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THE BIVARIATE MAXIMUM PROCESS AND QUASI-STATIONARY STRUCTURE OF BIRTH-DEATH PROCESSES

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Let $N(t)$ be a birth-death process on $\{0, 1, \ldots\}$ with state 0 reflecting and let $\boldsymbol{q}_K^T$ be the quasi-stationary distribution of the truncated process on $\{0, 1, \ldots, K\}$ with $\lambda_K > 0$. It is shown that the sequence $(\boldsymbol{q}_K^T)$ increases stochastically with K. The bivariate Markov chain $(M(t), N(t))$ where $M(t) = \max_{0 \leq t' \leq t} N(t')$ is studied as a stepping stone to the proof of the result.

birth-death processes * quasi-stationary structure * stochastic monotonicity * maximum process

0. Introduction

In a recent paper (Keilson and Ramaswamy [6]), it was shown that for an ergodic birth-death process on $\{0, 1, \ldots\}$ with ergodic distribution $\boldsymbol{e}_\infty^T$, the sequence of quasi-stationary distributions $\boldsymbol{q}_K^T$ [1, 2] on $\{0, 1, \ldots, K\}$ converges elementwise to $\boldsymbol{e}_\infty^T$, as $K \to \infty$. This result has since been extended to stochastically monotone Markov processes on $[0, \infty)$ (Pollak and Siegmund [9]). The following stochastic order inequality was made use of in [6]:

$$\boldsymbol{e}_K^{0T} < \boldsymbol{q}_K^T < \boldsymbol{e}_K^{RT}. \tag{0.1}$$

Here $\boldsymbol{e}_K^{0T}$ and $\boldsymbol{e}_K^{RT}$ are the ergodic distributions of the replacement processes obtained by replacing samples that reach state $K+1$ at states 0 and K respectively. As will be seen subsequently both $\boldsymbol{e}_K^{0T}$ and $\boldsymbol{e}_K^{RT}$ increase stochastically with K. It is then of some interest to enquire whether $\boldsymbol{q}_K^T$ also increases stochastically with K. We will show in this note that this is indeed so.

Let $N(t)$ be a birth-death process on $\{0, 1, \ldots\}$ (not necessarily ergodic) with governing transition rates λ_n and μ_n ($\mu_0 = 0$). Let $M(t)$ be the maximum of the process $N(t)$ up to time t, i.e. $M(t) = \max_{0 \leq t' \leq t} N(t')$ with $M(0) = 0$. As part of the proof of the stochastic increase of the sequence of quasi-stationary distributions

$\boldsymbol{q}_K^T$, the maximum process $M(t)$ is studied via examination of the bivariate Markov chain $[M(t), N(t)]$.

Let

$$q_{Kn}(t) = P[N(t) = n \mid M(t) \leq K], \quad 0 \leq n \leq K, \tag{0.2a}$$

$$d_{Kn}(t) = P[N(t) = n \mid M(t) = K], \quad 0 \leq n \leq K, \tag{0.2b}$$

and let $\boldsymbol{d}_K^T(t) = (d_{Kn}(t))$ and $\boldsymbol{q}_K^T(t) = (q_{Kn}(t))$ be the corresponding row vectors. Let $\boldsymbol{q}_K^T$ be the quasi-stationary distribution on $\{0, 1, \ldots, K\}$ (where $K+1$ is made absorbing). Then from the definition of quasi-stationarity, one has

$$\boldsymbol{q}_K^T(t) \to \boldsymbol{q}_K^T, \quad t \to \infty. \tag{0.3}$$

It will be shown subsequently as a lemma that $\boldsymbol{d}_K^T(t)$ also satisfies (0.3), i.e.

$$\boldsymbol{d}_K^T(t) \to \boldsymbol{q}_K^T, \quad t \to \infty. \tag{0.4}$$

Equation (0.4) is demonstrated by deriving explicitly the distribution of $\boldsymbol{d}_K^T(t)$ in terms of $s_{0K}(t)$, the density of the passage time T_{0K} from state 0 to state K.

It is known that T_{0K} increases locally (in the stochastic sense) with K (see e.g. Keilson and Sumita [7]; Whitt [10]), i.e. $s_{0K}(t)/s_{0K+1}(t)$ is non-increasing in t. The notation $T_{0K} <_l T_{0K+1}$ will be employed. This is essentially due to the fact that $s_{0K}(t)$ is log-concave (Keilson [5]). The local order is then used to demonstrate the stochastic increase of $\boldsymbol{d}_K^T(t)$ with K, i.e.,

$$\boldsymbol{d}_K^T(t) < \boldsymbol{d}_{K+1}^T(t) \ \text{ for each fixed } t > 0, \quad K \geq 1. \tag{0.5}$$

Since stochastic order is preserved under limits, one can use (0.4) and (0.5) to conclude that $\boldsymbol{q}_K^T < \boldsymbol{q}_{K+1}^T$ which proves the stochastic increase of the sequence of quasi-stationary distributions $(\boldsymbol{q}_K^T)$.

The stochastic increase with K of the ergodic distributions $\boldsymbol{e}_K^{RT}$ and $\boldsymbol{e}_K^{0T}$ is proved in Section 1. The bivariate maximum process is studied in Section 2. Equations (0.4) and (0.5) are established in Section 3.

1. Stochastic increase of the replacement distributions

Let $\pi_0 = 1$, $\pi_j = (\lambda_0 \cdots \lambda_{j-1})/(\mu_1 \cdots \mu_j)$, $j \geq 1$, be the potential coefficients. The ergodic distribution $\boldsymbol{e}_K^{RT}$ on $\{0, 1, \ldots, K\}$ for reflecting truncation at level K is just the vector of potential coefficients $(\pi_j, 0 \leq j \leq K)$ renormalized, as a consequence of detailed balance. Simple algebra then shows that $\boldsymbol{e}_K^{RT}$ increases stochastically with K.

The proof of the stochastic increases of $\boldsymbol{e}_K^{0T}$ with K is given next.

The elements of the probability vector e_K^{0T} are explicitly evaluated in [6] (see equations (3.8)-(3.15) in that reference). It is shown there that

$$e_K^0(j) = \pi_j e_K^0(0) - \theta_{j-1} i_K, \quad 1 \leq j \leq K, \tag{1.1a}$$

where $i_K = 1/E[T_{0K}]$ and θ_j is given by the recursive relation

$$\mu_{j+1}\theta_j = 1 + \lambda_j \theta_{j-1}, \quad 1 \leq j \leq K-1,$$
$$\theta_0 = \frac{1}{\mu_0}. \tag{1.1b}$$

The quantity $e_K^0(0)$, obtained from renormalization, is given by

$$e_K^0(0) = \frac{1}{E[T_{0K+1}]} \sum_0^K (\lambda_n \mu_n)^{-1}. \tag{1.1c}$$

Using equations (1.1), it is seen from elementary algebra that the ratio $e_K^0(j)/e_{K+1}^0(j)$ is nonincreasing with j. This shows that $e_K^{0T} <_l e_{K+1}^{0T}$, where $<_l$ denotes local order, which implies that the two vectors are stochastically ordered.

The reader should note the ease with which ordinary stochastic order is established by showing that the entities are ordered in the local sense, which is stronger. Surprisingly, the direct demonstration of ordinary stochastic increase of e_K^{0T} with K is elusive.

2. The bivariate maximum process

In this section the bivariate maximum process $[M(t), N(t)]$ is studied as a stepping stone to the proof of the stochastic increase of the quasi-stationary distributions q_K^T with K. In particular, the conditional probabilities $d_{Kn}(t)$ are derived explicitly in terms of the p.d.f. $s_{0K}(t)$ of the passage time T_{0K} from state 0 to state K. This will enable us to prove in Section 3 that the vector $d_K^T(t)$ of probabilities $d_{Kn}(t)$ increases stochastically with K for each t and that $d_K^T(t)$ converges to the quasi-stationary distribution q_K^T as $t \to \infty$.

Let $Z(t)$ be the two dimensional process $[M(t), N(t)]$. Note that $Z(t)$ is a bivariate Markov chain on $\{(m, n): 0 \leq n \leq m\}$. Further consider, for fixed K, the two 'vertical' sets $L_K = \{(K, n): 0 \leq n \leq K\}$ and $L_{K+1}^B = \{(K+1, n): 0 \leq n \leq K\}$ (B for bottom states). The vertical set L_{K+1} is similarly defined (see Figure 1).

Let the conditional distribution of the bivariate chain on the set L_{K+1}^B be given by

$$P[N(t) = n \mid M(t) = K+1, N(t) \leq K]. \tag{2.1}$$

One then has the following result which is easily proved using simple arguments involving conditional probabilities.

Lemma 2.1. *$d_{K+1}^T(t)$ is a mixture of the distribution U_{K+1}^T degenerate at $K+1$ and $d_{K+1}^{BT}(t)$.*

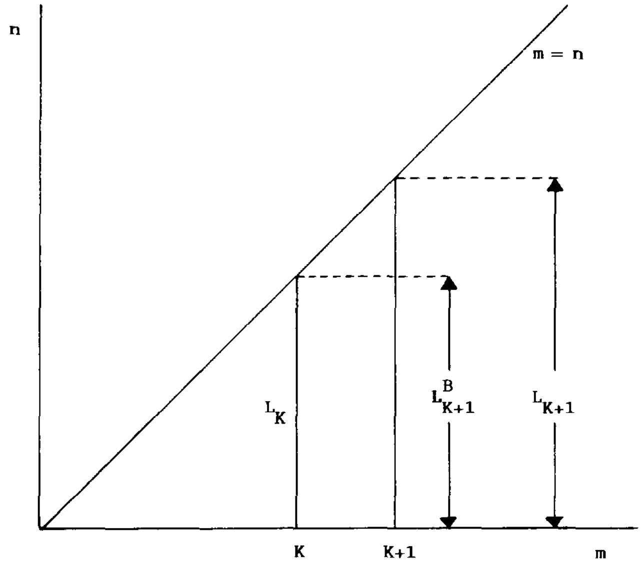

Fig. 1

The expressions for $\boldsymbol{d}_K^T(t)$ and $\boldsymbol{d}_{K+1}^{BT}(t)$ are obtained next. Note that the motion of the bivariate process $\boldsymbol{Z}(t)$ on the set L_K and on the set L_{K+1}^B can be thought of in terms of a birth-death process on states $\{0, 1, \ldots, K\}$ with loss from state K, i.e. $\lambda_K > 0$. Let the transition matrix for this loss process be $\boldsymbol{p}_K^*(t)$ whose (m, n)th element is given by $p_{K;mn}^*(t)$. Finally let $s_{0K}(t)$ be the p.d.f. of the passage time from state 0 to stake K with Laplace transform $\sigma_{0K}(s)$.

Lemma 2.2. *The distributions* $\boldsymbol{d}_K^T(t)$ *and* $\boldsymbol{d}_{K+1}^{BT}(t)$ *are given respectively by*

$$\boldsymbol{d}_K^T(t) = \frac{\int_0^t s_{0K}(\tau)\boldsymbol{U}_K^T\boldsymbol{p}_K^*(t-\tau)\,\mathrm{d}\tau}{\int_0^t s_{0K}(\tau)\boldsymbol{U}_K^T\boldsymbol{p}_K^*(t-\tau)\boldsymbol{1}\,\mathrm{d}\tau} \tag{2.2}$$

and

$$\boldsymbol{d}_{K+1}^{BT}(t) = \frac{\int_0^t s_{0K+2}(\tau)\boldsymbol{U}_K^T\boldsymbol{p}_K^*(t-\tau)\,\mathrm{d}\tau}{\int_0^t s_{0K+2}(\tau)\boldsymbol{U}_K^T\boldsymbol{p}_K^*(t-\tau)\boldsymbol{1}\,\mathrm{d}\tau}. \tag{2.3}$$

Proof. We will first prove (2.2). Observe that, under the condition $M(0)=0$, the event $[N(t)=n, M(t)=K]$ occurs at time t if a sojourn is initiated on L_K by a first passage from 0 to state K at some prior time $\tau \le t$ and if, subsequently, one has $N_K^*(t-\tau)=n$ for the loss process on $\{0, 1, \ldots, K\}$. These arguments imply that

$$P[N(t)=n, M(t)=K \mid M(0)=0] = \int_0^t s_{0K}(\tau)p_{K;Kn}^*(t-\tau)\,\mathrm{d}\tau, \quad 0 \le n \le K. \tag{2.4}$$

Summing over n in (2.4) one obtains that

$$P[M(t)=K]=\int_0^t s_{0K}(\tau)U_K^T p_K^*(t-\tau)\mathbf{1}\,\mathrm{d}\tau \tag{2.5}$$

where U_K^T is a probability vector with all mass at state K. Note that $U_K^T p_K^*(t)\mathbf{1}$ is the survival function of the passage time from state K to state $K+1$ so that (2.5) yields an expression for $P[M(t)=K]$ which might have been anticipated. Equation (2.2) is now a direct consequence of (2.4) and (2.5).

The proof of (2.3) is given next.

Let $I(t)$ be the inception time density on L_{K+1}^B, i.e. $I(t)\,\mathrm{d}t$ is the probability that a sojourn is initiated on L_{K+1}^B in $(t, t+\mathrm{d}t)$ to first order in $\mathrm{d}t$. One can then write, as in (2.4), that

$$P[M(t)=K+1, N(t)=n]=\int_0^t I(\tau)p_{K;Kn}^*(t-\tau)\,\mathrm{d}\tau, \quad 0\leq n\leq K. \tag{2.6}$$

It remains to evaluate $I(t)$. Note that the sequence of epochs at which sojourns on the set L_{K+1}^B are initiated form a delayed renewal process and that $I(t)$ is the renewal density for this renewal process. Since for the two-dimensional process there is permanent exit from the set L_{K+1} at state $(K+1, K+1)$ with hazard rate λ_{K+1}, the lifetime distribution of the renewal process is not honest. The (dishonest) p.d.f. of the time to the first renewal is seen to have Laplace transform $\theta(s)$ which is given by

$$\theta(s)=\sigma_{0K+1}(s)\frac{\nu_{K+1}}{\nu_{K+1}+s}q_{K+1} \tag{2.7}$$

where

$$\nu_n=\lambda_n+\mu_n, \qquad p_n=\frac{\lambda_n}{\nu_n}, \qquad q_n=\frac{\mu_n}{\nu_n}. \tag{2.8}$$

Now sojourns on the set L_{K+1}^B can be initiated only by an entry into state $(K+1, K)$. If a sojourn has just been initiated on the set L_{K+1}^B, then the time to the next initiation has Laplace transform $\eta(s)$ given by

$$\eta(s)==\sigma_K^+(s)\frac{\nu_{K+1}}{\nu_{K+1}+s}q_{K+1} \tag{2.9}$$

where $\sigma_K^+(s)$ is the Laplace transform of the passage time T_K^+ from state K to state $K+1$. Let

$$\tilde{I}(s)=\int_0^\infty \mathrm{e}^{-s\tau}I(\tau)\,\mathrm{d}\tau \tag{2.10}$$

be the Laplace transform of $I(t)$. The arguments above imply that

$$\tilde{I}(s)=\theta(s)[1+\eta(s)+\eta^2(s)+\cdots] \tag{2.11}$$

Substituting for $\theta(s)$ and $\eta(s)$ in (2.11), one obtains

$$\tilde{I}(s)=\sigma_{0K+1}(s)\left[\frac{\mu_{K+1}}{s+\nu_{K+1}-\mu_{K+1}\sigma_K^+(s)}\right]. \tag{2.12}$$

It is known that the passage time transforms $\sigma_n^+(s)$ satisfy the recurrence relation (Keilson [5])

$$\sigma_{K+1}^+(s)=\frac{\lambda_{K+1}}{s+\nu_{K+1}-\mu_{K+1}\sigma_K^+(s)}. \tag{2.13}$$

Using (2.13) in (2.12), it follows that

$$I(t)=\frac{\mu_{K+1}}{\lambda_{K+1}}s_{0K+2}(t). \tag{2.14}$$

Equation (2.3) now follows from (2.6) and (2.14).

Note that $\lim_{t\to\infty} s_{0K+2}(t)=0$. Hence from (2.14), one sees that $I(t)\to 0$ as $t\to\infty$. This is due to the fact that the renewal process under consideration has a dishonest underlying lifetime distribution.

3. Stochastic increase of the quasi-stationary distribution

In this section the stochastic increase with K of the sequence of quasi-stationary distributions $\boldsymbol{q}_K^T$ on $\{0, 1, \ldots, K\}$ will be demonstrated. To do so, it is first shown that $\boldsymbol{d}_K^T(t)\to\boldsymbol{q}_K^T$ as $t\to\infty$. It is then shown that $\boldsymbol{d}_K^T(t)$ increases stochastically with K. The fact that stochastic order is preserved under limits proves the desired result.

Before proceeding to the main results of this section, a few preliminary lemmas are needed.

The following simple result for a birth-death process is proved in Ledermann and Reuter [8] (see also Callaert and Keilson [1]).

Lemma 3.1. *Consider a birth-death process on $\{0, 1, \ldots, K\}$ with state 0 reflecting and loss from state K ($\lambda_K>0$). Let γ_K be its principal decay rate, i.e. let $-\gamma_K$ be the eigenvalue of the associated Q-matrix closest to zero. Then, for $K\geq 1$, one has $\gamma_{K-1}>\gamma_K$.*

As a consequence of Lemma 3.1 one has the following result which will be used subsequently in Lemma 3.3.

Lemma 3.2. *For any $K\geq 1$, one has*

$$e^{\gamma_K t}s_{0K}(t)\in L_1(0,\infty), \tag{3.1}$$

i.e., $\int_0^\infty e^{\gamma_K t}s_{0K}(t)\,dt<\infty$.

Proof. This follows at once from Lemma 3.1 and the fact that $s_{0K}(t) \sim e^{-\gamma_{K-1}t}$ as $t \to \infty$.

Lemma 3.3. *In the notation of* (0.2a) *and* (0.2b), *one has*

$$d_K^T(t) \to q_K^T, \quad t \to \infty. \tag{3.2}$$

Proof. It was proved in Lemma 2.2 that

$$d_K^T(t) = \frac{\int_0^t s_{0K}(\tau) U_K^T p_K^*(t-\tau)\, d\tau}{\int_0^t s_{0K}(\tau) U_K^T p_K^*(t-\tau) \mathbf{1}\, d\tau}. \tag{3.3}$$

Let γ_K be the principal decay rate for a birth-date process on $\{0, 1, \ldots, K\}$ with loss from state K and state 0 reflecting. As proved in Keilson [5], the transition matrix $p_K^*(t)$ has the representation

$$p_K^*(t) = e^{-\gamma_K t}(J_K + \varepsilon(t)). \tag{3.4}$$

where $\varepsilon(t) \to 0$, $t \to \infty$ elementwise. Also $J_K = r_K l_K^T / l_K^T r_K$ where l_K^T and r_K are the left and right eigenvectors of the Q-matrix Q_K^* associated with the eigenvalue $-\gamma_K$ closest to zero. One then has

$$\int_0^t s_{0K}(\tau) U_K^T p_K^*(t-\tau)\, d\tau = e^{-\gamma_K t} \int_0^t e^{\gamma_K \tau} s_{0K}(\tau) U_K^T J_K\, d\tau + e^{-\gamma_K t} \int_0^t e^{\gamma_K \tau} s_{0K}(\tau) U_K^T \varepsilon(t-\tau)\, d\tau. \tag{3.5}$$

Substituting (3.5) in (3.3) one obtains

$$d_K^T(t) = \frac{\int_0^t e^{\gamma_K \tau} s_{0K}(\tau) U_K^T J_K\, d\tau + \int_0^t e^{\gamma_K \tau} s_{0K}(\tau) U_K^T \varepsilon(t-\tau)\, d\tau}{\int_0^t e^{\gamma_K \tau} s_{0K}(\tau) U_K^T J_K \mathbf{1}\, d\tau + \int_0^t e^{\gamma_K \tau} s_{0K}(\tau) U_K^T \varepsilon(t-\tau) \mathbf{1}\, d\tau} \tag{3.6}$$

Since $\varepsilon(t) \to 0$ as $t \to \infty$ and $e^{\gamma_K t} s_{0K}(t) \in L_1(0, \infty)$ as shown in Lemma 3.2, it follows that

$$\int_0^t e^{\gamma_K \tau} s_{0K}(\tau) U_K^T \varepsilon(t-\tau)\, d\tau \to 0, \ t \to \infty. \tag{3.7}$$

Finally note that $U_K^T J_K / U_K^T J_K \mathbf{1} = q_K^T$. One then obtains the required result by applying (3.7) in (3.6).

One further lemma is needed.

Let $N(t)$ be a birth-death process on $\{0, 1, \ldots, K+1\}$ with state $K+1$ absorbing and transition matrix $P^A(t)$. Let the loss process on $\{0, 1, \ldots, K\}$ be governed by the (substochastic) transition matrix $p_K^*(t)$.

Lemma 3.4. *Let U_0^T and U_K^T be initial probability vectors concentrated on states 0 and K respectively. Then the family of probability vectors $q_K^{0T}(t)$ and $q_K^{KT}(t)$ given by*

$$q_K^{0T}(t)=\frac{U_0^T p_K^*(t)}{U_0^T p_K^*(t)\mathbf{1}}, \qquad q_K^{KT}(t)=\frac{U_K^T p_K^*(t)}{U_K^T p_K^*(t)\mathbf{1}} \tag{3.8}$$

respectively increase and decrease stochastically with t to the quasi-stationary distribution q_K^T.

Proof. Let $0<\tau<t$. Clearly

$$(U_0^T,0) <_l (U_0^T,0)p^A(t-\tau). \tag{3.9}$$

It is known that $P^A(t)$ is a TP_2 stochastic matrix for all $t>0$ (Karlin and McGregor [4]). Since local order is preserved under post-multiplication by TP_2 matrices, one has from (3.9) that

$$(U_0^T,0)p^A(\tau) <_l (U_0^T,0)p^A(t). \tag{3.10}$$

The definition of local order implies that ordinary stochastic order is preserved when the probability vectors on both sides of (3.10) are conditioned on $\{0,1,\ldots,K\}$. Note that $(U_0^T,0)p^A(t)$ conditioned on this set is just $q_K^{0T}(t)$. It follows then that $q_K^{0T}(t)$ increases stochastically to q_K^T. A similar argument shows that $q_K^{KT}(t)$ decreases stochastically with t to q_K^T.

It is shown next that $d_K^T(t)$ increases stochastically with K for each fixed t.

Theorem 3.5. *For each fixed $t>0$ and, for every $K\geq 1$,*

$$d_K^T(t)< d_{K+1}^T(t). \tag{3.11}$$

Proof. From Lemma 2.1, $d_{K+1}^T(t)$ is a mixture of the distribution U_{K+1}^T concentrated at state $K+1$ and $d_{K+1}^{BT}(t)$. Clearly $d_K^T(t)<U_{K+1}^T$. It suffices to show, therefore, that

$$d_K^T(t)< d_{K+1}^{BT}(t). \tag{3.12}$$

From (2.2) and (2.3), one has

$$d_K^T(t)=\frac{\int_0^t s_{0K}(\tau)U_K^T p_K^*(t-\tau)\,\mathrm{d}\tau}{\int_0^t s_{0K}(\tau)U_K^T p_K^*(t-\tau)\mathbf{1}\,\mathrm{d}\tau} \tag{3.13}$$

and

$$d_{K+1}^{BT}(t)=\frac{\int_0^t s_{0K+2}(\tau)U_K^T p_K^*(t-\tau)\,\mathrm{d}\tau}{\int_0^t s_{0K+2}(\tau)U_K^T p_K^*(t-\tau)\mathbf{1}\,\mathrm{d}\tau}. \tag{3.14}$$

Note that in the notation of Lemma 3.4, equation (3.13) may be rewritten as

$$d_K^T(t) = \int_0^t a_K(\tau) q_K^{KT}(t-\tau)\,\mathrm{d}\tau \tag{3.15}$$

where $a_K(\tau)$ is a p.d.f. with support on $(0, t)$ given by

$$a_K(\tau) = \frac{s_{0K}(\tau) U_K^T p_K^*(t-\tau)\mathbf{1}}{\int_0^t s_{0K}(\tau') U_K^T p_K^*(t-\tau')\mathbf{1}\,\mathrm{d}\tau'}, \quad 0 \leq \tau \leq t. \tag{3.16}$$

Similarly (3.14) can be rewritten as

$$d_{K+1}^{BT}(t) = \int_0^t a_{K+2}(\tau) q_K^{KT}(t-\tau)\,\mathrm{d}\tau \tag{3.17}$$

where $a_{K+2}(\tau)$ is a p.d.f. with support on $(0, t)$ given by

$$a_{K+2}(\tau) = \frac{s_{0K+2}(\tau) U_K^T p_K^*(t-\tau)\mathbf{1}}{\int_0^t s_{0K+2}(\tau') U_K^T p_K^*(t-\tau')\mathbf{1}\,\mathrm{d}\tau'}, \quad 0 \leq \tau \leq t. \tag{3.18}$$

Hence from (3.16) and (3.18), one has

$$\frac{a_K(\tau)}{a_{K+2}(\tau)} = C\frac{s_{0K}(\tau)}{s_{0K+2}(\tau)}, \quad 0 < \tau \leq t, \tag{3.19}$$

where C is dependent on t but independent of τ. For birth-death processes, it is known that for the passage time T_{0K} from state 0 to state K, one has $T_{0K} <_l T_{0K+2}$ (Keilson and Sumita [7]). This implies that the ratio $s_{0K}(\tau)/s_{0K+2}(\tau)$ is non-increasing in τ. From (3.19) one concludes that $a_K(\tau) <_l a_{K+2}(\tau)$ which implies that the mixing distributions in (3.15) and (3.17) are stochastically ordered. Moreover, from Lemma 3.4 it is seen that $q_K^{KT}(t-\tau)$ is stochastically increasing with τ, for $0 \leq \tau \leq t$. Since ordinary stochastic order is preserved under mixing, equation (3.12) and hence (3.11) follows.

Theorem 3.6. *For each $K > 1$,*

$$q_K^T < q_{K+1}^T. \tag{3.20}$$

Proof. This follows directly from Lemma 3.3 and Theorem 3.5 since stochastic order is preserved under limits.

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