uncertainty models. Clouds, encoded by a pair of fuzzy sets, were recently introduced by Neumaier [15] as a means to cope with imprecision while remaining computationally tractable, even in high dimensional spaces. A recent application [13] to space shuttle design problem demonstrate some of the potential of the representation. Moreover, as clouds are syntactically equivalent to interval-valued fuzzy sets with some boundary conditions, analyzing their connection with respect to other uncertainty frameworks also provides some insight about how interval-valued fuzzy sets can be interpreted by such frameworks. As we will see, generalized p-boxes, studied in the companion paper, constitute a bridge between clouds, possibility distributions and usual p-boxes.

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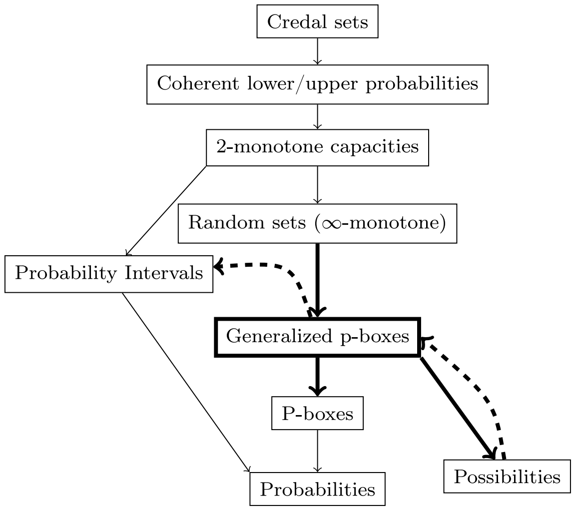

Fig. 1. Relationships among representations. A → B: A generalizes B. A - - → B: B is representable by A.

The paper is divided into four main sections: Section 2 studies the formalism of clouds and relates them to pairs of possibility distributions, to generalized p-boxes and to probability intervals. It is shown that generalized p-boxes are equivalent to a particular subfamily of clouds, named here comonotonic clouds. Section 3 studies non-comonotonic clouds. Since the lower probability they induce are in general even not two-monotone, simpler outer and inner approximations are proposed. Section 4 extends some of our results to the case of continuous models defined on the real line, since such models are often encountered in applications. The particular case of thin clouds, for which both upper and lower distributions coincide is emphasized, as they have non-empty credal sets in the continuous setting.

To make the paper easier to read, longer proofs have been moved to the Appendix. We will often refer to useful results from the companion paper [8], where basics about other representations and frameworks considered here can be found. Some definitions are recalled in footnotes. In the first four sections, we consider that our uncertainty concerns the value that a variable could assume on a finite set X containing n elements.

2. Clouds

Clouds were introduced by Neumaier [15] as a probabilistic generalizations of intervals.

Definition 2.1. A cloud $[\delta, \pi]$ is defined as a pair of mappings $\delta : X \to [0,1]$ and $\pi : X \to [0,1]$ from the set X to the unit interval $[0,1]$, such that:

- δ is pointwise less than or equal to π (i.e., $\delta \leqslant \pi$).
- $\pi(x) = 1$ for at least one element x in X.
- $\delta(y) = 0$ for at least one element y in X.

δ and π are, respectively, the lower and upper distributions of a cloud.

As mappings $\delta,\ \pi$ are mathematically equivalent to two nested fuzzy membership functions, a cloud $[\delta, \pi]$ is mathematically equivalent to an interval-valued fuzzy set (IVF)[19] with boundary conditions ($\pi(x) = 1$ and $\delta(y) = 0$). More precisely, it is mathematically equivalent to an interval-valued membership function whereby the membership value of each element x of X lies in $[\delta(x), \pi(x)]$. Since a cloud is equivalent to a pair of fuzzy membership functions, at most $2|X| - 2$ values (notwithstanding boundary constraints on δ and π) are needed to fully determine a cloud on a finite set. Two subcases of clouds considered by Neumaier [15] are the thin and fuzzy clouds. A *thin cloud* is defined as a cloud for which $\delta = \pi$, while a *fuzzy cloud* is a cloud for which $\delta = 0$.

Neumaier defines the credal set[1] $\mathscr{P}_{[\delta,\pi]}$ induced by a cloud $[\delta, \pi]$, as

$$\mathscr{P}_{[\delta,\pi]} = \{P \in \mathbb{P}_X | P(\{x \in X | \delta(x) \geqslant \alpha\}) \leqslant 1 - \alpha \leqslant P(\{x \in X | \pi(x) > \alpha\})\}, \tag{1}$$

where $\mathbb{P}_X$ is the set of probability measures on X. Interestingly enough, this definition gives a means to interpret IVF sets in terms of credal sets, or in terms of imprecise probabilities, eventually ending up with a behavioral interpretation of IVF by using Walley's [18] theory of coherent lower previsions.

[1] A credal set $\mathscr{P}$ is a closed convex set of probability distributions, here described by constraints on probabilities of some events.

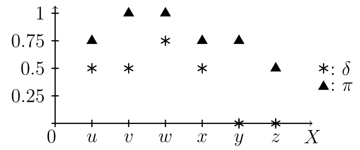

Fig. 2. Cloud $[\delta, \pi]$ of Example 2.2.

Let $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_M = 1$ be the ordered distinct values taken by both δ and π on elements of X, then denote the strong and regular cuts as

$$B_{\bar{\gamma}_i} = \{x \in X | \pi(x) > \gamma_i\} \quad \text{and} \quad B_{\gamma_i} = \{x \in X | \pi(x) \geqslant \gamma_i\} \tag{2}$$

for the upper distribution π and

$$C_{\bar{\gamma}_i} = \{x \in X | \delta(x) > \gamma_i\} \quad \text{and} \quad C_{\gamma_i} = \{x \in X | \delta(x) \geqslant \gamma_i\} \tag{3}$$

for the lower distribution δ. Note that in the finite case, $B_{\bar{\gamma}_i} = B_{\gamma_{i+1}}$ and $C_{\bar{\gamma}_i} = C_{\gamma_{i+1}}$, with $\gamma_{M+1} = 1$, and also

$$\emptyset = B_{\bar{\gamma}_M} \subset B_{\bar{\gamma}_{M-1}} \subseteq \cdots \subseteq B_{\bar{\gamma}_0} = X; \quad \emptyset = C_{\gamma_M} \subseteq C_{\gamma_{M-1}} \subseteq \cdots \subseteq C_{\gamma_0} = X,$$

and since $\delta \leqslant \pi$, this implies that $C_{\gamma_i} \subseteq B_{\gamma_i}$, hence $C_{\gamma_i} \subseteq B_{\bar{\gamma}_{i-1}}$, for all $i = 1, \ldots, M$. In such a finite case, a cloud is said to be discrete. In terms of constraints bearing on probabilities, the credal set $\mathscr{P}_{[\delta,\pi]}$ of a finite cloud is equivalently defined by the finite set of inequalities:

$$i = 0, \ldots, M, \quad P(C_{\gamma_i}) \leqslant 1 - \gamma_i \leqslant P(B_{\bar{\gamma}_i}), \tag{4}$$

under the above inclusion constraints. Note that some conditions, in addition to boundary ones advocated in Definition 2.1, must hold for $\mathscr{P}_{[\delta,\pi]}$ to be non-empty in the finite case. In particular, distribution δ cannot be equal to π (i.e., $\delta \neq \pi$). Otherwise, we have $C_{\gamma_i} = B_{\bar{\gamma}_{i-1}}$ $(= B_{\gamma_i})$, that is π and δ have a common γ_i-cut, and there is no probability distribution satisfying the constraint $1 - \gamma_{i-1} \leqslant P(C_{\gamma_i}) \leqslant 1 - \gamma_i$ since $\gamma_{i-1} < \gamma_i$. So, thin finite clouds induce empty credal sets.

Example 2.2. This example illustrates the notion of a cloud and will be used in the next sections to illustrate various results. Let us consider a space $X = \{u, v, w, x, y, z\}$ and the cloud $[\delta, \pi]$ pictured in Fig. 2 and whose values on X are summarized in Table 1.

The values γ_i corresponding to this cloud are

$$\begin{aligned} &0 \leqslant 0.5 \leqslant 0.75 \leqslant 1, \\ &\gamma_0 \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \gamma_3. \end{aligned}$$

Constraints associated to this cloud and corresponding to Eq. (4) are

$$\begin{aligned} &P(C_{\gamma_3} = \emptyset) \leqslant 1 - 1 \leqslant P(B_{\bar{\gamma}_3} = \emptyset), \\ &P(C_{\gamma_2} = \{w\}) \leqslant 1 - 0.75 \leqslant P(B_{\bar{\gamma}_2} = \{v, w\}), \\ &P(C_{\gamma_1} = \{u, v, w, x\}) \leqslant 1 - 0.5 \leqslant P(B_{\bar{\gamma}_1} = \{u, v, w, x, y\}), \\ &P(C_{\gamma_0} = X) \leqslant 1 - 0 \leqslant P(B_{\bar{\gamma}_0} = X). \end{aligned}$$

2.1. Clouds in the setting of possibility theory

To relate clouds with possibility distributions,[2] first consider the case of *fuzzy clouds* $[\delta, \pi]$. In this case, $\delta = 0$ and $C_{\gamma_i} = \emptyset$ for $i = 1, \ldots, M$, which means that constraints given by Eq. (4) reduce to $1 - \gamma_i \leqslant P(B_{\bar{\gamma}_i})$ for $i = 0, \ldots, M$ which, by using Proposition 2.5 of the companion paper [8], induces a credal set $\mathscr{P}_\pi$ equivalent to the one induced by the possibility distribution π. This shows that *fuzzy clouds* are equivalent to possibility distributions. The following proposition is a direct consequence of this observation:

Proposition 2.3. *Uncertainty modeled by a cloud $[\delta, \pi]$ is representable by the pair of possibility distributions $1 - \delta$ and π, and we have:*

$$\mathscr{P}_{[\delta,\pi]} = \mathscr{P}_\pi \cap \mathscr{P}_{1-\delta}.$$

Proof of Proposition 2.3. Consider a cloud $[\delta, \pi]$ and the constraints inducing the credal set $\mathscr{P}_{[\delta,\pi]}$. As for generalized p-boxes, these constraints can be split into two sets of constraints, namely, for $i = 0, \ldots, M$, $P(C_{\gamma_i}) \leqslant 1 - \gamma_i$ and $1 - \gamma_i \leqslant P(B_{\bar{\gamma}_i})$. Since $B_{\bar{\gamma}_i}$ are strong cuts of π, then by Proposition 2.5. in [8] we know that these constraints define a credal set equivalent to $\mathscr{P}_\pi$.

[2] A possibility distribution is a mapping $\pi : X \to [0, 1]$, with $\pi(x) = 1$ for at least one element, and inducing a credal set $\mathscr{P}_\pi$ such that $P \in \mathscr{P}_\pi$ iff $1 - \alpha \leqslant P(\{x \in X | \pi(x) > \alpha\})$ for all $\alpha \in [0, 1]$.

Table 1
Values of cloud distributions $[\delta, \pi]$ of Example 2.2

	u	v	w	x	y	z
π	0.75	1	1	0.75	0.75	0.5
δ	0.5	0.5	0.75	0.5	0	0

Table 2
Possibility distributions representing cloud of Example 2.2

	u	v	w	x	y	z
π	0.75	1	1	0.75	0.75	0.5
$1-\delta$	0.5	0.5	0.25	0.5	1	1

Note then that $P(C_{\gamma_i}) \leqslant 1-\gamma_i$ is equivalent to $P(C^c_{\gamma_i}) \geqslant \gamma_i$ (where $C^c_{\gamma_i} = \{x \in X | 1-\delta(x) > 1-\gamma_i\}$). By construction, $1-\delta$ is a normalized possibility distribution. Interpreting these inequalities in the light of Proposition 2.5. in [8], we see that they define the credal set $\mathscr{P}_{1-\delta}$. By merging the two set of constraints, we get $\mathscr{P}_{\delta,\pi} = \mathscr{P}_\pi \cap \mathscr{P}_{1-\delta}$. $\square$

This proposition shows that, as for generalized p-boxes, the credal set induced by a cloud is representable by a pair of possibility distributions [11]. This analogy between generalized p-boxes and clouds is studied in Section 2.3. This result also confirms that a cloud $[\delta, \pi]$ is equivalent to its *mirror* cloud $[1-\pi, 1-\delta]$ ($1-\pi$ becoming the lower distribution, and $1-\delta$ the upper one).

Example 2.4. Possibility distributions $\pi,\ 1-\delta$ representing the cloud of Example 2.2 are summarized in Table 2.

2.2. Clouds with non-empty credal sets

We now explore under which conditions a cloud $[\delta, \pi]$ induces a non-empty credal set $\mathscr{P}_{[\delta,\pi]}$. Using the fact that clouds are representable by pairs of possibility distributions, and applying Chateauneuf's [2] characteristic condition $(\forall A \subset X,\ \mathrm{Bel}_1(A) + \mathrm{Bel}_2(A^c) \leqslant 1)$ under which the credal sets associated to two belief functions Bel_1 and Bel_2 have a non-empty intersection, the following necessary and sufficient condition obtains:

Proposition 2.5. *A cloud $[\delta, \pi]$ has a non-empty credal set if and only if*

$$\forall A \subseteq X, \quad \max_{x\in A} \pi(x) \geqslant \min_{y \notin A} \delta(y).$$

Proof. Chateauneuf's condition applied to possibility distributions π_1 and π_2 reads $\forall A \subseteq X,\ \Pi_1(A) + \Pi_2(A^c) \geqslant 1$. Choose $\pi_1 = \pi$ and $\pi_2 = 1-\delta$. In particular $\Pi_2(A^c) = 1 - \min_{y\notin A}\delta(y)$. $\square$

A naive test for non-emptiness based on Proposition 2.5 would have exponential complexity, but in the case of clouds, it can be simplified as follows: suppose the space $X = \{x_1, \dots, x_n\}$ is indexed such that $\pi(x_1) \leqslant \pi(x_2) \leqslant \dots \leqslant \pi(x_n) = 1$ and consider an event A such that $\max_{x\in A}\pi(x) = \pi(x_i)$. The tightest constraint of the form $\max_{x\in A}\pi(x) = \pi(x_i) \geqslant \min_{y\notin A}\delta(y)$ is when choosing $A = \{x_1, \dots x_i\}$. Checking non-emptiness then comes down to checking the following set of $n-1$ inequalities:

$$j = 1, \dots, n-1, \quad \pi(x_i) \geqslant \min_{j>i} \delta(x_j). \tag{5}$$

This gives us an efficient tool to check the non-emptiness of a given cloud on a finite set, or to build a non-empty cloud from the knowledge of either δ or π. For instance, knowing δ, the cloud $[\delta, \pi]$ such that $\pi(x_i) = \min_{j>i}\delta(x_j),\ j = 1, \dots, n-1$ is the most restrictive non-empty cloud one may build, assuming the ordering $\pi(x_1) \leqslant \pi(x_2) \leqslant \dots \leqslant \pi(x_n) = 1$ (changing this assumption yields another non-empty cloud).

Now, consider the extreme case of a cloud for which $C_{\gamma_i} = B_{\bar{\gamma}_i}$ for all $i = 1, \dots, M$ in Eq. (4). In this case, $P(B_{\bar{\gamma}_i}) = P(C_{\gamma_i}) = 1-\gamma_i$ for all $i = 1, \dots, M$. Suppose distribution π takes distinct values on all elements of X. Ordering elements of X by increasing values of $\pi(x)$ $(\forall i,\ \pi(x_i) > \pi(x_{i-1}))$ enforces $\delta(x_i) = \pi(x_{i-1})$, with $\delta(x_1) = 0$. Let δ_π be this lower distribution. The (almost thin) cloud $[\delta_\pi, \pi]$ satisfies Eq. (5), and since $P(B_{\bar{\gamma}_i}) = 1-\gamma_i$, the induced credal set $\mathscr{P}_{[\delta_\pi,\pi]}$ contains the single probability measure P with distribution $p(x_i) = \pi(x_i) - \pi(x_{i-1})$ for all $x_i \in X$, with $\pi(x_0) = 0$. So if a finite cloud $[\delta, \pi]$ is such that if $\delta > \delta_\pi$, it induces an empty credal set $\mathscr{P}_{[\delta,\pi]}$; and if $\delta \leqslant \delta_\pi$, then the induced credal set $\mathscr{P}_{[\delta,\pi]}$ is not empty.

Eq. (5) can be extended to the case of any two possibility distributions $\pi_1,\ \pi_2$ for which we want to check whether $\mathscr{P}_{\pi_1} \cap \mathscr{P}_{\pi_2}$ is empty or not. This is meaningful because the setting of clouds does not cover all pairs $\pi_1,\ \pi_2$ such that $\mathscr{P}_{\pi_1} \cap \mathscr{P}_{\pi_2} \neq \emptyset$. To check it, first recall that for any two possibility distributions $\pi_1,\ \pi_2$, we do have $\mathscr{P}_{\min(\pi_1,\pi_2)} \subseteq \mathscr{P}_{\pi_1} \cap \mathscr{P}_{\pi_2}$, but, in general, the converse inclusion [10] does not hold. From this remark, we have:

- $\mathscr{P}_{\pi_1} \cap \mathscr{P}_{\pi_2} \neq \emptyset$ as soon as $\min(\pi_1, \pi_2)$ is a normalized possibility distribution.
- Not all pairs $\pi_1,\ \pi_2$ such that $\mathscr{P}_{\pi_1} \cap \mathscr{P}_{\pi_2} \neq \emptyset$ derive from a cloud $[1-\pi_2, \pi_1]$. Indeed, the normalization of $\min(\pi_1, \pi_2)$ does not imply that $1-\pi_2 \leqslant \pi_1$.

2.3. Generalized p-boxes as a special kind of clouds

We remind [8] that a generalized p-box $[\underline{F}, \overline{F}]$ is defined by two comonotonic mappings $\underline{F} : X \to [0,1]$, $\overline{F} : X \to [0,1]$ with $\underline{F} \leqslant \overline{F}$ and $\underline{F}(x) = \overline{F}(x) = 1$ for at least one element x of X. They induce a pre-order $\leqslant_{[\underline{F},\overline{F}]}$ on X such that $x \leqslant_{[\underline{F},\overline{F}]} y$ if $\underline{F}(x) \leqslant \underline{F}(y)$ and $\overline{F}(x) \leqslant \overline{F}(y)$, and elements of X are here indexed such that $i \leqslant j$ implies $x_i \leqslant_{[\underline{F},\overline{F}]} x_j$. A generalized p-box $[\underline{F}, \overline{F}]$ induces the following credal set:

$$\mathscr{P}_{[\underline{F},\overline{F}]} = \{P \in \mathbb{P}_X | i = 1, \dots, n, \ \alpha_i \leqslant P(A_i) \leqslant \beta_i\}, \tag{6}$$

where $A_i = \{x_1, \dots, x_i\}$, $\alpha_i = \underline{F}(x_i)$ and $\beta_i = \overline{F}(x_i)$ are lower and upper confidence bounds on set A_i. Note that $A_1 \subseteq \dots \subseteq A_n$, $\alpha_1 \leqslant \dots \leqslant \alpha_n$ and $\beta_1 \leqslant \dots \leqslant \beta_n$. The proposition below lays bare the nature of the relationship between such generalized p-boxes and clouds:

Proposition 2.6. *Let $[\delta, \pi]$ be a cloud defined on X. Then, the three following statements are equivalent:*

(i) *The cloud $[\delta, \pi]$ can be encoded as a generalized p-box $[\underline{F}, \overline{F}]$ such that $\mathscr{P}_{[\delta,\pi]} = \mathscr{P}_{[\underline{F},\overline{F}]}$.*
(ii) *δ and π are comonotonic ($\delta(x) < \delta(y) \Rightarrow \pi(x) \leqslant \pi(y)$).*
(iii) *Sets $\{B_{\overline{\gamma}_i}, C_{\gamma_j} | i, j = 0, \dots, M\}$ form a nested sequence (i.e., are completely (pre-)ordered with respect to inclusion).*

Proof of Proposition 2.6. We use a cyclic proof to show that statements (i)–(iii) are equivalent.

(i) ⇒ (ii) From the assumption, $\delta = 1 - \pi_{\underline{F}}$ and $\pi = \pi_{\overline{F}}$. Hence, using Proposition 3.3 in [8] and the definition of a generalized p-box, δ and π are comonotone, hence (i) ⇒ (ii).

(ii) ⇒ (iii) We will show that if (iii) does not hold, then (ii) does not hold either. Assume sets $\{B_{\overline{\gamma}_i}, C_{\gamma_j} | i, j = 0, \dots, M\}$ do not form a nested sequence, meaning that there exists two sets C_{γ_j}, $B_{\overline{\gamma}_i}$ with $j < i$ s.t. $C_{\gamma_j} \not\subset B_{\overline{\gamma}_i}$ and $B_{\overline{\gamma}_i} \not\subset C_{\gamma_j}$. This is equivalent to asserting $\exists x, y \in X$ such that $\delta(x) \geqslant \gamma_j$, $\pi(x) \leqslant \gamma_i$, $\delta(y) < \gamma_j$ and $\pi(y) > \gamma_i$. This implies $\delta(y) < \delta(x)$ and $\pi(x) < \pi(y)$, and that δ, π are not comonotonic.

(iii) ⇒ (i) Assume the sets $B_{\overline{\gamma}_i}$ and C_{γ_j} form a globally nested sequence whose current element is A_k. Then the set of constraints defining a cloud can be rewritten in the form $\alpha_k \leqslant P(A_k) \leqslant \beta_k$, where $\alpha_k = 1 - \gamma_i$ and $\beta_k = \min\{1 - \gamma_j | B_{\overline{\gamma}_i} \subseteq C_{\gamma_j}\}$ if $A_k = B_{\overline{\gamma}_i}$; $\beta_k = 1 - \gamma_i$ and $\alpha_k = \max\{1 - \gamma_j | B_{\overline{\gamma}_j} \subseteq C_{\gamma_i}\}$ if $A_k = C_{\gamma_i}$. Since $0 = \gamma_0 < \alpha_1 < \dots < \alpha_M = 1$, these constraints are equivalent to those describing a generalized p-box (Eq. (6)). This ends the proof. □

Proposition 2.6 indicates that only those clouds for which δ and π are comonotonic can be encoded by generalized p-boxes, and from now on, we will call such clouds *comonotonic*. Using Proposition 3.3 of the companion paper [8] and given a comonotonic cloud $[\delta, \pi]$, we can express this cloud as the following generalized p-box $\underline{F}$, $\overline{F}$ defined for any $x \in X$:

$$\overline{F}(x) = \pi(x) \quad \text{and} \quad \underline{F}(x) = \min\{\delta(y) | y \in X, \ \delta(y) > \delta(x)\}. \tag{7}$$

Conversely, note that any generalized p-box $[\underline{F}, \overline{F}]$ can be encoded by a comonotonic cloud, simply taking $\delta = 1 - \pi_{\underline{F}}$ and $\pi = \pi_{\overline{F}}$ (see Proposition 3.3 in [8]). This means that generalized p-boxes are special cases of clouds, and that comonotonic clouds and generalized p-boxes are equivalent representations. Also note that a comonotonic cloud $[\delta, \pi]$ and the equivalent generalized p-box $[\underline{F}, \overline{F}]$ induce the same complete pre-order on elements of X, that we note $\leqslant_{[\underline{F},\overline{F}]}$ to remain coherent with the notations of the companion paper [8]. We consider that elements x of X are indexed accordingly, as already specified.

In practice, this means that all the results that hold for generalized p-boxes also hold for comonotonic clouds, and conversely. In particular, comonotonic clouds are special cases of random sets,[3] in the sense that, for any comonotonic cloud $[\delta, \pi]$, there is a belief function Bel such that $\mathscr{P}_{[\delta,\pi]} = \mathscr{P}_{\mathrm{Bel}}$. Adapting Eq. (13) of the companion paper [8] to the case of a comonotonic cloud $[\delta, \pi]$, this random set is such that, for $j = 1, \dots, M$:

$$\begin{cases} E_j = \{x \in X | (\pi(x) \geqslant \gamma_j) \wedge (\delta(x) < \gamma_j)\}, \\ m(E_j) = \gamma_j - \gamma_{j-1}. \end{cases} \tag{8}$$

Note that in the formalism of clouds this random set can be expressed in terms of the sets $\{B_{\overline{\gamma}_i}, C_{\gamma_i} | i = 0, \dots, M\}$. Namely, for $j = 1, \dots, M$:

$$\begin{cases} E_j = B_{\overline{\gamma}_{j-1}} \setminus C_{\gamma_j} = B_{\gamma_j} \setminus C_{\gamma_j}, \\ m(E_j) = \gamma_j - \gamma_{j-1}. \end{cases} \tag{9}$$

Example 2.7. From the cloud in Example 2.2, $C_{\gamma_3} \subset C_{\gamma_2} \subset B_{\overline{\gamma}_2} \subset C_{\gamma_1} \subset B_{\overline{\gamma}_1} \subset B_{\overline{\gamma}_0}$, and the constraints defining $\mathscr{P}_{[\delta,\pi]}$ can be transformed into

[3] A random set is a non-negative mapping $m : \wp(X) \to [0,1]$ such that $\sum_{E \subset X} m(E) = 1$; $m(\emptyset) = 0$. It is also completely characterized by the belief function Bel such that $\forall A \subset X$, $\mathrm{Bel}(A) = \sum_{E \subseteq A} m(E)$. The credal set $\mathscr{P}_{\mathrm{Bel}}$ induced by such a random set is $\mathscr{P}_{\mathrm{Bel}} = \{P \in \mathbb{P}_X | \forall A \subseteq X, \ \mathrm{Bel}(A) \leqslant P(A)\}$.

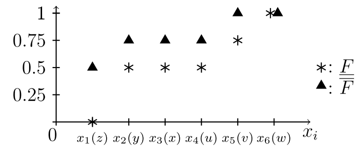

Fig. 3. Generalized p-box $[\underline{F}, \overline{F}]$ corresponding to cloud of Example 2.2.

Table 3
Generalized p-box equivalent to the cloud of Example 2.2

	u	v	w	x	y	z
$\overline{F}$	0.75	1	1	0.75	0.75	0.5
$\underline{F}$	0.5	0.75	1	0.5	0.5	0

$$\begin{aligned}
&0 \leqslant C_{\gamma_2} = \{w\} \leqslant 0.25, \\
&0.25 \leqslant B_{\bar{\gamma}_2} = \{v, w\} \leqslant 0.5, \\
&0.25 \leqslant C_{\gamma_1} = \{u, v, w, x\} \leqslant 0.5, \\
&0.5 \leqslant B_{\bar{\gamma}_1} = \{u, v, w, x, y\} \leqslant 1.
\end{aligned}$$

They are equivalent to the generalized p-box $[\underline{F}, \overline{F}]$ pictured on Fig. 3 and whose values are summarized in Table 3.

The following ordering is compatible with the two distributions (see Fig. 3):

$$z <_{[\underline{F},\overline{F}]} y <_{[\underline{F},\overline{F}]} x =_{[\underline{F},\overline{F}]} u <_{[\underline{F},\overline{F}]} v <_{[\underline{F},\overline{F}]} w,$$

and the corresponding random set, given by Eqs. (9) or (8), is:

$$m(\{x_5, x_6\}) = 0.25; \quad m(\{x_2, x_3, x_4, x_5\}) = 0.25; \quad m(\{x_1, x_2\}) = 0.5.$$

Comonotonic clouds being special cases of clouds, we may wonder if some of the results presented in this section extend to clouds that are not comonotonic (and called non-comonotonic). In particular, can uncertainty modeled by a non-comonotonic cloud be exactly modeled by an equivalent random set? Before addressing this issue in Section 3, we will discuss the relation between clouds and probability intervals, using above results to do so.

2.4. Clouds and probability intervals

As for generalized p-boxes and possibility distributions, there is no direct relationship between clouds and probability intervals [5]. Nevertheless, we can study how to transform a set of probability intervals into a cloud. Such transformations can be useful when one wishes to work with clouds but information is given in terms of probability intervals. There are mainly two paths that can be followed to do this transformation:

- The first consists in exploiting the fact that clouds are representable by pairs of possibility distributions (Proposition 2.3), and in extending existing transformations of probability intervals into a single possibility distribution.
- The second consists in exploiting the equivalence between generalized p-boxes and comonotonic clouds (Proposition 2.6).

The first path can be followed by considering the method developed by Masson and Denoeux [14] to build a possibility distribution π_L outer-approximating a given probability interval L. We have shown [7] that this method can be suitably extended in order to build a second possibility distribution π'_L, so that the pair $[1 - \pi'_L, \pi_L]$ form a cloud outer-approximating the probability interval L (i.e., $\mathscr{P}_L \subseteq \mathscr{P}_{[1-\pi'_L,\pi_L]}$).

Since generalized p-boxes and comonotonic clouds are equivalent representations, an alternative is to directly use transformations of probability intervals into generalized p-boxes (using Eq. (14) in [8]) to get an outer-approximating comonotonic cloud. Then, by Proposition 3.8 of the companion paper [8], we known that we can recover the information modeled by any probability interval L by means of at most $|X|/2$ clouds built by this method.

Using the first method, it is in general impossible to recover the information provided by the original probability interval. This shows that the first method can be very conservative. This is mainly due to the fact that even if it considers every possible ordering of elements, it is only based on the partial order induced by the probability interval. If a natural ordering of elements exists, the second method seems to be preferable. Otherwise, it is harder to justify the fact of considering one particular order rather than another one, and the first method should be applied. In this case, one has to be aware that a lot of information can be lost in the process. One may also find out the ordering inducing the most precise comonotonic cloud, but this question remains open.

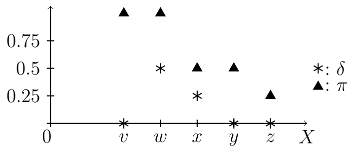

Fig. 4. Cloud $[\delta, \pi]$ of Example 3.1.

Table 4
Non-comonotonic cloud of Example 3.1

	v	w	x	y	z
π	1	1	0.5	0.5	0.25
δ	0	0.5	0.25	0	0

3. The nature of non-comonotonic clouds

When $[\delta, \pi]$ is a non-comonotonic cloud, Proposition 2.3 linking clouds and possibility distributions still holds, but Proposition 2.6 does not hold any longer. As we will see, non-comonotonic clouds appear to be less interesting, at least from a practical point of view, than comonotonic ones.

3.1. Characterization

One way of characterizing an uncertainty model is to find the maximal natural number n such that the lower probability[4] induced by this uncertainty model is always n-monotone (see [8] or Chateauneuf and Jaffray [3] for further details on n-monotonicity[5]). This is how we will proceed with non-comonotonic clouds: let $[\delta, \pi]$ be a non-comonotonic cloud, and $\mathscr{P}_{[\delta,\pi]}$ the induced credal set. The question is: what is the (minimal) n-monotonicity of the associated lower probability $\underline{P}$ induced by $\mathscr{P}_{[\delta,\pi]}$? To address this question, let us start with an example:

Example 3.1. Consider a set X with five elements $\{v, w, x, y, z\}$ and the following non-comonotonic cloud $[\delta, \pi]$ pictured on Fig. 4 and summarized in Table 4. This cloud is non-comonotonic, since $\pi(v) > \pi(x)$ and $\delta(v) < \delta(x)$. The credal set $\mathscr{P}_{[\delta,\pi]}$ can also be defined by the following constraints:

$$P(C_{\gamma_2} = \{w\}) \leqslant 1 - 0.5 \leqslant P(B_{\bar{\gamma}_2} = \{v, w\}),$$
$$P(C_{\gamma_1} = \{w, x\}) \leqslant 1 - 0.25 \leqslant P(B_{\bar{\gamma}_1} = \{v, w, x, y\})$$

with $\gamma_2 = 0.5$ and $\gamma_1 = 0.25$. Now, consider the events $B_{\bar{\gamma}_2},\ C^c_{\gamma_1},\ B_{\bar{\gamma}_2} \cap C^c_{\gamma_1}, B_{\bar{\gamma}_2} \cup C^c_{\gamma_1}$. We can check that:

$$\underline{P}(B_{\bar{\gamma}_2}) = 0.5; \quad \underline{P}(C^c_{\gamma_1}) = 0.25;$$
$$\underline{P}(B_{\bar{\gamma}_2} \cap C^c_{\gamma_1} = \{v\}) = 0; \quad \underline{P}(B_{\bar{\gamma}_2} \cup C^c_{\gamma_1} = \{v, w, y, z\}) = 0.5,$$

since at most a 0.5 probability mass can be assigned to x. Then the inequality $\underline{P}(B_{\bar{\gamma}_2} \cap C^c_{\gamma_1}) + \underline{P}(B_{\bar{\gamma}_2} \cup C^c_{\gamma_1}) < \underline{P}(B_{\bar{\gamma}_2}) + \underline{P}(C^c_{\gamma_1})$ holds, indicating that the lower probability induced by the cloud is not two-monotone.

This example shows that at least some non-comonotonic clouds induce lower probability measures that are not two-monotone. The following proposition gives a general characterization of a large family of such non-comonotonic clouds:

Proposition 3.2. *Let $[\delta, \pi]$ be a non-comonotonic cloud and assume there is a pair of events $B_{\bar{\gamma}_i},\ C_{\gamma_j}$ in the cloud s.t. $B_{\bar{\gamma}_i} \cap C_{\gamma_j} \notin \{B_{\bar{\gamma}_i}, C_{\gamma_j}, \emptyset\}$ and $B_{\bar{\gamma}_i} \cup C_{\gamma_j} \neq X$ (i.e., $B_{\bar{\gamma}_i},\ C_{\gamma_j}$ are just overlapping and do not cover the whole set X). Then, the lower probability measure of the credal set $\mathscr{P}_{\delta,\pi}$ is not two-monotone.*

The proof of Proposition 3.2 can be found in the Appendix. It comes down to showing that for any non-comonotonic cloud with a pair $B_{\bar{\gamma}_i},\ C_{\gamma_j}$ of events satisfying the proposition, the situation exhibited in the above example always occurs, namely the existence of two subsets of the form $B_{\bar{\gamma}_i}$ and $C^c_{\gamma_j}$ for which two-monotonicity fails. This indicates that random sets do not generalize such non-comonotonic clouds. It suggests that such non-comonotonic clouds are likely to be less tractable when processing uncertainty: for instance, simulation of such clouds via sampling methods will be difficult to implement, and the computation of lower/upper expectation too (since Choquet integral and lower expectation do not coincide when lower probability fails two-monotonicity).

Note that comonotonic clouds and clouds described by Proposition 3.2 cover a large number of possible discrete clouds, but that there remains some "small" subfamilies, i.e., those non-comonotonic clouds for which $\forall i, j,\ B_{\bar{\gamma}_i} \cap C_{\gamma_j} \in \{B_{\bar{\gamma}_i}, C_{\gamma_j}, \emptyset\}$, or $B_{\bar{\gamma}_i} \cup C_{\gamma_j} = X$. As such families are very peculiar, we do not consider them further here.

[4] The lower probability $\underline{P}$ induced by a credal set $\mathscr{P}$ is $\underline{P}(A) = \min_{P \in \mathscr{P}} P(A)$ for any $A \subseteq X$.

[5] Here we only need two-monotonicity: a set-function g with domain 2^X is two-monotone if and only if $\forall A, B \subseteq X,\ g(A) + g(B) \leqslant g(A \cup B) + g(A \cap B)$.

3.2. Outer approximation of a non-comonotonic cloud

We provide, in this section and the next one, some practical means to compute guaranteed outer and inner approximations of the exact probability bounds induced by a non-comonotonic cloud, eventually leading to an easier handling of such clouds.

Given a cloud $[\delta,\pi]$, we have proven that $\mathscr{P}_{[\delta,\pi]} = \mathscr{P}_{\pi} \cap \mathscr{P}_{1-\delta}$, where π and $1-\delta$ are possibility distributions. As a consequence, the upper and lower probabilities of $\mathscr{P}_{[\delta,\pi]}$ on any event can be bounded from above (resp. from below), using the possibility measures and the necessity measures induced by π and $\bar{\pi} = 1-\delta$. The following bounds, originally considered by Neumaier [15], provide, for all event A of X, an outer approximation of the range of $P(A)$:

$$\max(N_{\pi}(A), N_{1-\delta}(A)) \leqslant \underline{P}(A) \leqslant P(A) \leqslant \overline{P}(A) \leqslant \min(\Pi_{\pi}(A), \Pi_{1-\delta}(A)), \tag{10}$$

where $\underline{P}(A),\ \overline{P}(A)$ are the lower and upper probabilities induced by $\mathscr{P}_{[\delta,\pi]}$. Remember that probability bounds generated by possibility distributions alone are of the form $[0,\beta]$ or $[\alpha,1]$. Using a cloud and applying Eq. (10) lead to tighter bounds of the form $[\alpha,\beta] \subset [0,1]$, while remaining simple to compute. Nevertheless, these bounds are not, in general, the tightest ones enclosing $P(A)$ induced by $\mathscr{P}_{[\delta,\pi]}$, as the next example shows:

Example 3.3. Let $[\delta,\pi]$ be a cloud defined on a set X, such that distributions δ and π takes up to four different values on elements x of X (including 0 and 1). These values are such that $0=\gamma_0<\gamma_1<\gamma_2<\gamma_3=1$, and the distributions $\delta,\ \pi$ are such that:

$$\pi(x) = \begin{cases} 1 & \text{if } x \in B_{\bar{\gamma}_2}, \\ \gamma_2 & \text{if } x \in B_{\bar{\gamma}_1} \setminus B_{\bar{\gamma}_2}, \\ \gamma_1 & \text{if } x \notin B_{\bar{\gamma}_1}, \end{cases} \quad \delta(x) = \begin{cases} \gamma_2 & \text{if } x \in C_{\gamma_2}, \\ \gamma_1 & \text{if } x \in C_{\gamma_1} \setminus C_{\gamma_2}, \\ 0 & \text{if } x \notin C_{\gamma_1}. \end{cases}$$

Since $P(B_{\bar{\gamma}_1}) \geqslant 1-\gamma_1$ and $P(C_{\gamma_2}) \leqslant 1-\gamma_2$, from Eq. (4), we can check that $\underline{P}(B_{\bar{\gamma}_1} \setminus C_{\gamma_2}) = \underline{P}(B_{\bar{\gamma}_1} \cap C^c_{\gamma_2}) = \gamma_2-\gamma_1$. Now, by definition of a necessity measure, $N_{\pi}(B_{\bar{\gamma}_1} \cap C^c_{\gamma_2}) = \min(N_{\pi}(B_{\bar{\gamma}_1}), N_{\pi}(C^c_{\gamma_2})) = 0$ since $\Pi_{\pi}(C_{\gamma_2}) = 1$ because $C_{\gamma_2} \subseteq B_{\bar{\gamma}_1}$ and $\Pi_{\pi}(B_{\bar{\gamma}_1}) = 1$. Considering distribution δ, we can have $N_{1-\delta}(B_{\bar{\gamma}_1} \cap C^c_{\gamma_2}) = \min(N_{1-\delta}(B_{\bar{\gamma}_1}), N_{1-\delta}(C^c_{\gamma_2})) = 0$ since $N_{1-\delta}(B_{\bar{\gamma}_1}) = \Delta_{\delta}(B^c_{\bar{\gamma}_1}) = 0$ and $C_{\gamma_1} \subseteq B_{\bar{\gamma}_1}$ (which means that the elements x of X that are in $B^c_{\bar{\gamma}_1}$ are such that $\delta(x)=0$). Eq. (10) can thus result in a trivial lower bound (i.e., equal to 0), different from $\underline{P}(B_{\bar{\gamma}_1} \cap C^c_{\gamma_2})$.

Bounds given by Eq. (10) are the main motivation for clouds, after Neumaier [15]. Since these bounds are, in general, not optimal, Neumaier's claim that they are only vaguely related to Walley's previsions or to random sets is not surprising. Eq. (10) appears less useful in the case of comonotonic clouds, for which optimal lower and upper probabilities of events can be more easily computed (see Remark 3.7 in [8]).

3.3. Inner approximation of a non-comonotonic cloud

The previous outer approximation is easy to compute and allows to clarify some of Neumaier's claims. Nevertheless, it is still unclear how to practically use these outer bounds in subsequent treatments (e.g., propagation, fusion). The inner approximation of a cloud $[\delta,\pi]$ proposed now is a random set, which is easy to exploit in practice. This inner approximation is obtained as follows:

Proposition 3.4. *Let $[\delta,\pi]$ be a non-comonotonic cloud defined on X. Let us then define, for $j=1,\dots,M$, the following random set:*

$$\begin{cases} E_j = \{x \in X \mid (\pi(x) \geqslant \gamma_j) \wedge (\delta(x) < \gamma_j)\}, \\ m(E_j) = \gamma_j - \gamma_{j-1}, \end{cases}$$

where $0=\gamma_0<\cdots\gamma_j<\cdots<\gamma_M=1$ are the distinct values taken by $\delta,\ \pi$ on elements of X, E_j are the focal elements with masses $m(E_j)$ of the random set. This random set is an inner approximation of $[\delta,\pi]$, in the sense its credal set $\mathscr{P}_{\mathrm{Bel}}$ is included in $\mathscr{P}_{[\delta,\pi]}$.

In the case of non-comonotonic clouds satisfying Proposition 3.2, the inclusion is strict. This inner approximation appears to be a natural candidate, since on events of the type $\{B_{\bar{\gamma}_i},\ C_{\gamma_i},\ B_{\bar{\gamma}_i} \setminus C_{\gamma_j} \mid i=0,\dots,M;\ j=0,\dots,M;\ i \leqslant j\}$, it gives optimal bounds, and it is exact when the cloud $[\delta,\pi]$ is comonotonic.

4. Continuous clouds on the real line

It often happens that uncertainty is defined on the real line. It is thus important to know if results obtained so far can be extended to continuous settings. In the following, we consider clouds defined on a bounded interval $[\underline{r},\overline{r}]$.

First, let us recall that, as in the discrete case, a cloud $[\delta,\pi]$ defined on the real line is a pair of distributions such that, for any element $r \in \mathbb{R}$, $[\delta(r),\pi(r)]$ is an interval and there is an element r for which $\delta(r)=0$, and another r' for which $\pi(r')=1$. *Thin clouds* ($\pi=\delta$) and *fuzzy clouds* ($\delta=0$) have the same definition as in the case of finite set. The credal set $\mathscr{P}_{[\delta,\pi]}$ induced by a cloud on the real line is such that:

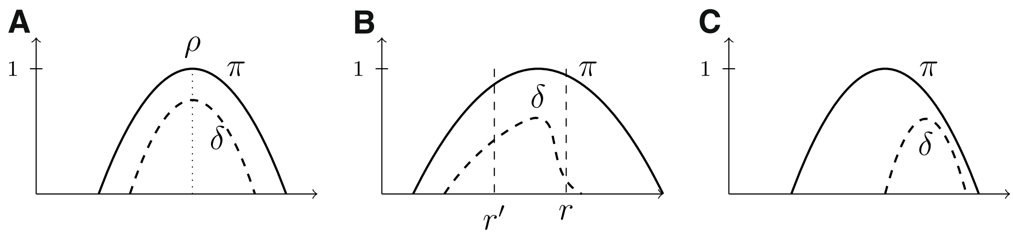

Fig. 5. Illustration of comonotonic (A), weakly comonotonic (B) and non-comonotonic clouds (C) on the real line.

$$\mathscr{P}_{[\delta,\pi]} = \{P|P(\{r \in \mathbb{R}, \delta(r) \geqslant \alpha\}) \leqslant 1 - \alpha \leqslant P(\{r \in \mathbb{R}, \pi(r) > \alpha\})\}, \tag{11}$$

where P is a σ-additive probability distribution.[6]

4.1. General results

As Proposition 2.5 in [8] has been proven for very general spaces [4], results whose proof is based on this proposition directly extend to models on the real line. Similarly, the proof of Proposition 2.6 extends directly to continuous models on the real line. Hence, the following statements still hold:

- If $[\delta, \pi]$ is a cloud, $1 - \delta,\ \pi$ are possibility distributions, and $\mathscr{P}_{[\delta,\pi]} = \mathscr{P}_{1-\delta} \cap \mathscr{P}_{\pi}$.
- If $[\underline{F}, \overline{F}]$ is a generalized p-box defined on the reals, then $\mathscr{P}_{[\underline{F},\overline{F}]} = \mathscr{P}_{\pi_{\underline{F}}} \cap \mathscr{P}_{\pi_{\overline{F}}}$ with, for all $r \in \mathbb{R}$:

  $$\pi_{\overline{F}}(r) = \overline{F}(r)$$

  and

  $$\pi_{\underline{F}}(r) = 1 - \sup\{\underline{F}(r')|r' \in \mathbb{R};\ \underline{F}(r') < \underline{F}(r)\}$$

  with $\pi_{\underline{F}}(\underline{r}) = 0$.
- *Generalized* p-boxes and comonotonic clouds are equivalent representation.

Note that, for clouds on the real line, we can define a weaker notion of comonotonicity: a (continuous) cloud $[\delta, \pi]$ is said to be *weakly* comonotonic if the sign of the derivative of distributions $\delta,\ \pi$ is the same in every point r of the real line $\mathbb{R}$. Being *weakly* comonotonic is not sufficient to be equivalent to a generalized p-box, since if π and δ are only weakly comonotonic, then it is possible to find two values r and r' such that $\delta(r) < \delta(r')$ and $\pi(r) > \pi(r')$. In this case, the (pre-)ordering jointly induced by the two distributions is not complete, and the definition of comonotonicity is not satisfied. Fig. 5A–C, respectively, illustrate the notion of comonotonic, non-comonotonic and weakly comonotonic clouds on the reals. Fig. 5A illustrates a comonotonic cloud (and, consequently, a generalized p-box) for which elements are ordered according to their distance to the mode ρ (i.e., for this particular cloud, two values $x,\ y$ in $\mathbb{R}$ are such that $x <_{[\underline{F},\overline{F}]} y$ if and only if $|\rho - x| > |\rho - y|$). Note that Fig. 5A is a good illustration of the potential use of a generalized p-box, as already noticed (see beginning of Section 3 in the companion paper [8]).

We can now extend the propositions linking clouds and generalized p-boxes with random sets. In particular, the following result extends Proposition 3.2 to the continuous case:

Proposition 4.1. *Let the distributions $[\delta, \pi]$ describe a continuous cloud on the reals and $\mathscr{P}_{[\delta,\pi]}$ be the induced credal set. Then, the random set defined by the Lebesgue measure on the unit interval $\alpha \in [0, 1]$ and the multimapping $\alpha \to E_\alpha$ such that:*

$$E_\alpha = \{r \in \mathbb{R}|(\pi(r) \geqslant \alpha) \wedge (\delta(r) < \alpha)\},$$

defines a credal set $\mathscr{P}_{\mathrm{Bel}}$ inner-approximating $\mathscr{P}_{\pi,\delta}$ ($\mathscr{P}_{\mathrm{Bel}} \subseteq \mathscr{P}_{\pi,\delta}$).

The proof can be found in the Appendix. It comes down to using sequences of discrete clouds outer- and inner-approximating $[\delta, \pi]$ and converging to it, and then to consider inner-approximations of those discrete clouds given by Proposition 3.4. This proposition has two corollaries:

Corollary 4.2. *Let $[\delta, \pi]$ be a comonotonic cloud with continuous distributions on the real line. Then the credal set $\mathscr{P}_{[\delta,\pi]}$ is also the credal set of a continuous random set with uniform mass density, whose focal sets are of the form, for $\alpha \in [0, 1]$:*

$$E_\alpha = \{r \in \mathbb{R}|(\pi(r) \geqslant \alpha) \wedge (\delta(r) < \alpha)\}.$$

To obtain the result, simply observe that the inner-approximation of Proposition 3.4 becomes exact for discrete comonotonic clouds, which are special cases of random sets. In particular, this is true for the sequences of discrete comonotonic clouds outer- and inner-approximating $[\delta, \pi]$ and converging to it. So, this sequence of random sets

[6] To avoid mathematical subtleties that would require special care, we restrict ourselves to σ-additive probability distributions rather than considering finitely additive ones.

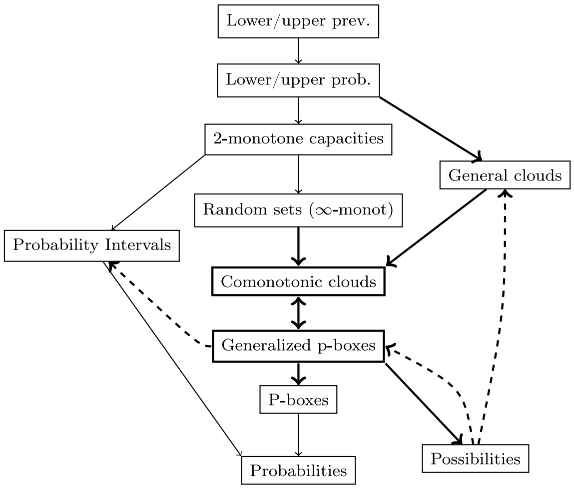

Fig. 6. Representation relationships: completed summary with clouds. A → B: B is a special case of A. A - - → B: B is representable by A.

converge to a continuous random set at the limit. Another interesting particular case is the one of uniformly continuous p-boxes.

Corollary 4.3. *The credal set $\mathscr{P}_{[\underline{F},\overline{F}]}$ described by two continuous and strictly increasing cumulative distributions $\underline{F}$, $\overline{F}$ forming a classical p-box on the reals is equivalent to the credal set described by the continuous random set with uniform mass density, whose focal sets are sets of the form $[x(\alpha), y(\alpha)]$ where $x(\alpha) = \overline{F}^{-1}(\alpha)$ and $y(\alpha) = \underline{F}^{-1}(\alpha)$.*

This is because strictly increasing continuous p-boxes are special cases of comonotonic clouds (or, equivalently, of generalized p-boxes). To check that, in this case, $E_\alpha = [x(\alpha), y(\alpha)]$, it suffices to consider the possibility distributions $\pi_{\underline{F}}$, $\pi_{\overline{F}}$ and to check that $\inf_r\{\pi_{\overline{F}}(r) \geqslant \alpha\} = x(\alpha)$ and that $\sup_r\{1 - \pi_{\underline{F}}(r) < \alpha\} = y(\alpha)$. The strict increasingness property can be relaxed to intervals where the cumulative functions are constant, provided one consider pseudo-inverses when building the continuous random set.

These results are interesting, for they can make the computation of lower and upper expectations over continuous generalized p-boxes easier. Another interesting point is that the framework developed by Smets [17] concerning belief functions on the reals can be applied to comonotonic clouds. Also note that above results extend and give alternative proofs to other results given by Alvarez [1] concerning continuous p-boxes.

4.2. Thin clouds

The case of thin clouds, for which $\pi = \delta$, is interesting. In this case, constraints (4) defining the credal set $\mathscr{P}_{[\delta,\pi]}$ reduce to $P(\pi(x) \geqslant \alpha) = P(\pi(x) > \alpha) = 1 - \alpha$ for all $\alpha \in (0, 1)$. As noticed earlier, when X is finite, thin clouds define empty credal sets, but is no longer the case when it is defined on the real line, as the following proposition shows:

Proposition 4.4. *If π is a continuous possibility distribution on the real line, then the credal set $\mathscr{P}_{[\pi,\pi]} = \mathscr{P}_\pi \cap \mathscr{P}_{1-\pi}$ is not empty.*

Proof of Proposition 4.4. Let $F(x) = \Pi((-\infty, x])$, with $x \in \mathbb{R}$. F is the distribution function of a probability measure P_π such that for all $\alpha \in [0, 1]$, $P_\pi(\{x \in \mathbb{R} | \pi(x) > \alpha\}) = 1 - \alpha$, where the sets $\{x \in \mathbb{R} | \pi(x) > \alpha\}$ form a continuous nested sequence (see [9, p. 285]). Such a probability lies in $\mathscr{P}_\pi$. Moreover,

$$P_\pi(\{x \in \mathbb{R} | \pi(x) > \alpha\}) = P_\pi(\{x \in \mathbb{R} | \pi(x) \geqslant \alpha\}),$$

due to uniform continuity of π. We also have $P_\pi(\{x \in \mathbb{R} | \pi(x) > \alpha\}) = 1 - \Pi(\{x \in \mathbb{R} | \pi(x) \geqslant \alpha\}^c) = 1 - \Delta(\{x \in \mathbb{R} | \pi(x) \geqslant \alpha\})$ again due to uniform continuity. Since $1 - \Delta(\{x \in \mathbb{R} | \pi(x) \geqslant \alpha\}) = \sup_{x|\pi(x) \geqslant \alpha} 1 - \pi(x)$, this means $P_\pi \in \mathscr{P}_{1-\pi}$. $\square$

A *thin* cloud is a particular case of comonotonic cloud. It induces a complete pre-ordering on the reals. If this pre-order is linear, it means that for any $\alpha \in [0, 1]$, there is only one value $r \in \mathbb{R}$ for which $\pi(r) = \alpha$, and that $\mathscr{P}_\pi \cap \mathscr{P}_{1-\pi}$ contains only one probability measure. In particular, if the order is the natural order of real numbers, this *thin* cloud reduces to an usual cumulative distribution. When the pre-order has ties, it means that for some $\alpha \in [0, 1]$, there are several values in $r \in \mathbb{R}$ such that $\pi(r) = \alpha$. Using Corollary 4.2, we can model the credal set $P_\pi \in \mathscr{P}_{1-\pi}$ by the random set with uniform mass density, whose focal sets are of the form

$$E_\alpha = \{r \in \mathbb{R} | \pi(r) = \alpha\}.$$

In this case, we can check that $\mathrm{Bel}(\{r \in \mathbb{R} | \pi(r) \geqslant \alpha\}) = 1 - \alpha$, in accordance with Eq. (4).

Finally, consider the specific case of a thin cloud modeled by an unimodal distribution π (formally, a fuzzy interval). In this case, each focal set associated to a value α is a doubleton $\{x(\alpha), y(\alpha)\}$ where $\{x | \pi(x) \geqslant \alpha\} = [x(\alpha), y(\alpha)]$. Noticeable probability distributions that are inside the credal set induced by such a *thin* cloud are the cumulative distributions F_+ and F_- such that for all α in $[0,1]$ $F_+^{-1}(\alpha) = x(\alpha)$ and $1 - F_-^{-1}(\alpha) = y(\alpha)$ (they respectively correspond to a mass density concentrated on values $x(\alpha)$ and $y(\alpha)$). All probability measures with cumulative functions of the form $\lambda \cdot F_+ + (1-\lambda) \cdot F_-$ also belong to the credal set (for $\lambda = \frac{1}{2}$, this distribution is obtained by evenly dividing mass density between elements $x(\alpha)$ and $y(\alpha)$). Other distributions inside this set are considered by Dubois et al. [9].

5. Conclusion

In this paper, Neumaier clouds are compared to other uncertainty representations, including generalized p-boxes introduced in the companion paper [8]. Properties of the cloud formalism are explained in the light of other representations. We are now ready to complete Fig. 1 with clouds. This completed picture is given by Fig. 6. New relationships and representations coming from this paper and its companion are in bold lines.

The next step is to explore computational aspects of each formalism as done by de Campos et al. [5] for probability intervals. In particular, we need to answer the following questions: how do we define operations of fusion, marginalization, conditioning or propagation for each of these models? Are the representations preserved after such operations, and under which assumptions? What is the computational complexity of these operations? Can the models presented here be easily elicited or integrated? If many results already exist for random sets, possibility distributions and probability intervals, few have been derived for generalized p-boxes or clouds, due to their novelty. The results presented in this paper and its companion can be helpful to perform such a study. Recent applications of clouds to engineering design problems [13] indicate that this model can be useful, and that such a study should be done to gain more insight about the potential of such models. In particular, the mathematical properties of comonotonic clouds appear to be quite attractive. Our study thus indicates how clouds and generalized p-boxes can be interpreted in other frameworks designed to handle uncertainty.

Another issue is to extend presented results to more general spaces, to general lower/upper previsions or to cases not considered here (e.g., continuous clouds with some discontinuities), possibly by using existing results [6,17].

Acknowledgements

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Appendix

We first recall a useful result by Chateauneuf [2] concerning the intersection of credal sets induced by random sets. Consider two random sets $\{(F_i, q_{i\cdot})|i = 1,\dots k\}$ and $\{(G_j, q_{\cdot j})|j = 1,\dots l\}$ on X, with $F_i,\ G_j$ the focal elements, $q_{i\cdot},\ q_{\cdot j}$ the corresponding masses and $\mathscr{P}_F$ and $\mathscr{P}_G$ the induced credal sets. Consider then the set $\mathscr{Q}$ of all random sets Q of the form $\{(F_i \cap G_j, q_{ij})|i = 1, \dots k;\ j = 1,\dots l\}$, with $F_i \cap G_j$ the focal sets and q_{ij} the masses such that $q_{i\cdot} = \sum_{j=1}^{l} q_{ij}$ and $q_{\cdot j} = \sum_{i=1}^{k} q_{ij}$ with the constraint that $q_{ij} = 0$ whenever $F_i \cap G_j = \emptyset$. Then the lower probability induced by the credal set $\mathscr{P}_F \cap \mathscr{P}_G$ is

$$\underline{P}(A) = \min_{P \in \mathscr{P}_1 \cap \mathscr{P}_2} P(A) = \min_{Q \in \mathscr{Q}} \mathrm{Bel}_Q(A) \quad \forall A \subseteq X,$$

where Bel_Q is the belief function induced by the random set Q.

Proof of Proposition 3.2. We first state a short lemma allowing us to emphasize the idea behind the proof of the latter proposition.

Lemma 5.1. *Let $(F_1, F_2),\ (G_1, G_2)$ be two pairs of sets such that $F_1 \subset F_2$, $G_1 \subset G_2$, $G_1 \not\subseteq F_2$ and $G_1 \cap F_1 \neq \emptyset$. Let also $\pi_F,\ \pi_G$ be two possibility distributions such that the corresponding belief functions are defined by mass assignments $m_F(F_1) = m_G(G_2) = \lambda$, $m_F(F_2) = m_G(G_1) = 1 - \lambda$. Then, the lower probability of the non-empty credal set $\mathscr{P} = \mathscr{P}_{\pi_F} \cap \mathscr{P}_{\pi_G}$ is not two-monotone.*

Proof of Lemma 5.1. Chateauneuf's result is applied to the possibility distributions defined in Lemma 5.1. The main idea is to exhibit two events and computing their lower probabilities, showing that two-monotonicity is violated. Consider the set $\mathscr{M}$ of matrices M of the form

$$\begin{array}{c|cc} & G_1 & G_2 \\ \hline F_1 & m_{11} & m_{12} \\ F_2 & m_{21} & m_{22} \end{array}$$

where

$$m_{11} + m_{12} = m_{22} + m_{12} = \lambda,$$
$$m_{21} + m_{22} = m_{21} + m_{11} = 1 - \lambda,$$
$$\sum m_{ij} = 1.$$

Each such matrix is a normalized (i.e., such that $m(\emptyset) = 0$) joint mass distribution for the random sets induced from possibility distributions $\pi_F,\ \pi_G$, viewed as marginal belief functions. Following Chateauneuf [2], for any event $E \subseteq X$, the lower probability $\underline{P}(E)$ induced by the credal set $\mathscr{P} = \mathscr{P}_{\pi_F} \cap \mathscr{P}_{\pi_G}$ is

$$\underline{P}(E) = \min_{M \in \mathscr{M}} \sum_{(F_i \cap G_j) \subset E} m_{ij}. \tag{12}$$

Now consider the four events $F_1,\ G_1,\ F_1 \cap G_1,\ F_1 \cup G_1$. Studying the relations between sets and the constraints on the values m_{ij}, we can see that:

$$\underline{P}(F_1) = \lambda; \quad \underline{P}(G_1) = 1 - \lambda; \quad \underline{P}(F_1 \cap G_1) = 0.$$

For $F_1 \cap G_1$, just consider the matrix $m_{12} = \lambda,\ m_{21} = 1 - \lambda$. To show that the lower probability is not even two-monotone, it is enough to show that $\underline{P}(F_1 \cup G_1) < 1$. To achieve this, consider the following mass distribution:

$$m_{11} = \min(\lambda, 1-\lambda); \quad m_{21} = 1 - \lambda - m_{11};$$
$$m_{12} = \lambda - m_{11}; \quad m_{22} = \min(\lambda, 1-\lambda).$$

It can be checked that this matrix is in the set $\mathscr{M}$, and yields

$$P(F_1 \cup G_1) = m_{12} + m_{11} + m_{21} = m_{11} + \lambda - m_{11} + 1 - \lambda - m_{11} = 1 - m_{11} = 1 - \min(\lambda, 1-\lambda) = \max(1-\lambda, \lambda) < 1,$$

since $(F_2 \cap G_2) \not\subseteq (F_1 \cup G_1)$ (due to the fact that $G_1 \not\subseteq F_2$). Then the inequality $\underline{P}(F_1 \cup G_1) + \underline{P}(F_1 \cap G_1) < \underline{P}(F_1) + \underline{P}(G_1)$ violates two-monotonicity. $\square$

To prove Proposition 3.2, we again use the result by Chateauneuf [2], and we exhibit a pair of events for which two-monotonicity fails. Chateauneuf results are applicable to clouds, since possibility distributions are equivalent to nested random sets. Consider a finite cloud described by Eq. (4) and the following matrix Q of weights q_{ij}

$$\begin{array}{c|ccccccc}
 & C^c_{\gamma_1} & \cdots & C^c_{\gamma_j} & \cdot & C^c_{\gamma_{i+1}} & \cdots & C^c_{\gamma_M} \\
\hline
B_{\overline{\gamma}_0} & q_{11} & \cdots & q_{1j} & \cdot & q_{1(i+1)} & \cdots & q_{1M} \\
\vdots & \vdots & \ddots & & & \vdots & & \vdots \\
B_{\overline{\gamma}_{j-1}} & q_{j1} & \cdots & \mathbf{q_{jj}} & \cdot & \mathbf{q_{j(i+1)}} & \cdots & q_{jM} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\
B_{\overline{\gamma}_i} & q_{(i+1)1} & \cdots & \mathbf{q_{(i+1)j}} & \cdot & \mathbf{q_{(i+1)(i+1)}} & \cdots & q_{(i+1)M} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\
B_{\overline{\gamma}_{M-1}} & q_{M1} & \cdots & q_{Mj} & \cdot & q_{M(i+1)} & \cdots & q_{MM}
\end{array}$$

Respectively, call Bel_1 and Bel_2 the belief functions equivalent to the possibility distributions, respectively, generated by the collections of sets $\{B_{\overline{\gamma}_i} | i = 0, \dots, M-1\}$ and $\{C^c_{\gamma_i} | i = 1, \dots, M\}$. Using the fact that possibility distributions can be mapped into random sets, we have $m_1(B_{\overline{\gamma}_i}) = \gamma_{i+1} - \gamma_i$ for $i = 0, \dots, M-1$, and $m_2(C^c_{\gamma_j}) = \gamma_j - \gamma_{j-1}$ for $j = 1, \dots, M$. As in the proof of Lemma 5.1, we consider every possible joint random set such that $m(\emptyset) = 0$ built from the two marginal belief functions $\mathrm{Bel}_1,\ \mathrm{Bel}_2$. Following Chateauneuf, let $\mathscr{Q}$ be the set of matrices Q s.t.

$$q_{i\cdot} = \sum_{j=1}^{M} q_{ij} = \gamma_i - \gamma_{i-1},$$
$$q_{\cdot j} = \sum_{i=1}^{M} q_{ij} = \gamma_j - \gamma_{j-1},$$
$$\text{if } i,\ j \text{ s.t. } B_{\overline{\gamma}_i} \cap C^c_{\gamma_j} = \emptyset \text{ then } q_{ij} = 0$$

and the lower probability of the credal set $\mathscr{P}_{[\delta,\pi]}$ on event E is such that:

$$\underline{P}(E) = \min_{Q \in \mathscr{Q}} \sum_{(B_{\overline{\gamma}_i} \cap C^c_{\gamma_j}) \subset E} q_{ij}. \tag{13}$$

Now, by hypothesis, there are at least two overlapping sets $B_{\overline{\gamma}_i}, C_{\gamma_j}$ $i > j$ that are not included in each other (i.e., $B_{\overline{\gamma}_i} \cap C_{\gamma_j} \notin \{B_{\overline{\gamma}_i}, C_{\gamma_j}, \emptyset\}$). Let us consider the four events $B_{\overline{\gamma}_i},\ C^c_{\gamma_j},\ B_{\overline{\gamma}_i} \cap C^c_{\gamma_j},\ B_{\overline{\gamma}_i} \cup C^c_{\gamma_j}$, which are all different by hypothesis. Considering Eq. (13), the matrix Q and the relations between sets, inclusions $B_{\overline{\gamma}_m} \subset \cdots \subset B_{\overline{\gamma}_0}$, $C^c_{\gamma_0} \subset \cdots \subset C^c_{\gamma_m}$ and, for $i = 0, \dots, m$, $C_{\gamma_i} \subset B_{\overline{\gamma}_i}$ imply:

$$\underline{P}(B_{\overline{\gamma}_i}) = 1 - \gamma_i; \quad \underline{P}(C^c_{\gamma_j}) = \gamma_j; \quad \underline{P}(B_{\overline{\gamma}_i} \cap C^c_{\gamma_j}) = 0.$$

For the last result, just consider the mass distribution $q_{kk} = \gamma_{k-1} - \gamma_k$ for $k = 1, \dots, m$.

Next, consider event $B_{\bar{\gamma}_i} \cup C^c_{\gamma_j}$ (which is different from X by hypothesis), and let them play the role of events F_1, G_1 in Lemma 5.1. Suppose all masses are such that $q_{kk} = \gamma_{k-1} - \gamma_k$, except for masses (in boldface in the matrix) q_{jj}, $q_{(i+1)(i+1)}$. Then, $C^c_{\gamma_j} \subset C^c_{\gamma_{i+1}}$, $B_{\bar{\gamma}_i} \subset B_{\bar{\gamma}_{j-1}}$, $C^c_{\gamma_j} \not\subseteq B_{\bar{\gamma}_{j-1}}$ by definition of a cloud and $B_{\bar{\gamma}_i} \cap C^c_{\gamma_j} \neq \emptyset$ by hypothesis. Finally, using Lemma 5.1, consider the mass distribution:

$$q_{(i+1)(i+1)} = \gamma_{i+1} - \gamma_i - q_{(i+1)j}; \quad q_{(i+1)j} = \min(\gamma_{i+1} - \gamma_i, \gamma_j - \gamma_{j-1});$$
$$q_{j(i+1)} = \min(\gamma_{i+1} - \gamma_i, \gamma_j - \gamma_{j-1}); \quad q_{jj} = \gamma_j - \gamma_{j-1} - q_{(i+1)j}.$$

It always gives a matrix in the set $\mathcal{Q}$. By considering every subset of $B_{\bar{\gamma}_i} \cup C^c_{\gamma_j}$, we thus get the following inequality:

$$\underline{P}(B_{\bar{\gamma}_i} \cup C^c_{\gamma_j}) \leqslant \gamma_{j-1} + 1 - \gamma_{i+1} + \max(\gamma_{i+1} - \gamma_i, \gamma_j - \gamma_{j-1}).$$

And, similarly to what was found in Lemma 5.1, we get:

$$\underline{P}(B_{\bar{\gamma}_i} \cup C^c_{\gamma_j}) + \underline{P}(B_{\bar{\gamma}_i} \cap C^c_{\gamma_j}) < \underline{P}(B_{\bar{\gamma}_i}) + \underline{P}(C^c_{\gamma_j}),$$

which shows that the lower probability is not two-monotone.

Proof of Proposition 3.4. First, we know that the random set given in Proposition 3.4 is equivalent to

$$\begin{cases} E_j = B_{\bar{\gamma}_{j-1}} \setminus C_{\gamma_j} = B_{\gamma_j} \setminus C_{\gamma_j}, \\ m(E_j) = \gamma_j - \gamma_{j-1}. \end{cases}$$

Now, if we consider the matrix given in the proof of Proposition 3.2, this random set comes down, for $i = 1, \dots, M$ to assign masses $q_{ii} = \gamma_i - \gamma_{i-1}$. Since this is a legal assignment, we are sure that for all events $E \subseteq X$, the belief function of this random set is such that $\mathrm{Bel}(E) \geqslant \underline{P}(E)$, where $\underline{P}$ is the lower probability induced by the cloud. The proof of Proposition 3.2 shows that this inclusion is strict for clouds satisfying the latter proposition (since the lower probability induced by such clouds is not two-monotone). □

Proof of Proposition 4.1. We build outer and inner approximations of the continuous random set that converge to the belief measure of the continuous random set, while the corresponding clouds of which they are inner approximations themselves converge to the uniformly continuous cloud.

First, consider a finite collection $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ of equidistant levels α_i ($\alpha_{i-1} - \alpha_i = 1/n$, $\forall i = 1, \dots, n$). Then, consider the following discrete non-comonotonic clouds $[\underline{\delta}_n, \underline{\pi}_n]$, $[\bar{\delta}_n, \bar{\pi}_n]$ that are, respectively, outer and inner approximations of the cloud $[\delta, \pi]$: for every value r in $\mathbb{R}$, do the following transformation:

$$\pi(r) = \alpha \text{ with } \alpha \in [\alpha_{i-1}, \alpha_i] \text{ then take } \underline{\pi}_n(r) = \alpha_i \text{ and } \bar{\pi}_n(r) = \alpha_{i-1},$$
$$\delta(r) = \alpha' \text{ with } \alpha' \in [\alpha_{j-1}, \alpha_j] \text{ then take } \underline{\delta}_n(r) = \alpha_{j-1} \text{ and } \bar{\delta}_n(r) = \alpha_j.$$

This construction is illustrated in Fig. 7 for the particular case when both π and δ are unimodal. In this particular case, for $i = 1, \dots, n$,

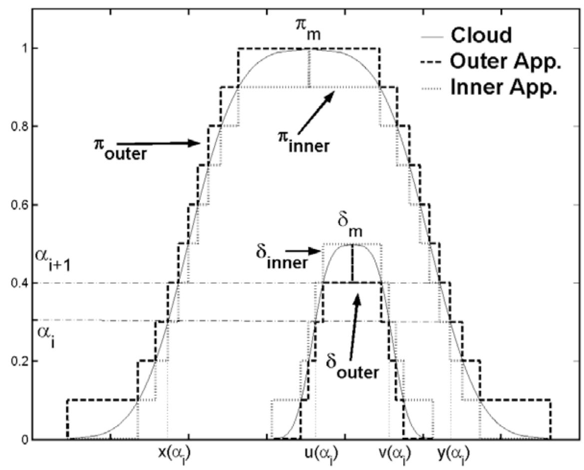

Fig. 7. Inner and outer approximations of a non-comonotonic clouds.

$$\{x \in \mathbb{R} | \underline{\pi}(x) \geqslant \alpha\} = [x(\alpha_{i-1}), y(\alpha_{i-1})] \text{ with } \alpha \in [\alpha_{i-1}, \alpha_i],$$
$$\{x \in \mathbb{R} | \underline{\delta}(x) > \alpha\} = [u(\alpha_i), v(\alpha_i)] \text{ with } \alpha \in [\alpha_{i-1}, \alpha_i],$$
$$\{x \in \mathbb{R} | \bar{\pi}(x) \geqslant \alpha\} = [x(\alpha_i), y(\alpha_i)] \text{ with } \alpha \in [\alpha_{i-1}, \alpha_i],$$
$$\{x \in \mathbb{R} | \bar{\delta}(x) > \alpha\} = [u(\alpha_{i-1}), v(\alpha_{i-1})] \text{ with } \alpha \in [\alpha_{i-1}, \alpha_i].$$

Given the above transformations, $\mathscr{P}(\underline{\pi}_n) \subset \mathscr{P}(\pi) \subset \mathscr{P}(\bar{\pi}_n)$, and $\lim_{n\to\infty}\mathscr{P}(\underline{\pi}_n) = \mathscr{P}(\pi)$ and also $\lim_{n\to\infty}\mathscr{P}(\bar{\pi}_n) = \mathscr{P}(\pi)$. Similarly, $\mathscr{P}(1-\underline{\delta}_n) \subset \mathscr{P}(1-\delta) \subset \mathscr{P}(1-\bar{\delta}_n)$, $\lim_{n\to\infty}\mathscr{P}(1-\bar{\delta}_n) = \mathscr{P}(1-\delta)$ and $\lim_{n\to\infty}\mathscr{P}(1-\underline{\delta}_n) = \mathscr{P}(1-\delta)$. Since the set of probabilities induced by the cloud $[\delta, \pi]$ is $\mathscr{P}(\pi) \cap \mathscr{P}(1-\delta)$, the two credal sets $\mathscr{P}(\underline{\pi}_n) \cap \mathscr{P}(1-\underline{\delta}_n)$ and $\mathscr{P}(\bar{\pi}_n) \cap \mathscr{P}(1-\bar{\delta}_n)$, are, respectively, inner and outer approximations of $\mathscr{P}(\pi) \cap \mathscr{P}(1-\delta)$. Moreover,

$$\lim_{n\to\infty} \mathscr{P}(\underline{\pi}_n) \cap \mathscr{P}(1-\underline{\delta}_n) = \mathscr{P}(\pi) \cap \mathscr{P}(1-\delta),$$
$$\lim_{n\to\infty} \mathscr{P}(\bar{\pi}_n) \cap \mathscr{P}(1-\bar{\delta}_n) = \mathscr{P}(\pi) \cap \mathscr{P}(1-\delta).$$

The random sets that are inner approximations (by Proposition 3.4) of the finite clouds $[\underline{\delta}_n, \underline{\pi}_n]$ and $[\underline{\delta}_n, \underline{\pi}_n]$ converge to the continuous random set defined by the Lebesgue measure on the unit interval $\alpha \in [0, 1]$ and the multimapping $\alpha \to E_\alpha$ such that:

$$E_\alpha = \{r \in \mathbb{R} | (\pi(r) \geqslant \alpha) \wedge (\delta(r) < \alpha)\}.$$

In the limit, it follows that this continuous random set is an inner approximation of the continuous cloud. □

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