Nano boundary layer equation with nonlinear Navier boundary condition

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This paper is dedicated to Professor William F. Ames on the occasion of his 80th birthday and for showing the way in nonlinear mathematics

Abstract

At the micro and nano scale the standard no slip boundary condition of classical fluid mechanics does not apply and must be replaced by a boundary condition that allows some degree of tangential slip. In this study the classical laminar boundary layer equations are studied using Lie symmetries with the no-slip boundary condition replaced by a nonlinear Navier boundary condition. This boundary condition contains an arbitrary index parameter, denoted by \( n > 0 \), which appears in the coefficients of the ordinary differential equation to be solved. The case of a boundary layer formed in a convergent channel with a sink, which corresponds to \( n = 1/2 \), is solved analytically. Another analytical but non-unique solution is found corresponding to the value \( n = 1/3 \), while other values of \( n \) for \( n > 1/2 \) correspond to the boundary layer formed in the flow past a wedge and are solved numerically. It is found that for fixed slip length the velocity components are reduced in magnitude as \( n \) increases, while for fixed \( n \) the velocity components are increased in magnitude as the slip length is increased.

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1. Introduction

The notion of a boundary layer was first introduced by Prandtl [18] over a hundred years ago to explain the discrepancies between the theory of inviscid fluid flow and experiment. In the inviscid limit it is assumed that the viscous forces are negligible as compared to inertial forces in the differential momentum conservation equations. This assumption in fact reduces the order of the differential momentum conservation equations and therefore fewer boundary conditions may be satisfied. In general the relaxed boundary condition in the inviscid limit is that for the tangential component of velocity at solid boundaries, which thus calls into question the uniqueness of inviscid solutions, and in a physical context allows for a finite but indeterminate slip velocity at solid boundaries which is not necessarily related to the form of slip assumed in this study. Another, much more significant difficulty with the inviscid approximation is that it implies that any body in a moving fluid stream experiences zero drag.

Prandtl observed that near solid boundaries viscous forces are of the same magnitude as inertial forces. This region of importance is usually confined to a thin layer adjacent to a solid boundary, which Prandtl referred to as the boundary layer. Using scaling arguments, Prandtl obtained a set of equations for the differential momentum conservation equations valid within the boundary layer, which were simpler that the full differential momentum conservation equations, but less simple than that of the differential momentum conservation equations in the inviscid limit. The applicability of these so-called boundary layer equations was demonstrated by Blasius [4] for the flow past a flat plate at zero incidence, which showed excellent agreement with experimental data.

A more detailed analysis of the boundary layer equations was later performed by Falkner and Skan [6] who examined similarity solutions with respect to an external velocity field proportional to a power of the length coordinate $x$, measured along a solid boundary downstream from the stagnation point. The resulting ordinary differential equation involves the power as a parameter, and the solutions of this equation were studied in detail by Hartree [8]. The similarity solutions obtained may be derived from a group invariant solution of the boundary layer equations by considering a linear combination of the Lie point symmetries, as recently shown by Mason [11] for the two-dimensional laminar jet. An outline of the Lie symmetries method may be found in either Hill [9] or Ames [2].

All theoretical investigations of the boundary layer equations have applied the no-slip boundary condition at the fluid-solid interface, which is a fundamental notion in fluid mechanics (Lamb [10]; Batchelor [3]; Slattery [22]), and assumes that the fluid velocity is zero relative to the solid boundary. For fluid flow at the micro and nano scale however, the no-slip boundary condition does not apply and a certain degree of tangential velocity slip must be allowed (see Gad-el-Hak [7]). Navier [15] introduced the linear boundary condition, later proposed by Maxwell [14], which remains the standard characterization of slip used today; namely the component of the fluid velocity tangential to the surface is assumed proportional to the tangential stress and the constant of proportionality is called the slip length (Matthews and Hill [12]). For the boundary layer equations along the solid boundary $y = 0$ the Navier boundary condition takes the form

$$|v_x| = \ell \left| \frac{\partial v_x}{\partial y} \right|,$$

(1.1)

where $\ell > 0$ is the constant slip length.

The Navier boundary condition has been extended to include the effects of multiple phases by Shikhmurzaev [21] and Qian et al. [19], and a generalized Navier boundary condition has been
proposed by Thompson and Troian [25] which is based on the results of molecular dynamical simulations, and indicates that the slip length is a nonlinear function of the shear rate, given by

\[ \ell = \alpha \left( 1 - \beta \left| \frac{\partial v_x}{\partial y} \right| \right)^{-\frac{1}{2}}, \]  

(1.2)

where \( \alpha \) and \( \beta \) are constants. We comment that the present authors (Matthews and Hill [13]) have recently established that for certain simple flows, the generalized Navier boundary condition, although highly nonlinear, still generates essentially unique solutions for the flow of a Newtonian fluid in simple geometries. However, we note that some multiplicity of solutions can arise from the two distinct cases \( \partial v_x/\partial y \) positive or negative, but the solution can be shown to be unique for each case.

Numerical results arising from Thompson and Troian [25] and modelled by Eqs. (1.1) and (1.2) can also be modelled by alternative nonlinear boundary conditions, and in this paper we study the similarity solutions of the laminar boundary layer equations using a nonlinear Navier boundary condition of the form

\[ |v_x| = \ell \left( \left| \frac{\partial v_x}{\partial y} \right| \right)^n, \]  

(1.3)

where \( \ell > 0 \) is the constant slip length and \( n > 0 \) is an arbitrary power parameter. In the following sections the laminar boundary layer equations are described. In the subsequent sections the classical similarity reductions of the boundary layer equations are derived. Then, analytical solutions are obtained where possible and numerical solutions for arbitrary values of \( n \) are obtained. Finally, we present a discussion of the results and concluding remarks. The method of the numerical solution used is outlined in Appendix A.

2. Boundary layer equations

With the assumption that viscosity is negligible in the bulk flow but important near a solid boundary, the steady two-dimensional boundary layer equations may be written in terms of a stream function \( \psi (x, y) \) defined via the following relation (Schlichting [20]):

\[ \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \ddot{v}_x \frac{d\ddot{v}_x}{dx} + \frac{\partial^3 \psi}{\partial y^3}, \]  

(2.1)

where

\[ v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}, \]  

(2.2)

and \( x \) is measured parallel to and \( y \) is measured perpendicular to a solid boundary located along \( y = 0 \). Finally, \( \ddot{v}_x \) is an assumed given external inviscid velocity field such that \( v_x \rightarrow \ddot{v}_x \) as \( y \rightarrow \infty \). For simplicity, only boundary layers without back-flow or separation are considered and we refer the reader to Schlichting [20] for a detailed discussion of these phenomena. The above partial differential equation is solved with suitable boundary conditions imposed at \( y = 0 \) and matching with the inviscid solution as \( y \rightarrow \infty \). The specific boundary conditions imposed in this study are

\[ y = 0: \quad |v_x| = \ell^n \left( \left| \frac{\partial v_x}{\partial y} \right| \right)^n; \quad v_y = 0, \quad \text{and} \quad y \rightarrow \infty: \quad v_x \equiv \ddot{v}_x = \alpha (x + \beta)^m, \]  

(2.3)
where $\alpha$, $\beta$, $m$ and $n$ are arbitrary constants with $n > 0$, $n \neq 2/3, 1$ and the reasons for these restrictions will be made clear later. The constant $\ell^* > 0$ is the slip length which has been scaled with $R^{n/2}$ where $R$ is the Reynolds number. In terms of the stream function the boundary conditions become

$$y = 0: \quad \frac{\partial \psi}{\partial y} = \ell^* \left( \left| \frac{\partial^2 \psi}{\partial y^2} \right| \right)^n; \quad \frac{\partial \psi}{\partial x} = 0, \quad \text{and} \quad y \to \infty: \quad \frac{\partial \psi}{\partial y} \to \alpha(x + \beta)^m. \quad (2.4)$$

Alternatively, the boundary condition at $y \to \infty$ may be complemented by

$$y \to \infty: \quad \frac{\partial \psi}{\partial x} \equiv \frac{\partial^2 \psi}{\partial y^2} \to 0, \quad (2.5)$$

since $\tilde{v}_x$ is a function of $x$ only.

### 3. Classical symmetry reductions

The Lie-group method of infinitesimal transformations is used for finding the symmetries of the boundary layer equations, and the general method is explained elsewhere [16]. The inviscid velocity terms in the governing partial differential equation may be eliminated by taking the $y$ derivative of both sides, so that

$$\frac{\partial \psi}{\partial y} \frac{\partial^3 \psi}{\partial x \partial y^2} - \frac{\partial \psi}{\partial x} \frac{\partial^3 \psi}{\partial y^3} = \frac{\partial^4 \psi}{\partial y^4}. \quad (3.1)$$

Using the MAPLE package DESOLV [5], it is found that this partial differential equation has the following Lie symmetry infinitesimals

$$X = (C_1 + C_4)x + C_2, \quad Y = C_4y + f(x), \quad \Psi = C_1\psi + C_3, \quad (3.2)$$

where $C_i$ are arbitrary constants and $f(x)$ is a sufficiently differentiable arbitrary function. Without loss of generality we may set $C_4 = 1$ and assume $C_1 \neq -1$, then the group invariant solution of the boundary layer equations is of the form

$$\psi(x, y) = \left( x + \frac{C_2}{C_1 + 1} \right) \frac{C_1}{C_1 + 1} F^*(\rho^*) - \frac{C_3}{C_1}, \quad (3.3)$$

where

$$\rho^* = \frac{y}{(x + \frac{C_2}{C_1 + 1})^{\frac{1}{C_1 + 2}}} - \frac{1}{C_1 + 1} \int \frac{f(x)\, dx}{(x + \frac{C_2}{C_1 + 1})^{\frac{C_1 + 2}{C_1 + 1}}}. \quad (3.4)$$

Since a stream function is determined up to an arbitrary additive constant we may set $C_3 = 0$. Also, to ensure that $y = 0$ corresponds to $\rho^* = 0$ we set $f(x) \equiv 0$. Substituting into the boundary layer equation yields an ordinary differential equation for $F^*(\rho^*)$

$$\frac{d^3 F^*}{d\rho^*} + \left( \frac{C_1}{C_1 + 1} \right) \frac{d}{d\rho^*} \left( F^* \frac{dF^*}{d\rho^*} \right) - \left( \frac{2C_1 - 1}{C_1 + 1} \right) \left( \frac{dF^*}{d\rho^*} \right)^2 + \left( x + \frac{C_2}{C_1 + 1} \right) \frac{3 - C_1}{C_1 + 1} \tilde{v}_x \frac{d\tilde{v}_x}{dx} = 0. \quad (3.5)$$

The slip boundary condition at $y = 0$ implies $C_1 = (2n - 1)/(n - 1)$, from which the condition $n \neq 1$ is required. Also, since $\tilde{v}_x = \alpha(x + \beta)^m$ the last term of the ordinary differential equation for $F^*(\rho^*)$ is $x$-absent provided $m = n/(3n - 2)$, $\beta = C_2/(C_1 + 1)$, from which the condition
\(n \neq 2/3\) is required. This condition also follows from the definition of \(C_1\) since it has been assumed that \(C_1 \neq -1\). Finally, the ordinary differential equation to be solved is

\[
\frac{d^3 F^*}{d\rho^*^3} + \left( \frac{2n - 1}{3n - 2} \right) \frac{d}{d\rho^*} \left( \frac{dF^*}{d\rho^*} \right) - \left( \frac{3n - 1}{3n - 2} \right) \left( \frac{dF^*}{d\rho^*} \right)^2 - \left( \frac{n}{3n - 2} \right) \alpha^2 = 0,
\]

which may alternatively be written

\[
\frac{d^3 F^*}{d\rho^*^3} + \left( \frac{2n - 1}{3n - 2} \right) \frac{dF^*}{d\rho^*} \left( \frac{d^2 F^*}{d\rho^*^2} \right) - \left( \frac{n}{3n - 2} \right) \left[ \left( \frac{dF^*}{d\rho^*} \right)^2 - \alpha^2 \right] = 0,
\]

which must be solved subject to

\[
\rho^* = 0: \left( \frac{dF^*}{d\rho^*} \right) = \ell^* \left( \frac{d^2 F^*}{d\rho^*^2} \right)^n; \quad F^* = 0, \quad \text{and} \quad \rho^* \to \infty: \frac{dF^*}{d\rho^*} \to \alpha,
\]

for \(C_1 \neq 0 \Rightarrow n \neq 1/2\). For \(n = 1/2\) the condition \(F^* = 0\) is absent and replaced by

\[
\rho^* \to \infty: \frac{d^2 F^*}{d\rho^*^2} \to 0.
\]

We also have

\[
\frac{v_x}{\bar{v}_x} = \frac{1}{\alpha} \frac{dF^*}{d\rho^*}, \quad (x + \beta) \frac{2n-1}{3n-2} \frac{v_y}{\bar{v}_x} = \frac{1}{\alpha} \left( \frac{n - 1}{3n - 2} \right) \left[ \rho^* \frac{dF^*}{d\rho^*} - \left( \frac{2n - 1}{n - 1} \right) F^* \right].
\]

We note that the parameter \(\alpha\) may be removed from the ordinary differential equation and boundary conditions by suitable transformations depending on the sign of \(\alpha\). Generally, \(\alpha > 0\) is the main interest describing flows away from the origin of \(x\), but for \(\alpha < 0\) the external stream flows towards the origin of \(x\), and one example of interest is that for the flow in a convergent channel with a sink, since it is one of the rare cases where a simple analytical solution may be found. This was first solved with the standard no-slip boundary condition by Pohlhausen [17] and corresponds to the case \(m = -1\), or \(n = 1/2\).

### 4. Analytical solution for \(n = 1/2\)

In this case we have \(\alpha < 0\) and

\[
\frac{d^3 F^*}{d\rho^*^3} + \left( \frac{dF^*}{d\rho^*} \right)^2 - \alpha^2 = 0,
\]

which must be solved subject to

\[
\rho^* = 0: \left( \frac{dF^*}{d\rho^*} \right) = \ell^* \left( \frac{d^2 F^*}{d\rho^*^2} \right)^{1/2}, \quad \text{and} \quad \rho^* \to \infty: \frac{dF^*}{d\rho^*} \to \alpha, \quad \frac{d^2 F^*}{d\rho^*^2} \to 0.
\]

Here we have

\[
\frac{v_x}{\bar{v}_x} = \frac{1}{\alpha} \frac{dF^*}{d\rho^*}, \quad \frac{v_y}{\bar{v}_x} = \frac{\rho^*}{\alpha} \frac{dF^*}{d\rho^*}.
\]

Hence we only need to solve for \(dF^*/d\rho^*\) in this case. Substituting the following transformations

\[
F^* = -\sqrt{-\alpha} F, \quad \rho^* = \frac{\rho}{\sqrt{-\alpha}}, \quad \ell^* = \frac{\ell}{(-\alpha)^{1/4}},
\]

Therefore the original problem is reduced to the study of a simple 

\[
\frac{d^3 F}{d\rho^3} + \left( \frac{2n - 1}{3n - 2} \right) \frac{1}{\rho} \frac{dF}{d\rho} - \left( \frac{3n - 1}{3n - 2} \right) \left( \frac{1}{\rho} \frac{dF}{d\rho} \right)^2 - \left( \frac{1}{3n - 2} \right) \alpha^2 = 0,
\]

which may alternatively be written

\[
\frac{d^3 F}{d\rho^3} + \left( \frac{2n - 1}{3n - 2} \right) \frac{1}{\rho} \frac{dF}{d\rho} \left( \frac{d^2 F}{d\rho^2} \right) - \left( \frac{1}{3n - 2} \right) \left[ \left( \frac{1}{\rho} \frac{dF}{d\rho} \right)^2 - \alpha^2 \right] = 0,
\]

which must be solved subject to

\[
\rho = 0: \left( \frac{1}{\rho} \frac{dF}{d\rho} \right) = \ell \left( \frac{d^2 F}{d\rho^2} \right)^n; \quad F = 0, \quad \text{and} \quad \rho \to \infty: \frac{1}{\rho} \frac{dF}{d\rho} \to \alpha,
\]

for \(C_1 \neq 0 \Rightarrow n \neq 1/2\). For \(n = 1/2\) the condition \(F = 0\) is absent and replaced by

\[
\rho \to \infty: \frac{d^2 F}{d\rho^2} \to 0.
\]

We also have

\[
\frac{v_x}{\bar{v}_x} = \frac{1}{\alpha} \frac{dF}{d\rho}, \quad \frac{v_y}{\bar{v}_x} = \frac{1}{\alpha} \left( \frac{n - 1}{3n - 2} \right) \left[ \rho \frac{dF}{d\rho} - \left( \frac{2n - 1}{n - 1} \right) F \right].
\]

We note that the parameter \(\alpha\) may be removed from the ordinary differential equation and boundary conditions by suitable transformations depending on the sign of \(\alpha\). Generally, \(\alpha > 0\) is the main interest describing flows away from the origin of \(x\), but for \(\alpha < 0\) the external stream flows towards the origin of \(x\), and one example of interest is that for the flow in a convergent channel with a sink, since it is one of the rare cases where a simple analytical solution may be found. This was first solved with the standard no-slip boundary condition by Pohlhausen [17] and corresponds to the case \(m = -1\), or \(n = 1/2\).

### 4. Analytical solution for \(n = 1/2\)

In this case we have \(\alpha < 0\) and

\[
\frac{d^3 F^*}{d\rho^*^3} + \left( \frac{dF^*}{d\rho^*} \right)^2 - \alpha^2 = 0,
\]

which must be solved subject to

\[
\rho^* = 0: \left( \frac{dF^*}{d\rho^*} \right) = \ell^* \left( \frac{d^2 F^*}{d\rho^*^2} \right)^{1/2}, \quad \text{and} \quad \rho^* \to \infty: \frac{dF^*}{d\rho^*} \to \alpha, \quad \frac{d^2 F^*}{d\rho^*^2} \to 0.
\]

Here we have

\[
\frac{v_x}{\bar{v}_x} = \frac{1}{\alpha} \frac{dF^*}{d\rho^*}, \quad \frac{v_y}{\bar{v}_x} = \frac{\rho^*}{\alpha} \frac{dF^*}{d\rho^*}.
\]

Hence we only need to solve for \(dF^*/d\rho^*\) in this case. Substituting the following transformations

\[
F^* = -\sqrt{-\alpha} F, \quad \rho^* = \frac{\rho}{\sqrt{-\alpha}}, \quad \ell^* = \frac{\ell}{(-\alpha)^{1/4}},
\]
Fig. 1. Velocity profiles corresponding to Eq. (4.11) for various values of \( \ell \). The values of \( \lambda \) corresponding to largest positive root of Eq. (4.10) are also shown.

Fig. 2. Velocity profiles corresponding to Eq. (4.11) for various values of \( \ell \) with \( \alpha = -1 \).

gives

\[
\frac{d^3F}{d\rho^3} - \left( \frac{dF}{d\rho} \right)^2 + 1 = 0,
\]

which must be solved subject to

\[
\rho = 0: \quad \left| \frac{dF}{d\rho} \right| = \ell \left( \left| \frac{d^2F}{d\rho^2} \right| \right)^{\frac{1}{2}}, \quad \text{and} \quad \rho \to \infty: \quad \frac{dF}{d\rho} \to 1, \quad \frac{d^2F}{d\rho^2} \to 0.
\]
Also
\[ \frac{v_x}{\bar{v}_x} = \frac{dF}{d\rho}, \quad \sqrt{-\alpha} \frac{v_y}{\bar{v}_x} = \rho \frac{dF}{d\rho}. \]  \hspace{1cm} (4.7)

Multiplying both sides of Eq. (4.5) by \(2 dF/d\rho\) and integrating once with the boundary conditions at \(\rho \to \infty\) we obtain
\[ \left( \frac{d^2 F}{d\rho^2} \right)^2 = \frac{2}{3} \left( \frac{dF}{d\rho} - 2 \right) \left( 1 - \frac{dF}{d\rho} \right)^2. \]  \hspace{1cm} (4.8)

This is a separable first-order differential equation for \(dF/d\rho\), which may be integrated to yield
\[ \rho = \sqrt{2} \tanh^{-1} \left( \frac{1}{\sqrt{3}} \frac{dF}{d\rho} + \frac{2}{3} \right) - C_1. \]  \hspace{1cm} (4.9)

Substituting into the slip boundary condition we find \(C_1 = \tanh^{-1}(\lambda)\), where \(\lambda\) is the largest positive root of
\[ 9\lambda^4 + 3\sqrt{2}\ell^2\lambda^3 - 12\lambda^2 - 3\sqrt{2}\ell^2\lambda + 4 = 0. \]  \hspace{1cm} (4.10)

Hence
\[ \frac{v_x}{\bar{v}_x} = 3 \tanh^2 \left( \frac{\rho}{\sqrt{2}} + \tanh^{-1}(\lambda) \right) - 2, \]
\[ \sqrt{-\alpha} \frac{v_y}{\bar{v}_x} = \rho \left\{ 3 \tanh^2 \left( \frac{\rho}{\sqrt{2}} + \tanh^{-1}(\lambda) \right) - 2 \right\}. \]  \hspace{1cm} (4.11)

These profiles are illustrated in Figs. 1–3 for \(\ell\) equal to 0, 0.1, 0.5 and 1.
5. Solutions for $\alpha > 0$

For $\alpha > 0$ substituting the following transformations

$$F^* = \sqrt{\alpha} F, \quad \rho^* = \frac{\rho}{\sqrt{\alpha}}, \quad \ell^* = \frac{\ell}{\alpha^{3n - 1}},$$

yields

$$\frac{d^3 F}{d\rho^3} + \left(\frac{2n - 1}{3n - 2}\right) \frac{d}{d\rho} \left( F \frac{dF}{d\rho} \right) - \left(\frac{3n - 1}{3n - 2}\right) \left( \frac{dF}{d\rho} \right)^2 + \left(\frac{n}{3n - 2}\right) = 0,$$

which must be solved subject to

$$\rho = 0: \quad \frac{dF}{d\rho} = \ell \left( \frac{d^2 F}{d\rho^2} \right)^n; \quad F = 0, \quad \text{and} \quad \rho \to \infty: \quad \frac{dF}{d\rho} \to 1.$$  (5.3)

We also have

$$\frac{v_y}{v_x} = \frac{dF}{d\rho}, \quad \sqrt{\alpha(x + \beta)} \frac{2n - 1}{3n - 2} \frac{v_y}{v_x} = \left(\frac{n - 1}{3n - 2}\right) \left[ \frac{dF}{d\rho} - \left(\frac{2n - 1}{n - 1}\right) F \right].$$  (5.4)

5.1. Analytic solution for $n = 1/3$

In this case we have

$$\frac{d^3 F}{d\rho^3} + \frac{1}{3} \left[ \frac{d}{d\rho} \left( F \frac{dF}{d\rho} \right) - 1 \right] = 0,$$  (5.5)

subject to

$$\rho = 0: \quad \frac{dF}{d\rho} = \ell \left( \frac{d^2 F}{d\rho^2} \right)^{1/3}; \quad F = 0, \quad \text{and} \quad \rho \to \infty: \quad \frac{dF}{d\rho} \to 1.$$  (5.6)

Here we have

$$\frac{v_y}{v_x} = \frac{dF}{d\rho}, \quad \sqrt{\alpha(x + \beta)} \frac{1}{3} \frac{v_y}{v_x} = 2\rho \frac{dF}{d\rho} - F.$$  (5.7)

Eq. (5.5) may be integrated twice

$$\frac{d^2 F}{d\rho^2} + \frac{1}{3} \left( F \frac{dF}{d\rho} - \rho \right) + C_1 = 0, \quad \frac{dF}{d\rho} + \frac{1}{6} \left( f^2 - \rho^2 \right) + C_1 \rho + C_2 = 0.$$  (5.8)

The nonlinear Navier boundary condition implies $|C_2| = \ell |C_1|^{1/3}$, where $C_1 \neq 0$. The first order differential equation obtained above is a Ricatti differential equation. Making the substitution

$$F = \frac{6}{y} \frac{dy}{d\rho},$$  (5.9)

we have

$$\frac{d^2 y}{d\rho^2} - \frac{1}{6} \left( \frac{1}{6} \rho^2 - C_1 \rho - C_2 \right) y = 0.$$  (5.10)

This is a second-order linear differential equation, and letting
\[ y(\rho) = u(\rho) \exp \left( \frac{\delta}{12} \rho^2 - \frac{C_1}{2\delta} \rho \right), \quad (5.11) \]

where \( \delta \pm 1 \), we have
\[ \frac{d^2 u}{d\rho^2} + \frac{\delta}{3}(\rho - 3C_1) \frac{du}{d\rho} + \left( \frac{C_1^2}{4} + \frac{C_2}{6} + \frac{\delta}{6} \right) u = 0. \quad (5.12) \]

Finally, letting
\[ t = -\frac{\delta}{6}(\rho - 3C_1)^2, \quad (5.13) \]
we obtain a second-order differential equation of a known form
\[ t \frac{d^2 u}{dt^2} + \left( \frac{1}{2} - t \right) \frac{du}{dt} - \left( \frac{3C_1^2}{8\delta} + \frac{C_2}{4\delta} + \frac{1}{4} \right) u = 0. \quad (5.14) \]

This is Kummer’s differential equation, which has general solution
\[ u(t) = C_1^* M \left( \frac{3C_1^2}{8\delta} + \frac{C_2}{4\delta} + \frac{1}{4}, \frac{1}{2}, t \right) + C_2^* U \left( \frac{3C_1^2}{8\delta} + \frac{C_2}{4\delta} + \frac{1}{4}, \frac{1}{2}, t \right), \quad (5.15) \]

where \( M(a, 1/2, t) \) and \( U(a, 1/2, t) \) are Kummer functions of the first and second kind [1]. We note that the parameter \( \delta \) is equal to \( \pm 1 \), and it is tempting to take \( \delta = +1 \) so that the functional argument of the Kummer functions will be negative, and the Kummer function of the second kind will be complex valued unless the first argument is a negative integer, and we may set \( C_2^* = 0 \). However, if we take \( \delta = -1 \) the boundary condition at infinity is automatically satisfied. To see this, note that
\[ F = 2(\rho - 3C_1) \frac{1}{u} \frac{du}{dt} - \rho + 3C_1, \quad (5.16) \]
which in the limit \( t \to \infty \), which corresponds to \( \rho \to \infty \), implies
\[ F \sim 2(\rho - 3C_1) \left( \frac{1}{u} \frac{du}{dt} \right)_{t \to \infty} - \rho. \quad (5.17) \]

Now, from the solution for \( u(t) \) we have
\[ u \sim e^{t}t^{a-\frac{1}{2}}, \quad t \to \infty \quad \Rightarrow \quad \frac{du}{dt} \sim e^{t}t^{a-\frac{1}{2}}, \quad t \to \infty \quad \Rightarrow \quad \frac{1}{u} \frac{du}{dt} \sim 1, \quad t \to \infty, \quad (5.18) \]
which implies \( F \sim \rho \) and \( dF/d\rho \sim 1 \) as \( \rho \to \infty \). Therefore for \( \delta = -1 \) we have
\[ F = \frac{6 \frac{d}{d\rho} \left[ \exp(-\rho^2) + \frac{C_1^2}{12} (M(a, \frac{1}{2}, \frac{1}{6}(\rho - 3C_1)^2) + C_3 U(a, \frac{1}{2}, \frac{1}{6}(\rho - 3C_1)^2)) \right] \exp(-\rho^2) + \frac{C_1^2}{12} (M(a, \frac{1}{2}, \frac{1}{6}(\rho - 3C_1)^2) + C_3 U(a, \frac{1}{2}, \frac{1}{6}(\rho - 3C_1)^2)) \right]}{\exp(-\rho^2) + \frac{C_1^2}{12} (M(a, \frac{1}{2}, \frac{1}{6}(\rho - 3C_1)^2) + C_3 U(a, \frac{1}{2}, \frac{1}{6}(\rho - 3C_1)^2))}, \quad (5.19) \]
where \( a = 1/4 - 3C_1^2/8 - C_2/4 \) and \( C_3 = C_2^*/C_1^* \). Now, since the boundary condition at infinity is already satisfied and the slip boundary condition is satisfied by \( |C_2| = \ell|C_1|^{1/3} \) there is only one boundary condition, namely \( F(0) = 0 \), to determine the two remaining integration constants \( C_1 \) and \( C_3 \), and the solution is therefore not unique. It should be noted that \( n = 1/3 \) implies \( m = -1/3 \), and the non-uniqueness of solutions of the Falkner–Skan equations for \( m < 0 \) has been established by Stewartson [24].
Consider first the case $\ell = 0$, which implies $C_2 = 0$. It is reasonable to suppose that $C_1 < 0$ so that the functional arguments of the Kummer functions do not pass through zero. Solving $F(0) = 0$ gives $C_3$ uniquely as a function of $C_1$. Figs. 4 and 5 plot the velocity profiles for these solutions for two values of $C_1$, which show that the velocity profile in the $x$ direction possesses a velocity excess with the limit as $\rho \to \infty$ approached from above rather than from below. These types of solutions correspond to a wall jet in a retarded outer flow, and were studied extensively by Steinheuer [23].
Next consider the case $\ell = 0.1$. Solving $F(0) = 0$ with $|C_2| = \ell|C_1|^{1/3}$ again gives $C_3$ uniquely as a function of $C_1$. Figs. 6 and 7 plot the velocity profiles for these solutions for two values of $C_1$. We find that as $\ell$ increases so does $(dF/d\rho)_{\rho=0}$ and eventually pass unity. Although the physical meaning of these profiles is not clear, we note that to the authors knowledge it is the only exact (although non-unique) analytic closed form solution of the boundary layer equations, besides the case of the flow in a convergent channel discussed earlier.
5.2. Numerical solutions for $n > 1/2$

As demonstrated by Falkner and Skan [6] the family of inviscid flows with $\tilde{v}_x \sim x^m$ corresponds to flow past a wedge with angle $\pi \gamma$, where $\gamma = 2m/(m + 1)$ and as such the analysis applies to the boundary layer formed near the wedge. This same analogy may be applied in this analysis, since in the inviscid limit the nonlinear Navier boundary condition is dropped. Since in this analysis $m = n/(3n - 2)$ we have $\gamma = n/(2n - 1)$. Now, since we are concerned only with boundary layers without back-flow or separation [20], we require $\gamma > 0$ which implies that $n > 1/2$. 

Fig. 8. Velocity profiles corresponding to Eq. (5.4) for $\ell = 0.5$ and various values of $n$.

Fig. 9. Velocity profiles corresponding to Eq. (5.4) for $\ell = 0.5$ and various values of $n$ with $\alpha = 1$. 

\begin{equation}
(x + \beta)^{2n-1} \left( \frac{v_y}{\tilde{v}_x} \right)^{3n-2}
\end{equation}
Equation (5.2) with the boundary conditions given by Eq. (5.3) are solved numerically for the cases $n = 2, 3, 4$ and 5 and $\ell = 0.5$, and for $\ell = 0, 0.1, 0.5$ and 1 and $n = 3$. An outline of the numerical method is given in Appendix A. The profiles for the $x$ and $y$ components of velocity given by Eq. (5.4) are illustrated in Figs. 8–11.
6. The case \( n = 2/3 \)

When \( n = 2/3 \) we have \( C_1 = -1 \) and the group invariant solutions obtained above are not valid. Returning to the Lie symmetry infinitesimals

\[
X = (C_1 + C_4)x + C_2, \quad Y = C_4 y + f(x), \quad \Psi = C_1 \psi + C_3,
\]

we set \( C_4 = -C_1 \), then the group invariant solution of the boundary layer equations is of the form

\[
\psi(x, y) = \exp \left( \frac{C_1}{C_2} x \right) F^*(\rho^*) - \frac{C_3}{C_1},
\]

where

\[
\rho^* = y \exp \left( \frac{C_1}{C_2} x \right) - \frac{1}{C_2} \int f(x) \exp \left( \frac{C_1}{C_2} x \right) dx.
\]

Since a stream function is determined up to an arbitrary additive constant we may set \( C_3 = 0 \). Also, to ensure that \( y = 0 \) corresponds to \( \rho^* = 0 \) we set \( f(x) \equiv 0 \). Substituting into the boundary layer equation yields an ordinary differential equation for \( F^*(\rho^*) \)

\[
\frac{d^3 F^*}{d\rho^3} + \left( \frac{m}{2} \right) \frac{d}{d\rho^*} \left( F^* \frac{dF^*}{d\rho^*} \right) - \left( \frac{3m}{2} \right) \left( \frac{dF^*}{d\rho^*} \right)^2 + \alpha^2 m = 0.
\]

Here we assume that \( \tilde{v}_x = \alpha e^{mx} \), which is a limiting case of \( \tilde{v}_x = \alpha (x + \beta)^m \), and the last term of the ordinary differential equation for \( F^*(\rho^*) \) is \( x \)-absent provided \( m = 2C_1/C_2 \). Finally, the ordinary differential equation to be solved is

\[
\frac{d^3 F^*}{d\rho^3} + \left( \frac{m}{2} \right) \frac{d}{d\rho^*} \left( F^* \frac{dF^*}{d\rho^*} \right) - \left( \frac{3m}{2} \right) \left( \frac{dF^*}{d\rho^*} \right)^2 + \alpha^2 m = 0,
\]

which may alternatively be written

\[
\frac{d^3 F^*}{d\rho^3} + \left( \frac{m}{2} \right) F^* \frac{d^2 F^*}{d\rho^*} - m \left[ \left( \frac{dF^*}{d\rho^*} \right)^2 - \alpha^2 \right] = 0,
\]

which must be solved subject to

\[
\rho^* = 0: \quad \frac{dF^*}{d\rho^*} = \ell^* \left( \frac{d^2 F^*}{d\rho^*} \right)^{\frac{3}{2}}; \quad F = 0, \quad \text{and} \quad \rho^* \to \infty: \quad \frac{dF^*}{d\rho^*} \to \alpha.
\]

We also have

\[
\frac{v_y}{v_x} = \frac{1}{\alpha} \frac{dF^*}{d\rho^*}, \quad \exp \left( \frac{m}{2} x \right) \frac{v_y}{v_x} = -\frac{m}{2\alpha} \left( \rho^* \frac{dF^*}{d\rho^*} + F^* \right).
\]

As before, the parameter \( \alpha \) may be removed from the ordinary differential equation and boundary conditions by suitable transformations depending on the sign of \( \alpha \). For \( \alpha < 0 \) substituting the following transformations

\[
F^* = -\sqrt{-\alpha} F, \quad \rho^* = \frac{\rho}{\sqrt{-\alpha}}, \quad \ell^* = \frac{\ell}{(-\alpha)^{\frac{3\alpha}{2} - 1}};
\]

gives
Fig. 12. Velocity profiles corresponding to Eq. (6.16) for $\ell = 0.5$ and various values of $m$.

Fig. 13. Velocity profiles corresponding to Eq. (6.16) for $\ell = 0.5$ and various values of $m$ with $\alpha = 1$.

\[
\frac{d^3 F}{d\rho^3} - \left(\frac{m}{2}\right) \frac{d}{d\rho} \left( F \frac{dF}{d\rho} \right) + \left(\frac{3m}{2}\right) \left(\frac{dF}{d\rho}\right)^2 - m = 0,
\]  
which must be solved subject to

\[
\rho = 0: \quad \left|\frac{dF}{d\rho}\right| = \ell \left(\frac{d^2 F}{d\rho^2}\right)^{\frac{3}{2}}; \quad F = 0, \quad \text{and} \quad \rho \to \infty: \quad \frac{dF}{d\rho} \to 1.
\]  

We also have

\[
\frac{v_x}{\bar{v}_x} = \frac{dF}{d\rho}, \quad \sqrt{-\alpha} \exp\left(\frac{m}{2} x\right) \frac{v_y}{\bar{v}_x} = m \left(\frac{\rho}{d\rho} + F\right).
\]
For $\alpha > 0$ substituting the following transformations

$$F^* = \sqrt{\alpha} F, \quad \rho^* = \frac{\rho}{\sqrt{\alpha}}, \quad \ell^* = \frac{\ell}{\alpha^{\frac{3m}{2}} - 1},$$

(6.13)
yields

$$\frac{d^3 F}{d\rho^3} + \left( \frac{m}{2} \right) \frac{d}{d\rho} \left( F \frac{dF}{d\rho} \right) - \left( \frac{3m}{2} \right) \left( \frac{dF}{d\rho} \right)^2 + m = 0,$$

(6.14)
which must be solved subject to
Fig. 16. Blow-up of Fig. 15 near the origin for various values of \( \ell \) with \( \alpha = 1 \).

\[
\rho = 0: \quad \left| \frac{dF}{d\rho} \right| = \ell \left( \left| \frac{d^2F}{d\rho^2} \right| \right)^{\frac{3}{2}}; \quad F = 0, \quad \text{and} \quad \rho \to \infty: \quad \frac{dF}{d\rho} \to 1. \quad (6.15)
\]

We also have

\[
\frac{v_x}{\bar{v}_x} = \frac{dF}{d\rho}, \quad \sqrt{\alpha} \exp\left( \frac{m}{2} x \right) \frac{v_y}{\bar{v}_x} = -\frac{m}{2} \left( \rho \frac{dF}{d\rho} + F \right). \quad (6.16)
\]

It may be noticed that the results for \( \alpha > 0 \) for \( m > 0 \) are identical for \( \alpha < 0 \) and \( m < 0 \), so that we only consider the case \( \alpha > 0, m > 0 \). Equation (6.14) with the boundary conditions given by Eq. (6.15) are solved numerically for the cases \( m = 1, 2, 3 \) and \( 4 \) and \( \ell = 0, 0.1, 0.5 \) and \( 1 \) and \( n = 2 \). The numerical method used is similar to that given in Appendix A. The profiles for the \( x \) and \( y \) components of velocity given by Eq. (6.16) are illustrated in Figs. 12–16.

7. Discussion and concluding remarks

In this study the classical laminar boundary layer equations are solved with the nonlinear Navier boundary condition Eq. (1.3) which contains the arbitrary power index \( n \). Solutions are obtained via classical Lie point symmetry reductions. Analytical solutions are obtained for the cases \( n = 1/2 \) and \( n = 1/3 \), while numerical solutions are obtained for \( n = 2/3 \) and \( n > 1 \).

For \( n = 1/2 \) the analytical solution demonstrates that as the slip length increases the rate of change of the tangential velocity through the boundary layer decreases, since the velocity at the solid surface is no longer zero and slips with a velocity which increases as \( \ell \) increases. However, there is very little effect on the normal velocity through the boundary layer. For \( n > 1 \) only numerical solutions are possible, and it is found that for fixed \( \ell \) the velocity components are reduced in magnitude as \( n \) increases. For fixed \( n \) the velocity components are increased in magnitude as the slip length is increased, which is also to be expected.

For \( n = 1/3 \) another analytical solution is obtained, which predicts a velocity excess and the limit at infinity is approached from above instead of from below. Solutions in this regime have only been found numerically to date, and although the analytical solution obtained is not unique,
to the authors' knowledge, it is the only exact analytical closed form solution of the boundary layer equations, besides the case of the flow in a convergent channel discussed earlier, and first obtained for no-slip by Pohlhausen [17].

For the special case $n = 2/3$, which corresponds to an inviscid flow with $\tilde{v}_x \sim e^{mx}$, the results demonstrate that for fixed $\ell$ the velocity components are increased in magnitude as $m$ increases, and for fixed $m$ the velocity components are increased in magnitude as the slip length increases.

Somewhat ironically, this analysis demonstrates that there is no similarity solution for either the standard Navier boundary condition (that is, $n = 1$), which implies that there is no solution for stagnation point flow (that is, $m = 1 \Rightarrow n = 1$), or for the flow past a flat plate (that is, $m = 0 \Rightarrow n = 0$). This could possibly be an indication that the scaling arguments used to obtain the boundary layer equations may be altered for a boundary layer involving tangential fluid slip. Further studies are required to ascertain whether the traditional boundary layer scaling arguments are in fact completely valid in micro and nano scale systems.

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Appendix A. Numerical method

Here we describe the method used to obtain the numerical solutions implemented in MAPLE. The ordinary differential equation to be solved is

$$\frac{d^3 F}{d\rho^3} + \left(\frac{2n-1}{3n-2}\right) F \frac{d^2 F}{d\rho^2} - \left(\frac{n}{3n-2}\right) \left[\left(\frac{dF}{d\rho}\right)^2 - 1\right] = 0,$$

subject to

$$\rho = 0: \left|\frac{dF}{d\rho}\right| = \ell \left|\left(\frac{d^2 F}{d\rho^2}\right)\right|^n; \quad F = 0, \quad \text{and} \quad \rho \rightarrow \infty: \frac{dF}{d\rho} \rightarrow 1.$$

We now outline how this is solved in MAPLE below for the case $n = 2$ and $\ell = 1$.

```maple
> restart;
> n := 2;
> L := 1;
> rho[inf] := 10;

Here we have defined the values of $n$ and $\ell \equiv L$, as well as a new variable $\rho_{\text{inf}}$ whose value is chosen to approximate the limit $\rho \rightarrow \infty$. We now enter the ordinary differential equations and boundary conditions.

```
Now, although it appears we have four boundary conditions for a third order ordinary differential equation, the inclusion of the as-yet unknown parameter $\lambda$ necessitates an extra boundary condition. The reason for this is that the nonlinear Navier boundary condition has been split into two boundary conditions with $\lambda$ acting as a ‘separation constant’ whose value is to be determined. The $\lambda^2$ and $\lambda^{2n}$ take care of the absolute values in the nonlinear Navier boundary condition. Finally, the parameter $\beta$ is a ‘continuation parameter’ which allows the solution of a boundary value problem via a continuous transformation from an easier problem, corresponding to $\beta = 0$, to the actual problem to be solved, corresponding to $\beta = 1$. In essence, the solution corresponding to $\beta = 0$ is used as an initial solution profile for $\beta = 1$. We are now in a position to solve the ordinary differential equation and boundary conditions using MAPLE’s internal boundary value problem numerical routines.

The argument $\text{continuation} = \beta$ instructs MAPLE’s numerical routines to regard $\beta$ as the ‘continuation parameter’. The return is a procedure that may be used to obtain the solution values if given the value of the independent variable. For example

$$
\left[ \begin{array}{c} \rho = 0., \ f(\rho) = 0., \ \frac{d}{d\rho} f(\rho) = 0.390219150265731674, \\ d^2 f(\rho)/d\rho^2 = 0.624675246532377048, \quad \lambda = 0.790363999109142790 \end{array} \right].$$

$$
\left[ \begin{array}{c} \rho = 5., \ f(\rho) = 4.53000141173967030, \ \frac{d}{d\rho} f(\rho) = 0.999993753925616624, \\ d^2 f(\rho)/d\rho^2 = 0.0000239379206308468368, \quad \lambda = 0.790363999109142346 \end{array} \right].
$$

Note that the value of the parameter $\lambda$ is also returned, which (as expected) has a constant value taking into account rounding errors. This solution method was first used for the case $n = 1/2$ and compared to the analytical solution obtained, which showed complete agreement. The solution method for $n = 2/3$ is analogous to the above method, with an altered ordinary differential equation.

References