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A Cell Population Model Described by Impulsive PDEs—Existence and Numerical Approximation

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Abstract—This paper considers a cell population model described by impulsive PDEs. Existence and numerical approximation of solutions are investigated. By showing that the related operator $a(x)\frac{d}{dx}$ on $C_0(-\infty,\infty)$ generates a strongly conitunous semigroups under general assumption on $a(x)$, we prove the existence of the solution of the nonlinear problem with impulses and given approximation of the solution of the homogenuous equation. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In recent years, the partial differential equation of the form

$$\frac{\partial u(x,t)}{\partial t} + c(x)\frac{\partial u(x,t)}{\partial x} = f(x,u), \qquad 0 \le x \le 1, \quad 0 \le t < \infty, \tag{1.1}$$

$$u(x,0) = v(x), \qquad 0 \le x \le 1, \tag{1.2}$$

has been studied intensively (see, for example, [1–3]). Under certain hypotheses on c and f, this equation models the dynamics of a self-reproductive cell population. The equation of maturity structured model (see [3,4]) of cell population dynamics is

$$\frac{\partial w(x,t)}{\partial t} + \frac{\partial(v(x)w(x,t))}{\partial x} = 0, \tag{1.3}$$

$$v(x_0)w(x_0,t) = 2\,v(1)\,w(1,t), \qquad t \ge 0, \tag{1.4}$$

$$w(x,0) = \phi(x), \qquad x_0 \le x \le 1. \tag{1.5}$$

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Problem (1.1),(1.2) was studied under assumption that $c(x)$ is continuously differentiable. Problem (1.3)–(1.5) was studied for the special case that $v(x) = x$. In order to further study these problems, the assumption on $c(x)$ and $v(x)$ should be general. If at certain moments of time, the processes of cell's maturing experience are subject to short time perturbations whose duration is negligible in comparison with the duration of the process, it is natural to assume that these perturbations are in the form of impulses (see [5–7]). Thus, we have the following impulsive PDEs.

$$\frac{\partial u(x,t)}{\partial t} + a(x)\,\frac{\partial u(x,t)}{\partial x} = f(x,\,u), \qquad t > 0, \quad x \in I, \tag{1.6}$$

$$u(x,0) = \phi(x), \qquad x \in I, \tag{1.7}$$

$$\Delta\, u(x,t_k) = \psi_k(u(x,t_k)), \qquad k = 1,\,2,\,\dots, \tag{1.8}$$

where I is an interval contained in the real line, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \to +\infty$, as $k \to \infty$, $\psi_K \in C(Y,\,Y)$, and $Y = C_0(I)$, where $C_0(I)$ is a Banach functions space.

In this paper, we will study the existence of solutions for problem (1.6)–(1.8) under the assumption that $a(x)$ is continuous and locally integrable (but not Lipschitzian). We will obtain a numerical approximation of the classical solution for problem (1.6)–(1.8) with special case, i.e., $f \equiv 0$.

It is obvious that problem (1.6)–(1.8) is related with operator $a(x)\frac{d}{dx}$ on $C(I)$, where $C(I)$ is the set of all continuous functions on I. This operator is considered in many places (see [8–10]). This operator is related not only with our problem (1.6)–(1.8), but also with the question of characterizing all flow on I whose generator extends $a(x)\frac{d}{dx}$, where $a \in C(I)$ and $\frac{d}{dx}$ acts on $C^1(I)$, where $C^1(I)$ is the set of all continuously differential functions on I. The motivation for this is the fact [11] that all closed quasi-well-behaved derivations on $C(I)$ are equivalent, via a homeomorphism of I, to the closure of the operator $a(x)\frac{d}{dx}$. So far, the most recent results on the operator $a(x)\frac{d}{dx}$ is that it generates a strongly continuous semigroup in the subspace of $C_0(-\infty, \in fty)$ containing all even functions and $a(x)$ is an odd function (see [10]).

In this paper, we will consider the operator $a(x)\frac{d}{dx}$ in whole space $C_0(-\infty,\infty)$ and show that it generates a strongly continuous semigroup without assumpting that $a(x)$ is an odd function. Actually, our results can be used to give the characterization for the generation of continuous groups of $*$-automorphisms on $C_0(-\infty,\infty)$ by presenting necessary and sufficient conditions on $a(x)$ that make (not the closure of) $a(x)\frac{d}{dx}$ a generator.

It is known that the main conditions [10] in which the operator generates a strongly continuous semigroup seems contradictory to previous ones (see [8,9]). The reason, which was pointed out in [10], is the symmetry of the functions and different domain of the operator than usual. In this paper, we will study the operator on a space of functions without the symmetry.

The properties of interest are whether there is a strongly continuous semigroup whose generator is the operator $a(x)\frac{d}{dx}$ (or $-a(x)\frac{d}{dx}$). In Section 2, we will use the method in [10,12] to show the necessary and sufficient so conditions on $a(x)$ for operator $a(x)\frac{d}{dx}$ (and $-a(x)\frac{d}{dx}$) to generate a strongly continuous semigroup, and in Section 3, show that nonlinear problem (1.6)–(1.8) has an unique global mild solution. Finally, we will give a numerical approximation of the solution for the problem with special case.

2. PROPERTIES OF OPERATORS $a(x)\frac{d}{dx}$ AND $-a(x)\frac{d}{dx}$

In this paper, we take $I = (-\infty,\,\infty)$ in problem (1.6) and (1.8). Define $Y = C_0(I) \equiv \{\phi \mid \phi \in C(I)$ and $\lim_{x\to\pm\infty}\phi = 0\}$ with norm $\|\phi\| = \sup_{-\infty<x<\infty}|\phi(x)|$. In this section, we shall assume that $a(x)$ is a continuous function satisfying the following conditions:

(i) $a(x)$ is continuous on I;

(ii) $m(\{x \mid a(x) = 0\}) = 0$, here m is Lebesgue measure;

(iii) $(1/a(x)) \in L^1_{\text{loc}}(I)$.

We define an operator H on Y. Let

$$D(H) = \Big\{\phi \in Y \mid \phi'(x) \text{ exists and continuous at } x \text{ when } a(x) \neq 0; \\ \lim_{x \to x_0} a(x)\phi'(x) \text{ exists when } a(x_0) = 0; \text{ and } \lim_{x \to \pm\infty} \hat{\phi}(x) = 0\Big\}, \tag{2.1}$$

where

$$\hat{\phi}(x) = \begin{cases} a(x)\phi(x), & a(x) \neq 0, \\ \lim_{x \to x_0} a(x)\phi(x), & a(x_0) = 0, \end{cases}$$

and

$$(H\phi)(x) = \begin{cases} a(x)\dfrac{d\phi(x)}{dx}, & \text{if } a(x) \neq 0, \\ \lim_{x \to x_0} a(x)\,\dfrac{d\phi(x)}{dx}, & \text{if } a(x_0) = 0. \end{cases} \tag{2.2}$$

LEMMA 2.1. (HILLE-YOSIDA THEOREM). *(See [12,13].) A is the generator of a C_0-contraction semigroup iff A is closed, densely defined, and for each $\lambda > 0$, $\lambda \in \rho(A)$ and $\|\lambda(\lambda - A)^{-1}\| \leq 1$.*

Now we can prove the following result.

THEOREM 2.1. *If (i)–(iii) are satisfied and $a(x)$ is nonnegative, then $-H$ generates a strongly continuous contraction semigroup if and only if $1/a(x)$ is not locally integrable at $+\infty$, i.e., $\int_0^{+\infty} (1/a(x))\,dx = +\infty$.*

PROOF. Let conditions (i)–(iii) be satisfied and $1/a(x)$ is not locally integrable at $+\infty$. It is easy to see that $D(-H)$ is a dense subset of Y since $C^1_c(I) \subset D(-H)$, where $C^1_c(I) = \{\phi \mid \phi \in C^1(I)$ and $\operatorname{supp}\phi$ is compact set in $I\}$. The Lemma 1 in [10] can be used here to show that $-H$ is a closed operator.

We need show that the image $\operatorname{Im}(\lambda I + H)$ of $\lambda I + H$ is dense in Y for $\lambda > 0$. For a given $g \in C_c(I)$, define

$$\phi(x) = e^{-\int_0^x (\lambda/a(s))\,ds} \int_{-\infty}^{x} \frac{g(t)}{a(t)}\, e^{\int_0^t (\lambda/a(s))\,ds}\, dt. \tag{2.3}$$

It is clear that $\phi(x)$ is a continuous function on I and $\lim_{x \to -\infty} \phi(x) = 0$. By assumption $\int_0^{+\infty} (1/a(x))\,dx = +\infty$, we see that $\lim_{x \to +\infty} \phi(x) = 0$. If $a(x) \neq 0$, then

$$\phi'(x) = -\frac{\lambda}{a(x)}\,\phi(x) + \frac{g(x)}{a(x)}. \tag{2.4}$$

If $a(x_0) = 0$, the limit $\lim_{x \to x_0} a(x)\phi'(x) = \lim_{x \to x_0}[-\lambda\,\phi(x) + g(x)]$ exists. Therefore, $(\lambda I + H)\phi = g$ and $\operatorname{Im}(\lambda I + H)$ is dense in Y for $\lambda > 0$. From (2.3), we also see

$$\|\phi\| \leq \frac{\|g\|}{\lambda}. \tag{2.5}$$

Since $(\lambda I + H)\phi = 0$ for $\lambda > 0$ implies that $e^{\lambda \int_0^x (1/a(s))\,ds}\,\phi(x) = \text{const}$, it follows the assumption that $\phi(x) = 0$. So, $(\lambda I + H)$ is injective for $\lambda > 0$. The $(\lambda I + H)$ is also surjective for $\lambda > 0$. In fact, for a given $\psi \in Y$, there are $\phi_n \in C_c(I)$, $n = 1, 2, \dots$, such that $(\lambda I + H)\phi_n \to \psi$, as $n \to \infty$. Since $-H$ is closed operator, by (2.5) there exists a ϕ such that $\phi_n \to \phi$, as $n \to \infty$, and $(\lambda I + H)\phi = \psi$.

The above argument shows that $(\lambda I + H)^{-1}$ exists and $\|(\lambda I + H)^{-1}\| \le 1/\lambda$ for $\lambda > 0$. By Lemma 2.1, we see that $-H$ generates a strongly continuous contraction semigroup on Y.

Conversely, if $-H$ generates a C_0-contraction semigroup on Y, then there is a $\lambda > 0$ with $\lambda \in \rho(-H)$ such that for each given $g \in C_c(I)$, there exists $\phi \in D(-H)$ such that (2.3) is satisfied. We can choose the g such that $\operatorname{supp} g \in (-M, M)$ and $\int_{-M}^{M} (g(t)/a(t))\, e^{\int_0^t (\lambda/a(s))\,ds}\, dt > 0$, where $M > 0$. Since $\lim_{x\to+\infty} \phi(x) = 0$, there must be $\int_0^{+\infty} (1/a(s))\, ds = +\infty$. This completes the proof.

REMARK 2.1. The function space Y here is different from the one given in [10]. The function space in [10] is a subspace of Y containing all even functions. The assumptions for $a(x)$ are also different. Here, $a(x)$ is not necessary an odd function.

REMARK 2.2. The domain of operator H defined by (2.5) is similar to that defined in [9,10]. But we would like to pointed out that the operator H defined here is more reasonable than the operator defined in [10, p. 1].

Similar to the proof of the Theorem 2.1, replacing (2.3) by

$$\phi(x) = e^{\int_0^x (\lambda/a(s))\,ds} \int_x^{+\infty} \frac{g(t)}{a(t)}\, e^{-\int_0^t (\lambda/a(s))\,ds}\, dt,$$

we get the following result about operator H.

THEOREM 2.2. *If a(x) is nonnegative and satisfies conditions (i)–(iii), then operator H generates a strongly continuous contraction semigroup if and only if $1/a(x)$ is not locally integrable at $-\infty$, i.e., $\int_{-\infty}^{0} (1/a(x))\, dx = +\infty$.*

The assumption that $a(x)$ is nonnegative is only used in showing that $\ker(\lambda I \pm H) = \{0\}$. If this assumption is replaced by

(iv) $\lim_{r\to-\infty} \int_r^0 (1/a(x))\, dx \ne 0$, or the limit does not exist;
(iv)′ $\lim_{r\to+\infty} \int_0^r (1/a(x))\, dx \ne 0$, or the limit does not exist; then Theorem 2.1 and Theorem 2.2 are still valid.

THEOREM 2.3. *If conditions (i)–(iv)′ are satisfied, then $-H$ (H) generates a strongly continuous contraction semigroup if and only if $1/a(x)$ is not locally integrable at $+\infty$ ($-\infty$), i.e., $\int_0^{+\infty} (1/a(x))\, dx = +\infty$ ($\int_{-\infty}^{0} (1/a(x))\, dx = +\infty$).*

3. EXISTENCE

In this section, we will study the existence of solution of problem (1.6)–(1.8). First, we will give some abstract results.

Let X be a Banach space. Consider an evolution process described by

$$\frac{du(t)}{dt} = A\,u(t), \qquad t \in J, \quad t \ne t_k, \tag{3.1}$$

$$\Delta\, u|_{t=t_k} = I_k(u(t_k)), \qquad k = 1, 2, \dots, \tag{3.2}$$

$$u(0) = u_0, \tag{3.3}$$

where $u : [0,\infty) \longmapsto X$, $J = (0,\infty)$, A is a generator of a C_0-semigroup $\{T(t)\}_{t\ge0}$ on X, $0 < t_1 < t_2 < \cdots$, $t_k \to \infty$ as $k \to \infty$, $I_k \in C(X,X)$ for each k, $u_0 \in X$. The $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e., $\Delta u|_{t=t_k} = u(t_k{}^+) - u(t_k{}^-)$, where $u(t_k{}^+)$ and $u(t_k{}^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively. Similar to the notations in [14], let $PC[J,X] = \{u \mid u$ is a map from J into X such that $u(t)$ is continuous at $t \ne t_k$, and left continuous at $t = t_k$ and $u(t_k{}^+)$ exists for $k = 1, 2, \dots\}$.

DEFINITION 3.1. *An X-valued function $u(t)$ is said to be a solution of (3.1)–(3.3) if*

(a) *$u(t) \in PC[J, X]$; and*

(b) *$u(t)$ is continuously differentiable and $u(t) \in D(A)$ for $t > 0$ and $t \neq t_k$, $k = 1, 2 \dots$ and (3.1)–(3.3) are satisfied.*

As to solution of (3.1)–(3.3), we have the following result.

THEOREM 3.1. *If the domain $D(A)$ of the operator A is invariant under each I_k, that is, I_k maps $D(A)$ to $D(A)$, $k = 1, 2, \dots$, and $u_0 \in D(A)$, then problem (3.1)–(3.3) has a unique solution $u(t)$ given by*

$$u(t) = \begin{cases} T(t)u_0, & 0 \leq t \leq t_1, \\ T(t)u_0 + \sum\limits_{i=1}^{k} T(t - t_i)\, I_i(u(t_i)), & t_k < t \leq t_{k+1}, \quad k = 1, 2, \dots. \end{cases} \tag{3.4}$$

PROOF. It follows from [12] that the function $u(t)$ defined by (3.4) satisfies (3.1) for $0 < t < t_1$ and $u(t_1^-) = T(t_1)u_0 = u(t_1)$ and such $u(t)$ is unique. From the assumption, we also have $\Delta u(t_1) = I_1(u(t_1)) \in D(A)$. If $u(t)$ defined by (3.4) satisfies equation (3.1) for $t_i < t < t_{i+1}$, $i = 1, 2, \dots, k$ and

$$\begin{aligned} u(t_{i+1}^-) &= u(t_{i+1}) = T(t_{i+1})u_0 + \sum_{j=1}^{i} T(t_{i+1} - t_j)\, I_j(u(t_j)), \qquad i = 1, 2, \dots, k, \\ \Delta u(t_i) &= I_i(u(t_i)) \in D(A), \end{aligned}$$

then for $t_{k+1} < t < t_{k+2}$, we have

$$\begin{aligned} \frac{du(t)}{dt} &= A\,T(t)u_0 + \sum_{i=1}^{k+1} A\,T(t - t_i)\, I_i(u(t_i)) = A\,u(t), \\ u\left(t_{k+2}^-\right) &= u(t_{k+2}) = T(t_{k+2})u_0 + \sum_{i=1}^{k+1} T(t_{i+1} - t_i)\, I_i(u(t_i)), \end{aligned}$$

and

$$\begin{aligned} \Delta u(t_{k+1}) &= u\left(t_{k+1}{}^+\right) - u(t_{k+1}) \\ &= T(t_{k+1})u_0 + \sum_{j=1}^{k+1} T(t_{k+1} - t_j)\, I_j(u(t_j)) \\ &\quad - T(t_{k+1})u_0 + \sum_{j=1}^{k} T(t_{k+1} - t_j)\, I_j(u(t_j)) \\ &= I_{k+1}(u(t_{k+1})) \in D(A). \end{aligned}$$

Thus, the theorem is proved by induction.

Note that $T(t) : X \longmapsto D(A)$ for $t > 0$ if A generates an analytic semigroup $\{T(t)\}_{t \geq 0}$, we have the following theorem.

THEOREM 3.2. *If $\{T(t)\}_{t \geq 0}$ is an analytic semigroup and A is its generator and $I_k \in C(X, X)$, $k = 1, 2, \dots$, then problem (3.1)–(3.3) has a unique solution $u(t)$ which can be expressed by (3.4).*

We can give another expression of the solution of (3.1)–(3.3). First, we define an operator-valued function $\Psi : [0, \infty) \longmapsto L(X, X)$ by the formula

$$\Psi(t) = \begin{cases} T(t), & t \in [0, t_1], \\ T(t - t_k)\,(I_k + I)\,T(t_k - t_{k-1}), \dots (I_1 + I)\,T(t_1), & t \in (t_k, t_{k+1}],\ k = 1, 2, \dots, \end{cases} \tag{3.5}$$

where $\{T(t)\}_{t\geq 0}$ is C_0-semigroup which has an infinitesimal generator A, I is the identity operator and

$$(I_k + I)(x) = I_k(x) + x$$

for $x \in X$. We further define function $u(t)$ by the relation

$$u(t) = \Psi(t)\, u_0. \tag{3.6}$$

COROLLARY 3.1. *The $u(t)$, defined by (3.6), is the solution of problem (3.1)–(3.3).*

We now turn to problem (1.6)–(1.8). The problem without impulsive actions was considered in many places. When $a : R \longmapsto R$ is locally Libschitzien, the problem without impulses was discussed in [2,15] by the classical method of characteristics and a Trotter product formula approach was given in [16]. But when a is merely continuous, the classical theory of characteristics cannot be applied. Using the results obtained in Section 2, we can prove the following theorem.

THEOREM 3.3. *Suppose that*

(i) *$a(x)$ is nonnegative and satisfies (i)–(iii) in Section 2, $\int_0^{+\infty}(1/a(x))\,dx = +\infty$ and $\lim_{x\to\pm\infty} a(x) = 0$;*
(ii) *$f \in C^1(R\times R)$, there is positive number M such that $|\frac{\partial f(x,u)}{\partial x}| \leq M$ and $|\frac{\partial f(x,u)}{\partial u}| \leq M$ for each $(x,u) \in R\times R$, $\lim_{x\to\pm\infty} f(x,u) = 0$ for each $u \in R$;*
(iii) *$\psi_k \in L(X,X)$ and $\psi_k(D(-H)) \subset D(-H)$.*

Then problem (1.6)–(1.8) has a unique global mild solution on $[0,\infty)$.

PROOF. First, problem (1.6)–(1.8) can be written in an abstract form,

$$\frac{du(t)}{dt} + H\,u(t) = g(u(t)), \tag{3.7}$$

$$u(0) = u_0, \tag{3.8}$$

$$\Delta u(t_k) = \psi_k(u(t_k)), \tag{3.9}$$

where $u(t) = u(t,\cdot)$, $g(u(t)) = f(\cdot, u(t,\cdot))$, $u_0 = \phi(x)$, and H is defined by (2.1) and (2.2).

Define

$$\Phi(t) = \begin{cases} T(t), & t \in [0,t_1], \\ T(t-t_k)\,(\psi_k+I)\,T(t_k-t_{k-1}),\dots,(\psi_1+I)\,T(t_1), & t\in(t_k,t_{k+1}],\ k=1,2,\dots, \end{cases} \tag{3.10}$$

where $T(t)$ is the semigroup generated by $-H$.

Taking $u_1(t) = u_0$, then $g(u_1(t)) \in D(-H)$ and $\Phi(t)\,g(u_1(t))$ is continuous in t excepting at t_k, it has discontinuities of first kind at t_k and integrable in t. Therefore, we can define

$$u_2(t) = \Phi(t)\,u_0 + \int_0^t \Gamma(t,\tau)\,g(u_1(\tau))\,d\tau,$$

where

$$\Gamma(t,\tau) = \begin{cases} T(t-t_k)\,(\psi_k+I)\,T(t_k-t_{k-1}), & \\ \quad \dots(\psi_1+I)\,T(t_1-\tau), & \text{if } t\in(t_k,t_{k+1}],\ \tau\in[0,t_1], \\ T(t-t_k)\,(\psi_k+I)\,T(t_k-t_{k-1}), & \\ \quad \dots(\psi_l+I)\,T(t_l-\tau), & \text{if } k\geq l \text{ and } t\in(t_k,t_{k+1}],\ \tau\in(t_{l-1},t_l], \\ T(t-\tau), & \text{if } t\geq\tau \text{ and } t,\ \tau\in(t_k,t_{k+1}] \text{ or} \\ & \quad t,\tau\in[0,t_1],\ k=1,2,\dots,l=2,3,\dots. \end{cases} \tag{3.11}$$

According to Corollary 3.1 and [12], it is easy to check that $u_2(t) \in D(-H)$ if $t \neq t_k$, and satisfies following problem:

$$\frac{dw(t)}{dt} + H\,w(t) = g(u_1(t)), \qquad t \neq t_k,\ k = 1, 2, \ldots,$$
$$w(0) = u_0,$$
$$\Delta w(t_k) = \psi_k(w(t_k)), \quad k = 1, 2, \ldots.$$

In the similar way, we can define

$$u_n(t) = \Phi(t)\,u_0 + \int_0^t \Gamma(t,\tau)\,g(u_{n-1}(\tau))\,d\tau, \tag{3.12}$$

for $n = 2, 3, \ldots$, and $u_n(t)$ satisfies

$$\frac{dw(t)}{dt} + H\,w(t) = g(u_{n-1}(t)), \qquad t \neq t_k, \quad k = 1, 2, \ldots, \tag{3.13}$$
$$w(0) = u_0, \tag{3.14}$$
$$\Delta w(t_k) = \psi_k(w(t_k)), \qquad k = 1, 2, \ldots. \tag{3.15}$$

For a given $S > 0$, let $k_0 = \max\{k \mid t_k \in [0, S]\}$ and $N = \max\{\|\psi_k\| \mid \|\psi_k\| \geq 1$, and $1 \leq k \leq k_0\}$, we can get

$$\|u_{n+1}(t) - u_n(t)\|_Y \leq N\,M \int_0^t \|u_n(\tau) - u_{n-1}(\tau)\|_Y\,d\tau, \tag{3.16}$$

for $n = 2, 3, \ldots$.

Therefore,

$$\|u_{n+1}(t) - u_n(t)\|_Y \leq (N\,M)^{n-1}\,\frac{t^{n-1}}{n!} \max_{0 \leq \tau \leq S} \|u_2(\tau) - u_1(\tau)\|, \tag{3.17}$$

for $n = 2, 3, \ldots$. It follows from (3.17) that there exists Y-valued function $u(t)$ on $[0, S]$ such that $u_n(t)$ converges uniformly to $u(t)$ on $[0, S]$. Note that $u_n(t)$ satisfies (3.13)–(3.15), by (3.12) and assumption on f, we see that $u(t)$ satisfies

$$u(t) = \Phi(t)\,u_0 + \int_0^t \Gamma(t,\tau)\,g(u(\tau))\,d\tau.$$

The argument above shows that (3.7)–(3.9) has a mild solution on $[0, S]$ for each $S > 0$. If problem (3.7)–(3.9) has another mild solution $\tilde{u}(t)$ on $[0, S]$, then

$$\|u(t) - \tilde{u}(t)\|_Y \leq M \int_0^t \|\Gamma(t,\tau)\|\,\|u(\tau) - \tilde{u}(\tau)\|_Y\,d\tau.$$

By Gronwall's inequality, $u(t) = \tilde{u}(t)$ on $[0, S]$. So, we see that problem (3.7)–(3.9) has a unique global mild solution on $[0, \infty)$. We have proved the theorem.

4. APPROXIMATION

In the following, we will give numerical approximation of solution of problem (1.6)–(1.8) with $f \equiv 0$.

For each given positive integer number n and $\tau_n > 0$. Let $Y_n = l_0(-\infty, \infty)$, i.e.,

$$Y_n = \left\{ b = \{b_m\}_{m=-\infty}^{\infty} \mid b_m \in R,\ \lim_{m \to \pm\infty} b_m = 0 \right\},$$

with norm $\|b\|_n = \sup_{-\infty<m<\infty} |b_m|$, then Y_n is a Banach space. Define

$$P_n\phi = \left\{\phi\left(\frac{m}{n}\right)\right\}_{m=-\infty}^{\infty} \tag{4.1}$$

for $\phi \in Y$, then $P_n : Y \longmapsto Y_n$ and P_n is a linear operator with norm $\|P_n\| \le 1$.

Taking for E_n the linear operator which assign to a sequence $b = \{b_m\}_{m=-\infty}^{\infty}$ the function $\phi(x)$ which is defined as follows:

$$E_n\, b = \phi(x) = \begin{cases} b_m, & x = \dfrac{m}{n}, \\ n(b_{m+1} - b_m)\left(x - \dfrac{m}{n}\right) + b_m, & x \in \left(\dfrac{m}{n}, \dfrac{m+1}{n}\right). \end{cases} \tag{4.2}$$

We also define

$$\begin{aligned} u_{0,m} &= \phi\left(\frac{m}{n}\right), \\ u_{l+1,m} &= \begin{cases} -a\left(\dfrac{m}{n}\right)(\tau_n\, n\, u_{l,m+1} - \tau_n\, n\, u_{l,m}) + u_{l,m}, \\ \quad 0 < (l+1)\tau_n < t_1 \text{ or } t_k < l\tau_n < (l+1)\tau_n \le t_{k+1}; \\ \left\{\left(I + \tilde{\psi}_k\right)\{u_{l,m}\}_{-\infty}^{\infty}\right\}_m, \\ \quad l\tau_n \le t_k < (l+1)\tau_n \le t_{k+1}; \end{cases} \end{aligned} \tag{4.3}$$

where $l = 0, 2, \dots,\ m = 0, \pm1, \pm2, \dots,\ \phi \in Y$, and

$$\left(I + \tilde{\psi}_k\right)\{u_{l,m}\}_{-\infty}^{\infty} = \{u_{l,m}\}_{-\infty}^{\infty} + P_n\,\psi_k(E_n\left(\{u_{l,m}\}_{m=-\infty}^{\infty}\right).$$

We recall the following result.

LEMMA 4.1. *(See [12].) Let $G(\rho_n)$ be a sequence of bounded linear operators from Banach space X_n into X_n satisfying*

$$\|G(\rho_n)^k\| \le M\, e^{\omega\rho_n k}, \qquad k = 1, 2, \dots$$

and

$$A_n = \rho_n^{-1}(G(\rho_n) - I) \to\to A.$$

If $D(A)$ is dense in X and if there is a λ_0 with Re $\lambda_0 > \omega$ *such that the range of $\lambda_0 I - A$ is dense in X then $\bar{A}$, the closure of A, is the infinitesimal generator of C_0 semigroup $S(t)$ on X. Moreover, if $k_n\,\rho_n \to t$ as $n \to \infty$ then*

$$G(\rho_n)^{k_n} \to\to S(t),$$

where $D(S(t)) = X$.

Note that $A_n \to\to A$ means that for every $x \in D(A)$

$$\|A_n\, Q_n x - Q_n\, Ax\|_{X_n} \to 0, \qquad \text{as } n \to \infty,$$

where $Q_n : X \to X_n$ is a bounded linear operator and $\|Q_n x\|_{X_n} \to \|x\|$ as $n \to \infty$ for every $x \in X$ (see [12] for detail).

Now, we can prove the following theorem.

THEOREM 4.1. *Suppose that $a(x)$ is positive and bounded and satisfies (i)–(iii) in Section 2. If $\int_0^{+\infty} (1/a(x))\,dx = +\infty$ and $\phi \in D(-H)$, then problem (3.7)–(3.9) with $f \equiv 0$ has a unique solution and the u_{l_n,m_n} defined by (4.3) converge to the solution at (t,x) where $l_n\tau_n \to t$ if $t \neq t_k$, $l_n\tau_n \to t^-$ if $t = t_k$, and $(m_n/n) \to x$, as $n \to \infty$.*

PROOF. The existence and uniqueness of the solution are guaranteed by Theorem 2.1 and Theorem 3.1. We will only show the numerical approximation of the solution.

For each given positive integer number n, we define an operator $F(1/n^2)$ mapping $Y_n \longmapsto Y_n$ as follows:

$$F\left(\frac{1}{n^2}\right)\{b_m\}_{m=-\infty}^{\infty} = \{b'_m\}_{m=-\infty}^{\infty}, \tag{4.4}$$

where $b'_m = -a(m/n)((1/n)\,b_{m+1} - (1/n)\,b_m) + b_m$. It is clear that $F(1/n^2)$ is a linear bounded operator and

$$\left\|F\left(\frac{1}{n^2}\right)^k\right\|_{L(Y_n,Y_n)} \leq e^{\omega\,(k/n^2)}, \qquad k = 1,\,2,\dots, \tag{4.5}$$

where $\omega = \max_{-\infty<x<\infty} |a(x)|^2$.

We now define

$$H_n = n^2\left(F\left(\frac{1}{n^2}\right) - I\right),$$

then $H_n\colon Y_n \longmapsto Y_n$ is a bounded linear operator.

Taking $\phi \in D(-H)$, then

$$\|H_n P_n\phi - P_n(-H)\phi\|_{Y_n} = \sup_m\left|a\left(\frac{m}{n}\right)\frac{\phi(m+1/n) - \phi(m/n)}{1/n} - (H\phi)\left(\frac{m}{n}\right)\right|. \tag{4.6}$$

We will show that

$$\|H_nP_n\phi - P_n(-H)\phi\|_{Y_n} \to 0, \qquad \text{as } n \to \infty. \tag{4.7}$$

If (4.7) is not valid, then there exists $\epsilon_0 > 0$, n_k and m_k, $k = 1,\,2,\dots$, such that $n_k \to \infty$ (as $k \to \infty$) and

$$\left|a\left(\frac{m_k}{n_k}\right)\frac{\phi(m_k+1/n_k) - \phi(m_k/n_k)}{1/n_k} - a\left(\frac{m_k}{n_k}\right)\phi'\left(\frac{m_k}{n_k}\right)\right| \geq \epsilon_0, \tag{4.8}$$

for $k = 1,\,2,\dots.$ Note that

$$\begin{aligned}&\left|a\left(\frac{m_k}{n_k}\right)\frac{\phi(m_k+1/n_k) - \phi(m_k/n_k)}{1/n_k} - a\left(\frac{m_k}{n_k}\right)\phi'\left(\frac{m_k}{n_k}\right)\right|\\ &\quad\leq n_k\int_{m_k/n_k}^{m_k+1/n_k}\left|(H\phi)(x) - a\left(\frac{m_k}{n_k}\right)\phi'\left(\frac{m_k}{n_k}\right)\right|dx\\ &\qquad+ n_k\int_{m_k/n_k}^{m_k+1/n_k}\left|a\left(\frac{m_k}{n_k}\right) - a(x)\right||\phi'(x)|\,dx.\end{aligned} \tag{4.9}$$

If $\{m_k/n_k\}_{k=1}^{\infty}$ is bounded, then we can assume that $\lim_{k\to\infty}(m_k/n_k) = x_0$, (otherwise, choose subsequence of $\{m_k/n_k\}_{k=1}^{\infty}$). Since $a(x)$ and $(H\phi)(x)$ are continuous function, we see that

$$n_k\int_{m_k/n_k}^{m_k+1/n_k}\left|(H\phi)(x) - a\left(\frac{m_k}{n_k}\right)\phi'\left(\frac{m_k}{n_k}\right)\right|dx + n_k\int_{m_k/n_k}^{m_k+1/n_k}\left|a\left(\frac{m_k}{n_k}\right) - a(x)\right||\phi'(x)|\,dx \to 0,$$

as $k \to \infty$, it follows from (4.9) that it contradicts (4.8).

If $\{m_k/n_k\}_{m=-\infty}^{\infty}$ is unbounded, then there exists a large k such that

$$\begin{aligned} n_k \int_{m_k/n_k}^{m_k+1/n_k} \left|(H\phi)(x) - a\left(\frac{m_k}{n_k}\right)\phi'\left(\frac{m_k}{n_k}\right)\right| dx < \frac{\epsilon_0}{2} \\ n_k \int_{m_k/n_k}^{m_k+1/n_k} \left|a\left(\frac{m_k}{n_k}\right) - a(x)\right| |\phi'(x)|\, dx < \frac{\epsilon_0}{2}, \end{aligned} \tag{4.10}$$

because that $(H\phi)(x)$ is uniformly continuous and $\lim_{x\to\pm\infty}(H\phi)(x) = 0$. The (4.10) still contradicts (4.8). These contradictions show that (4.7) is valid.

Taking $\tau_n = 1/n^2$. For a given (t,x), if $0 < t \le t_1$, choose l_n and m_n such that $l_n\tau_n \le t_1$ and $l_n\tau_n \to t$ and $(m_n/n) \to x$. By Theorem 2.1, (4.7) and [12, Theorem 3.6.7], we see that

$$\left\|F(\tau_n)^{l_n} P_n\phi - P_n T(t)\phi\right\|_{Y_n} = \left\|\{u_{l_n,m}\}_{m=-\infty}^{\infty} - P_n T(t)\phi\right\|_{Y_n} \to 0, \tag{4.11}$$

as $n \to \infty$. The argument above also shows that

$$\left\|F(\tau_n)^{l_n} P_n\phi - P_n T(t)\phi\right\|_{Y_n} \to 0 \quad (n \to \infty),$$

only if $\tau_n l_n \to t$ as $n \to \infty$. Since

$$|u_{l_n,m_n} - (T(t)\phi)(x)| \le |u_{l_n,m_n} - (T(t)\phi)(m_n)| + |(T(t)\phi)(m_n) - (T(t)\phi)(x)|, \tag{4.12}$$

and

$$\begin{aligned} &|u_{l_n,m_n} - (T(t)\phi)(m_n)| \\ &\quad\le |u_{l_n,m_n} - (E_nP_n T(t)\phi)(m_n)| + |(E_nP_n T(t)\phi)(m_n) - (T(t)\phi)(m_n)| \\ &\quad\le \|E_n\{u_{l_n,m}\}_{m=-\infty}^{\infty} - E_nP_n(T(t)\phi)\|_X + \|E_nP_nT(t)\phi - T(t)\phi\|_X \\ &\quad\le \|E_n\|\|\{u_{l_n,m}\}_{m=-\infty}^{\infty} - P_n T(t)\phi\|_{Y_n} + \|E_nP_nT(t)\phi - T(t)\phi\|_X. \end{aligned} \tag{4.13}$$

Notice that $\|E_n\| \le 1$ and $E_nP_n \to I$ as $n \to \infty$, by (4.11)–(4.13) we see

$$\lim_{n\to\infty} u_{l_n,m_n} = (T(t)\phi)(x). \tag{4.14}$$

Let $t_1 \le t \le t_2$, $t_1 \le l_n\tau_n \le t_2$, and $l_n\tau_n \to t$ as $n \to \infty$, by (4.3) we have

$$\{u_{l_n,m}\}_{m=-\infty}^{\infty} = F(\tau_n)^{l_n} P_n\phi + F(\tau_n)^{l_n-l_{n_1}}\tilde{\psi}_1\left(F(\tau_n)^{l_{n_1}} P_n\phi\right), \tag{4.15}$$

where $\tau_n l_{n_1} \le t_1$ and $\tau_n(l_{n_1}+1) \ge t_1$. There must be $\tau_n l_{n_1} \to t_1$ as $n \to \infty$, since $\tau_n \to 0$ as $n \to \infty$. Therefore,

$$\begin{aligned} &|u_{l_n,m_n} - [(T(t)\phi)(m_n) + (T(t-t_1)\phi)(m_n)\psi_1(T(t_1)\phi)(m_n)]| \\ &\quad\le \|E_n\{u_{l_n,m}\}_{m=-\infty}^{\infty} - [(T(t)\phi) + T(t-t_1)\psi_1(T(t_1)\phi)]\| \\ &\quad\le \left\|E_n\left(F(\tau_n)^{l_n} P_n\phi\right) - (T(t)\phi)\right\|_Y + \left\|E_n\left(F(\tau_n)^{l_n-l_{n_1}}\tilde{\psi}_1\left(F(\tau_n)^{l_{n_1}}\right) P_n\phi\right)\right\|_Y. \end{aligned} \tag{4.16}$$

Since ψ_1 is continuous and $E_n F(\tau_n)^{l_n} P_n\phi \to T(t)\phi$ as $n \to \infty$, it follows from (4.16) that

$$|u_{l_n,m_n} - [(T(t)\phi)(m_n) + T(t-t_1)\psi_1(T(t_1)\phi)(m_n)]| \to 0, \qquad \text{as } n \to \infty.$$

So, we get

$$\lim_{n\to\infty} u_{l_n,m_n} = (T(t)\phi)(x) + (T(t-t_1)\psi_1(T(t_1)\phi))(x). \tag{4.17}$$

From (4.16), we also see

$$\lim_{n\to\infty} E_n\{u_{l_n,m}\}_{m=-\infty}^{\infty} = T(t-t_1)(I+\psi_1)T(t_1)\phi. \tag{4.18}$$

Let $t_2 \le t \le t_3$, $t \le l_n\tau_n \le t_3$, and $l_n\tau_n \to t$ as $n \to \infty$, it follows from (4.3) that

$$\begin{aligned}\{u_{l_n,m}\}_{m=-\infty}^{\infty} = F(\tau_n)^{l_n - l_{n_2}} \Big\{ & F(\tau_n)^{l_{n_2}} P_n\phi + F(\tau_n)^{l_{n_2}-l_{n_1}} \tilde{\psi}_1 \left(F(\tau_n)^{l_{n_1}} P_n\phi\right) \\ & + \tilde{\psi}_2 \left(F(\tau_n)^{l_{n_2}} P_n\phi + F(\tau_n)^{l_{n_2}-l_{n_1}} \tilde{\psi}_1 \left(F(\tau_n)^{l_{n_1}} P_n\phi\right)\right)\Big\},\end{aligned} \tag{4.19}$$

where $l_{n_2}\tau_n \le t_2$ and $\tau_n(l_{n_2} + 1) > t_2$, so $l_{n_2}\tau_n \to t_2$ as $n \to \infty$. Similar to the proof of (4.18), we can get

$$\lim_{n\to\infty} E_n\{u_{l_n,m}\}_{m=-\infty}^{\infty} = T(t - t_2)(I + \psi_2)T(t_2 - t_1)(I + \psi_1)T(t_1)\phi. \tag{4.20}$$

If $k > 2$, $t_k \le t \le t_{k+1}$, $t_k \le l_n\tau_n \le t_{k+1}$, and $l_n\tau_n \to t$ as $n \to \infty$, similar to the proof (4.20), we can also get

$$\begin{aligned}&\lim_{n\to\infty} E_n\{u_{l_n,m}\}_{m=-\infty}^{\infty} \\ &\quad = T(t - t_k)(I + \psi_k)T(t_k - t_{k-1}), \dots, (I + \psi_2)T(t_2 - t_1)(I + \psi_1)T(t_1)\phi.\end{aligned} \tag{4.21}$$

It is follows from (4.21), (3.5), and Corollary 3.1 that the theorem has been proved.

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