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Deformation of a smooth Deligne–Mumford stack via differential graded Lie algebra

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ABSTRACT

For a smooth Deligne–Mumford stack over \mathbb{C} , we define its associated Kodaira–Spencer differential graded Lie algebra and show that the deformation functor of the stack is isomorphic to the deformation functor of the Kodaira–Spencer algebra if the stack is proper over \mathbb{C} .

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Introduction

Grothendieck and his followers established a general method to deal with the *deformation theory*, which was initiated by Kodaira and Spencer. Grothendieck's method can be applied to the deformation of almost every algebro-geometric (or analytico-geometric) object.

Recently, many people believe that a deformation theory over a field of characteristic 0 should be 'controlled' by a *differential graded Lie algebra* (DGLA in short). This principle seems to have come from the researches concerning homotopy theory, quantization, mirror symmetry, etc. (see, for example, [K]).

One prototype example to this principle is the deformation theory of compact complex manifold via Maurer–Cartan equation on the vector field valued $(0, 1)$ -forms. This is the *Newlander–Nirenberg theorem* (or rather Kuranishi's proof of the existence of Kuranishi space). If we restrict to infinitesimal deformations, we can describe the situation as a bijection between

$$\underbrace{\left\{ \text{Maurer–Cartan solutions in } KS_X^1 \otimes m_A \right\}}_{\text{gauge equivalence}} \cong \underbrace{\left\{ \begin{array}{l} \text{deformations of a compact} \\ \text{complex manifold } X \text{ over } A \end{array} \right\}}_{\text{isomorphisms}} \quad (1)$$

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where A is a local artinian \mathbb{C} -algebra and $KS_X^\bullet = (A_X^{0,\bullet}(\Theta_X), \bar{\delta}, [-, -])$ the Kodaira–Spencer algebra on X (see Theorem 1.10). This isomorphism is functorial in A . The left-hand side is the deformation functor associated to the Kodaira–Spencer DGLA KS_X^\bullet , denoted by Def_{KS_X} , and the right-hand side is the usual deformation functor Def_X of X .

Although the correspondence (1) was originally based on highly analytic arguments by Newlander–Nirenberg, the statement itself concerns only infinitesimal deformations, therefore it is algebraic in nature. Actually, Iacono [Ia] recently gave a purely algebraic proof (i.e., it involves no analysis of differential equations).

With a view toward this situation, it is quite reasonable to expect that the correspondence (1) can be generalized to the case of smooth Deligne–Mumford stacks. We can get some flavor from the case where \mathcal{X} is given by a global quotient $[X/G]$ of a proper smooth algebraic variety X by an action of a finite group G . Giving a deformation of $\mathcal{X} = [X/G]$ should be equivalent to giving a deformation of X on which the G -action lifts. Therefore, the deformations of \mathcal{X} are given by the G -invariant part $\text{Def}_X(A)^G$ of the deformations of X . Since the correspondence (1) is G -equivariant, if we take $(KS_X^\bullet)^G$ as the DGLA, we get a functorial bijection

$$\text{Def}_{(KS_X)^\bullet}^G(A) \xrightarrow{\sim} \text{Def}_X(A)^G,$$

which describes the infinitesimal deformations of the stack $\mathcal{X} = [X/G]$ via a DGLA $(KS_X^\bullet)^G$.

In this article, we prove the following theorem:

Main Theorem (Theorem 4.4). *Let \mathcal{X} be a smooth separated analytic Deligne–Mumford stack. Then we can associate the Kodaira–Spencer differential graded Lie algebra $KS_{\mathcal{X}}^\bullet$ and there is a natural isomorphism of deformation functors*

$$\Gamma : \text{Def}_{KS_{\mathcal{X}}^\bullet} \rightarrow \text{Def}_{\mathcal{X}}.$$

By a standard GAGA type argument (Proposition 2.6), we also have a corresponding statement for a proper smooth (algebraic) Deligne–Mumford stack over \mathbb{C} .

Corollary. *Let X be a proper smooth Deligne–Mumford stack over \mathbb{C} . Then we also have the isomorphism $\text{Def}_{KS_X^\bullet} \xrightarrow{\sim} \text{Def}_X$, where the right-hand side is the deformation functor of algebraic deformations of \mathcal{X} .*

One obvious application of our main theorem is the deformation of a proper algebraic variety with only isolated quotient singularities, because the isolated quotient singularity is rigid if the dimension is not less than three [Sch2].

Corollary. *Let X be a proper algebraic variety with only isolated quotient singularities over \mathbb{C} of dimension not less than three. Then the deformation functor of X is isomorphic to the deformation functor of the Kodaira–Spencer algebra on the canonical covering stack $\mathcal{X} \rightarrow X$.*

More generally, our main theorem describes the locus of equisingular deformations in the deformation space of a proper normal variety with only quotient singularities, i.e., a complex V-manifold.

Our proof of the main theorem is parallel to the proof in [Ia]. We mention some reasons why one can transplant the proof to the case of DM stacks. One reason is, of course, the algebraic nature of the arguments of the proof in [Ia]. Another is the recent development of the deformation theory on algebraic stacks due to Aoki and Olsson [AO1]. In particular, it is crucial for our argument that Aoki’s theorem on the equivalence of the deformation functor of an algebraic stack and the deformation functor of the simplicial space associated to the stack.

The article goes as follows. In Section 1, we review some general results on deformation functors and DGLA fixing our notation. The next section treats the deformation theory of stacks. We review Aoki’s result and deduce some consequences in the case of DM stacks. We also treat GAGA type arguments for DM stacks as far as we need. In Section 3, we define the Kodaira–Spencer algebra on a

smooth DM stack and prove a Dolbeault type theorem, which is essentially due to Behrend [B]. After these preparations, we verify that the proof in [Ia] works completely in the case of our main theorem.

1. Deformation theory and differential graded Lie algebras

In this section, we review the theory of deformation functors of differential graded Lie algebras and its application to the deformation theory of a compact complex manifold. The references are [M] and [Ia, Chapter I]. For Section 1.3, see also [F-M].

1.1. Let **Art** be the category of local artinian \mathbb{C} -algebra A such that $A/m_A \cong \mathbb{C}$, where m_A is the maximal ideal of A . We mean by a *functor of artinian rings* [Sch1] a covariant functor

$$D : \mathbf{Art} \rightarrow \mathbf{Set}$$

such that $D(\mathbb{C})$ is the one-point set. The *tangent space* t_D to a functor of artinian rings D is defined by

$$t_D = D(\mathbb{C}[\varepsilon]),$$

where $\mathbb{C}[\varepsilon]$ is the ring of dual numbers $\mathbb{C}[x]/(x^2)$.

1.2. Let A, B, C be local artinian \mathbb{C} -algebras and

$$\eta : D(B \times_A C) \rightarrow D(B) \times_{D(A)} D(C)$$

be the natural map. We call a functor of artinian rings D a *deformation functor* if it satisfies (i) if $B \rightarrow A$ is surjective, so is η , and (ii) If $A = \mathbb{C}$, η is bijective [M, Definition 2.5]. We remark that these conditions are closely related to Schlessinger’s criterion of existence of a *hull* (see Remark to Definition 2.7 in [F-M]).

1.3. Most deformation functors are described (in an implicit way) by obstruction classes to the existence of a lifting of a small extension and the space which parametrizes the isomorphism classes of liftings in case the obstruction vanishes. Fantechi and Manetti abstracted these “obstruction theories” in the framework of functors of artinian rings ([F-M, Definition 3.1], see also [M, Definition 2.12]). An *obstruction theory* of a functor of artinian rings D is a pair $(V, \text{ob}(-))$ consisting of a \mathbb{C} -vector space V , the *obstruction space*, and a map $\text{ob}(\alpha) : D(\bar{A}) \rightarrow V \otimes I$, the *obstruction map*, for every small extension

$$\alpha: 0 \rightarrow I \rightarrow A \rightarrow \bar{A} \rightarrow 0,$$

i.e., an extension with $I \cdot m_A = 0$, satisfying the following conditions:

1. If $\bar{x} \in D(\bar{A})$ lifts to $D(A)$, $\text{ob}(\alpha)(\bar{x}) = 0$.
2. For any morphism φ of small extensions

$$\begin{array}{ccccccccc} \alpha_1: & 0 & \longrightarrow & I_1 & \longrightarrow & A_1 & \longrightarrow & \bar{A}_1 & \longrightarrow & 0 \\ & & & \varphi_1 \downarrow & & \varphi \downarrow & & \bar{\varphi} \downarrow & & \\ \alpha_2: & 0 & \longrightarrow & I_2 & \longrightarrow & A_2 & \longrightarrow & \bar{A}_2 & \longrightarrow & 0, \end{array}$$

we have the compatibility $\text{ob}(\alpha_2)(\bar{\varphi}_*(\bar{x})) = (\text{id}_V \otimes \varphi_1)(\text{ob}(\alpha_1)(\bar{x}))$ for every $\bar{x} \in D(\bar{A}_1)$.

Moreover, if $\text{ob}(\alpha)(\bar{x}) = 0$ implies the existence of a lifting of \bar{x} to $D(A)$, the obstruction theory is called *complete*.

Proposition 1.4. (See [M, Proposition 2.17].) *Let D_1 and D_2 be deformation functors and $\varphi : D_1 \rightarrow D_2$ a morphism of functors, (V_1, ob_{D_1}) and (V_2, ob_{D_2}) obstruction theories for D_1 and D_2 , respectively. Assume that*

- (i) φ induces a surjection (resp. bijection) on the tangent spaces $t_{D_1} \rightarrow t_{D_2}$.
- (ii) There is an injective linear map between obstruction spaces $\psi : V_1 \rightarrow V_2$ such that $\text{ob}_{D_2} \circ \varphi = \psi \circ \text{ob}_{D_1}$.
- (iii) The obstruction theory (V_1, ob_{D_1}) is complete.

Then, the morphism φ is smooth (resp. étale).

1.5. Now we recall the definition of a differential graded Lie algebra. Let $L^\bullet = \bigoplus_{i \in \mathbb{Z}} L_i$ be a graded \mathbb{C} -vector space. A triple $(L^\bullet, d, [-, -])$ is a *differential graded Lie algebra* (DGLA in short) if it satisfies (i) $d = \sum d_i$ is a homogeneous differential of degree 1, i.e., $d_i : L^i \rightarrow L^{i+1}$ and $d \circ d = 0$, (ii) the bracket $[-, -]$ is a homogeneous, graded skew-symmetric bilinear form on L^\bullet , i.e., $[L^i, L^j] \subset L^{i+j}$ and $[a, b] + (-1)^{\text{deg } a \cdot \text{deg } b} [b, a] = 0$, (iii) the graded Jacobi identity $[a, [b, c]] = [[a, b], c] + (-1)^{\text{deg } a \cdot \text{deg } b} [b, [a, c]]$ holds for the bracket, and (iv) the graded Leibniz rule $d[a, b] = [da, b] + (-1)^{\text{deg } a} [a, db]$ holds.

1.6. Given a DGLA L^\bullet , we can associate its Maurer–Cartan functor as follows. Let L^\bullet be a DGLA and A a local artinian \mathbb{C} -algebra. We define the *Maurer–Cartan functor* $MC_L : \mathbf{Art} \rightarrow \mathbf{Set}$ associated to L^\bullet by

$$MC_L(A) = \left\{ x \in L^1 \otimes m_A \mid dx + \frac{1}{2}[x, x] = 0 \right\},$$

where d and $[-, -]$ is the DGLA structure on $L^\bullet \otimes m_A$ induced in an obvious way by L^\bullet . We call an element of $MC_L(A)$ a *Maurer–Cartan solution*.

1.7. In some cases, the space of Maurer–Cartan solutions $MC_L(A)$ is ‘too big.’ In such a case, we get a more reasonable deformation functor considering the gauge action. Let L^\bullet be a DGLA and $a \in L^0 \otimes m_A$ where A is a local artinian \mathbb{C} -algebra. For $x \in L^1 \otimes m_A$, the *gauge action* of a is given by

$$e^a * x = x + \sum_{n=0}^{\infty} \frac{[a, -]^n}{(n+1)!} ([a, x] - da),$$

where $[a, -]^n(y)$ is the operator $[a, -]$ applied to y n -times recursively: $[a, [a, [\dots, [a, y] \dots]]]$. Note that the gauge action preserves the space of the Maurer–Cartan solutions $MC_L(A)$. Two Maurer–Cartan solutions $x, y \in MC_L(A)$ are said to be *gauge equivalent* if there exists $a \in L^0 \otimes m_A$ such that $y = e^a * x$. We define the *deformation functor* $\text{Def}_L : \mathbf{Art} \rightarrow \mathbf{Set}$ of a DGLA L^\bullet by $\text{Def}_L(A) = MC_L(A)/(\text{gauge equivalence})$. One can easily check the following facts (see, for example, [Ia, §1.3.5]):

- (i) The functor Def_L is a deformation functor in the sense of Definition 1.2.
- (ii) The tangent space to the deformation functor Def_L is canonically isomorphic to $H^1(L^\bullet, d)$.
- (iii) There is a natural complete obstruction theory of Def_L with the obstruction space $H^2(L^\bullet, d)$.

1.8. One of the DGLA’s which appear naturally in a geometric context is the Kodaira–Spencer algebra. Let X be a complex manifold and $KS_X^p = A_X^{0,p}(\Theta_X)$ be the space of C^∞ -differential forms of type $(0, p)$ with (holomorphic) vector field coefficients. Then, $(KS_X^\bullet, \bar{\partial}, [-, -])$ is a DGLA in a natural way, where $[-, -]$ is a bracket induced by the Lie bracket on Θ_X . We call this DGLA the *Kodaira–Spencer algebra*.

1.9. We can reformulate the classical deformation theory of a compact complex manifold by Kodaira–Spencer, Newlander–Nirenberg and Kuranishi, using the framework of DGLA and its deformation functor for infinitesimal deformations.

Let X be a scheme (resp. an analytic space) and A be a local artinian \mathbb{C} -algebra. A *lifting* of X to A is a pair (\tilde{X}, φ) where \tilde{X} is a scheme (resp. an analytic space) flat over A and φ is an isomorphism $\tilde{X} \times_{\text{Spec } A} \text{Spec } \mathbb{C} \xrightarrow{\sim} X$. Two liftings (\tilde{X}_1, φ_1) and (\tilde{X}_2, φ_2) are isomorphic if there exists an A -isomorphism $\tilde{f} : \tilde{X}_1 \rightarrow \tilde{X}_2$ that is compatible with the marking isomorphisms, i.e. $\varphi_1 = \varphi_2 \circ (\tilde{f} \otimes_A \mathbb{C})$. We define the *deformation functor* $\text{Def}_X : \mathbf{Art} \rightarrow \mathbf{Set}$ of X by $\text{Def}_X(A) = \{\text{isomorphism class of liftings } \tilde{X} \text{ of } X \text{ to } A\}$.

Theorem 1.10. (See [1a, Theorem II.7.3].) *Let X be a complex manifold and KS_X^\bullet the associated Kodaira–Spencer algebra. Then we have an isomorphism $\gamma : \text{Def}_{KS_X} \rightarrow \text{Def}_X$ between the deformation functors.*

2. Deformation of a Deligne–Mumford stack

Recently, Aoki [A] explored the deformation theory of an algebraic stack (in the sense of Artin), whose work depends on the preceding work by Olsson [O1,O2]. In the first half of this section, we review some results from [A].

We can define a deformation functor $\text{Def}_{\mathcal{X}}$ of an algebraic stack \mathcal{X} just as in (1.9) for an algebraic stack \mathcal{X} , namely, for a local artinian \mathbb{C} -algebra A , we define $\text{Def}_{\mathcal{X}}(A)$ to be the set of isomorphism classes of liftings $\tilde{\mathcal{X}}$ of \mathcal{X} to A (here we note that we will only consider 1-isomorphism classes of liftings, so that our deformation functor is coarser than the one in [A, Definition 1.1]).

We will write groupoid space for internal groupoid in the category of algebraic spaces [A, Definition 2.1.2]. To an algebraic stack \mathcal{X} , we can associate a choice of a smooth atlas $U \rightarrow \mathcal{X}$ and a groupoid space

$$T = R_s \times_U R \xrightarrow{m} R \begin{matrix} \xrightarrow{s} \\ \xleftarrow{t} \\ \xrightarrow{e} \end{matrix} U \quad (2)$$

(Note: The diagram shows a curved arrow labeled 'i' from R to R, and a curved arrow labeled 's' from R to U.)

such that s and t are smooth and $(s, t) : R \rightarrow U \times U$ is quasi-compact and separated. Conversely, if we are given a groupoid space $R \rightrightarrows U$ with these properties, we can recover the algebraic stack \mathcal{X} .

For a groupoid space $R \rightrightarrows U$, we can naturally define its associated deformation functor $\text{Def}_{(R \rightrightarrows U)} : \mathbf{Art} \rightarrow \mathbf{Set}$ in an obvious way. Aoki [A] showed that the correspondence between algebraic stacks and groupoid spaces induces an isomorphism of these deformation functors.

Theorem 2.1. (See [A, Proposition 3.2.5].) *Let X be an algebraic stack and $R \rightrightarrows U$ be an associated groupoid space. Then there is a natural isomorphism of functors*

$$C : \text{Def}_{(R \rightrightarrows U)} \rightarrow \text{Def}_{\mathcal{X}} .$$

As a corollary, we get the following results.

Corollary 2.2. (See [A, Corollary 3.2.6].) *Let $A \rightarrow \bar{A}$ be an extension with square zero ideal $I = \text{Ker}(A \rightarrow \bar{A})$ and \mathcal{X} be an algebraic stack flat over $S = \text{Spec } \bar{A}$. Denote by $p : X_\bullet \rightarrow S$ the associated simplicial space over S to a groupoid representation $X_0 = U \rightarrow \mathcal{X}$. Then, (i) there is an obstruction class $\omega \in \text{Ext}^2(L_{X_\bullet/S}, p^*I)$ to the existence of a lifting of \mathcal{X} to A , and (ii) if ω vanishes, the set of isomorphism classes of liftings of \mathcal{X} to A is a torsor under $\text{Ext}^1(L_{X_\bullet/S}, p^*I)$, where $L_{X_\bullet/S}$ is the cotangent complex associated to the ringed étale topoi on X_\bullet [11] and Ext groups are also computed on the étale site.*

Corollary 2.3. (See [A, §4.2].) *The functor $\text{Def}_{\mathcal{X}}$, and therefore $\text{Def}_{(R \rightrightarrows U)}$, is a deformation functor in the sense of (1.2).*

If we restrict to a Deligne–Mumford stack (DM stack in short), we can take an étale atlas $U \rightarrow \mathcal{X}$ so that all the projections of the simplicial scheme X_\bullet are étale. The transitivity of the cotangent complex [II, II.2.1.5.6] and the vanishing of the cotangent complex for étale morphisms [II, Proposition III.3.1.1] imply that $L_{X_\bullet/S}$ descends to the cotangent complex $L_{\mathcal{X}/S}$ on the étale site of \mathcal{X} . This implies the following corollary:

Corollary 2.4. *For a DM stack \mathcal{X} , the liftings of $f : \mathcal{X} \rightarrow S$ with square zero ideal I are controlled by the Ext groups on the étale site $\text{Ext}^i(L_{\mathcal{X}/S}, f^*I)$, ($i = 1, 2$). In particular, the deformation functor $\text{Def}_{\mathcal{X}}$ is isomorphic to the deformation functor of the structure sheaf $\mathcal{O}_{\mathcal{X}}$, i.e., the functor*

$$\text{Def}_{\mathcal{O}_{\mathcal{X}}}(A) = \left\{ \begin{array}{l} \text{isomorphism class of a sheaf of algebras } \mathcal{S} \text{ flat over } A \\ \text{on the étale site over } \mathcal{X} \text{ such that } \mathcal{S} \otimes_A \mathbb{C} \cong \mathcal{O}_{\mathcal{X}} \end{array} \right\}.$$

In the rest of this section, we review some comparison results of GAGA type as much as we need.

In the definition of algebraic stacks, if we replace the category of schemes by the category of analytic spaces, we get the concept of *analytic stacks* in Artin’s sense, or Deligne–Mumford’s sense. Given an algebraic stack \mathcal{X} , we have the associated analytic stack \mathcal{X}_{an} ; one way to see this is to take a groupoid $R \rightrightarrows U$ representing \mathcal{X} and take the associated groupoid of analytic spaces $R_{an} \rightrightarrows U_{an}$, which induces \mathcal{X}_{an} . Similarly, we have the concept of liftings of an analytic stack to a local artinian ring and if we have a lifting $\tilde{\mathcal{X}}$ of an algebraic stack \mathcal{X} to A , we have the corresponding lifting of analytic stack $\tilde{\mathcal{X}}_{an}$ of \mathcal{X}_{an} to A , i.e., we have a natural transformation

$$\alpha : \text{Def}_{\mathcal{X}} \rightarrow \text{Def}_{\mathcal{X}_{an}}.$$

Note that the proof of Theorem 2.1 holds true for analytic stacks by the same proof as in [A,II,O1]; therefore, Corollary 2.4 also holds true for analytic DM stacks.

Proposition 2.5. *Let \mathcal{X} be a DM stack proper over an affine \mathbb{C} -scheme S and \mathcal{F} be a coherent sheaf on \mathcal{X} . Then we have a natural isomorphism*

$$H^p(\mathcal{X}, \mathcal{F}) \cong H^p(\mathcal{X}_{an}, \mathcal{F}_{an}).$$

Proof. This is standard. Chow’s Lemma for DM stacks [L-M, Corollaire 16.6.1] and the dévissage technique of Grothendieck ([EGA3, §3], or [G]) reduces the proof to the classical case [Se]. \square

Proposition 2.6. *The natural transformation $\alpha : \text{Def}_{\mathcal{X}} \rightarrow \text{Def}_{\mathcal{X}_{an}}$ is an isomorphism if \mathcal{X} is a smooth DM stack proper over \mathbb{C} .*

Proof. As we assumed that \mathcal{X} is smooth, the infinitesimal deformation space and the obstruction space in Corollary 2.4 are given by $H^i(\mathcal{X}, \Theta_{\mathcal{X}})$ ($i = 1, 2$). We also have the same statement for \mathcal{X}_{an} . If we apply Proposition 2.5 for $\mathcal{F} = \Theta_{\mathcal{X}}$, we get our proposition. \square

We remark that as a corollary of the proposition, we get an isomorphism of the deformation functors $\text{Def}_{(R \rightrightarrows U)} \cong \text{Def}_{(R_{an} \rightrightarrows U_{an})}$. The obstruction theory in the proof of the proposition above is in fact a complete obstruction theory in the sense of Definition 1.3.

Proposition 2.7. *Let \mathcal{X} be a smooth (algebraic or analytic) DM stack and $\text{Def}_{\mathcal{X}}$ its deformation functor. There is a natural complete obstruction theory for $\text{Def}_{\mathcal{X}}$ with obstruction space $H^2(\mathcal{X}, \Theta_{\mathcal{X}})$. If \mathcal{X} is proper over \mathbb{C} , the obstruction theory is compatible with the operation of taking the associated analytic stack.*

Proof. This is also implied by [A, Corollary 3.2.6]. The compatibility with base change (ii) in Definition 1.3 goes back to the definition of the obstruction class in [II, Théorème 2.1.7]. \square

3. Kodaira–Spencer algebra associated to a smooth DM stack

The observation in the introduction suggests that the deformation theory of a smooth Deligne–Mumford stack should also be controlled by a DGLA something like the Kodaira–Spencer algebra, and such a DGLA should be realized as a DGL sub-algebra of the Kodaira–Spencer algebra of an atlas of the stack.

For a Deligne–Mumford stack \mathcal{X} , the cotangent sheaf $\Omega_{\mathcal{X}}$ is a well-defined $\mathcal{O}_{\mathcal{X}}$ -coherent sheaf and it is locally free if \mathcal{X} is smooth. Therefore, the tangent sheaf $\Theta_{\mathcal{X}}$ is also a locally free sheaf for a smooth \mathcal{X} . Similarly, on the étale site over \mathcal{X}_{an} , we have the sheaf of C^∞ -differentials $\mathcal{A}_{\mathcal{X}_{an}}^{p,q}$.

In the rest of the article, we will work in the analytic category. Moreover, we always assume that a DM stack is locally compact and second-countable, i.e., we assume that every analytic space in the étale site of a DM stack (in particular an atlas) is locally compact and second-countable.

Now take an étale atlas $U \rightarrow \mathcal{X}$. Let $R = U \times_{\mathcal{X}} U$ be the “space of relations” and $s, t : R \rightarrow U$ the first and second projections, respectively. Since U is a smooth space, we have the associated Kodaira–Spencer algebra $KS_U^\bullet = A_U^{0,p}(\Theta_U)$. The étale morphism $s : R \rightarrow U$ induces a map $s^* : A_U^{0,p}(\Theta_U) \rightarrow A_R^{0,p}(\Theta_R)$ and the same holds for t .

Proposition–Definition 3.1. *Let \mathcal{X} be a smooth DM stack. Define $KS_{\mathcal{X}}^\bullet$ by*

$$KS_{\mathcal{X}}^p = \{x \in A_U^{0,p}(\Theta_U) \mid s^*x = t^*x\} \subset A_U^{0,p}(\Theta_U).$$

Then,

- (i) $KS_{\mathcal{X}}^p$ does not depend on the choice of an atlas $U \rightarrow \mathcal{X}$. More precisely, $KS_{\mathcal{X}}^p$ is the space of global sections $\Gamma(\mathcal{X}, \mathcal{A}_{\mathcal{X}}^{0,p}(\Theta_{\mathcal{X}}))$.
- (ii) $KS_{\mathcal{X}}^\bullet = \bigoplus_p KS_{\mathcal{X}}^p$ is a differential graded Lie sub-algebra of KS_U^\bullet .

We call $KS_{\mathcal{X}}$ the Kodaira–Spencer algebra of the DM stack \mathcal{X} .

Proof. (i) is nothing but the descent property of $\mathcal{A}_{\mathcal{X}}^{0,p}(\Theta_{\mathcal{X}})$ on the analytic-étale site on \mathcal{X} . For (ii), it is enough to show that $KS_{\mathcal{X}}^\bullet$ is closed under $\bar{\partial}$ and the bracket $[-, -]$ of KS_U^\bullet . This is equivalent to saying that $\bar{\partial}$ and $[-, -]$ commute with s^* and t^* . But this is obvious because s and t are étale. \square

Example 3.2. For a global quotient DM stack $[X/G]$, we can take $X \rightarrow [X/G]$ as an atlas. The proposition immediately implies that $KS_{[X/G]}^\bullet = (KS_X^\bullet)^G$.

Given the Kodaira–Spencer algebra $KS_{\mathcal{X}}^\bullet$ for a smooth DM stack, we can, of course, consider its deformation functor $\text{Def}_{KS_{\mathcal{X}}}$. Its tangent space and complete obstruction space (1.7) can be computed by the following Dolbeault type theorem.

Theorem 3.3. *Let \mathcal{X} be a smooth separated DM stack over \mathbb{C} . Then there is an isomorphism $H^p(KS_{\mathcal{X}}^\bullet, \bar{\partial}) \xrightarrow{\sim} H^p(\mathcal{X}, \Theta_{\mathcal{X}})$ for all p .*

Proof. Take an étale atlas $U \rightarrow \mathcal{X}$ with U smooth and Stein and consider the associated Čech–Dolbeault complex as usual:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Gamma(U_p, \Theta) & \longrightarrow & \Gamma(U_p, \mathcal{A}^{0,0}(\Theta)) & \longrightarrow & \cdots \longrightarrow \Gamma(U_p, A^{0,q}(\Theta)) \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \Gamma(U_1, \Theta) & \longrightarrow & \Gamma(U_1, \mathcal{A}^{0,0}(\Theta)) & \longrightarrow & \cdots \longrightarrow \Gamma(U_1, A^{0,q}(\Theta)) \longrightarrow \cdots \\
 & & \uparrow t^* - s^* & & \uparrow t^* - s^* & & \uparrow t^* - s^* \\
 0 & \longrightarrow & \Gamma(U_0, \Theta) & \longrightarrow & \Gamma(U_0, \mathcal{A}^{0,0}(\Theta)) & \longrightarrow & \cdots \longrightarrow \Gamma(U_0, A^{0,q}(\Theta)) \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & \longrightarrow & KS_{\mathcal{X}}^0 & \longrightarrow & \cdots \longrightarrow KS_{\mathcal{X}}^q \longrightarrow \cdots \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
 \end{array} \tag{3}$$

where U_p is the $(p + 1)$ -fold self fiber product of U over \mathcal{X} . The vanishing of cohomologies of coherent sheaves on a Stein space implies that the complex appeared in the leftmost column calculates $H^p(\mathcal{X}, \Theta_{\mathcal{X}})$. The Dolbeault theorem on a usual complex manifold U_p implies that the rows but the bottom one is exact. On the other hand, Behrend [B] showed that if we replace U by its ‘refinement,’ we have a partition of unity associated with U [B, Definition 22], and this immediately implies that the columns except the leftmost one are exact [B, Proposition 23]. A standard double complex argument leads to our theorem. \square

4. Infinitesimal Newlander–Nirenberg theorem for smooth DM stack

In this section, we complete the proof of our main theorem. After the preparations in Sections 2 and 3, our proof is an honest transplantation of the proof of Theorem 1.10 found in [Ia, Chapter II] to the context of smooth DM stacks. We begin with the following theorem.

Theorem 4.1. *Let \mathcal{X} be a smooth separated analytic Deligne–Mumford stack and $KS_{\mathcal{X}}^{\bullet}$ be the associated Kodaira–Spencer algebra. Then, there exists a natural injective morphism between the deformation functors*

$$\Gamma : \text{Def}_{KS_{\mathcal{X}}} \rightarrow \text{Def}_{\mathcal{X}}$$

in the analytic category.

Proposition 4.2. *Let A be a local artinian \mathbb{C} -algebra with the maximal ideal m_A and $x \in MC_{KS_{\mathcal{X}}}(A)$ be a Maurer–Cartan solution of $KS_{\mathcal{X}}$. Then, to x we can associate a lifting $\tilde{\mathcal{X}} \in \text{Def}_{\mathcal{X}}(A)$.*

Proof. Corollary 2.4 implies that it is enough to construct a sheaf of algebras \mathcal{S}_x on \mathcal{X} flat over A such that $\mathcal{S}_x \otimes_A \mathbb{C} \cong \mathcal{O}_{\mathcal{X}}$. Take a smooth Stein space U as an atlas $U \rightarrow \mathcal{X}$. The tangent space to the deformation functor Def_{KS_U} associated to the Kodaira–Spencer algebra KS_U^{\bullet} of U is isomorphic to $H^1(U, \Theta_U)$, which is in fact trivial, for U is smooth and Stein. This means that Def_{KS_U} is trivial [Ia, Lemma II.7.1]. In other words, for any Maurer–Cartan solution $x \in MC_{KS_{\mathcal{X}}}(A) \subset MC_{KS_U}(A)$, there is a C^{∞} -vector field $a \in A_U^{0,0}(\Theta_U) \otimes m_A$ such that $e^a * x = 0$. We define the operator $\mathbf{I}_x : \mathcal{A}_U^{0,0} \otimes A \rightarrow \mathcal{A}_U^{0,1} \otimes A$ for $x = \sum_{i,j} x_{ij} d\bar{z}_i \frac{\partial}{\partial z_j}$ by the contraction

$$\mathbf{I}_x(f) = - \sum_{i,j} x_{ij} \frac{\partial f}{\partial z_j} d\bar{z}_i,$$

where z_i are local holomorphic coordinates on U . By an explicit calculation of the gauge action [Ia, Lemma II.5.5], we have

$$e^a \circ (\bar{\partial} + \mathbf{I}_x) \circ e^{-a} = \bar{\partial} + e^a * \mathbf{I}_x = \bar{\partial},$$

where the last equality follows from $e^a * x = 0$. This means that the diagram

$$\begin{array}{ccccc} \mathcal{S}_x = \text{Ker}(\bar{\partial} + \mathbf{I}_x) & \longrightarrow & \mathcal{A}_U^{0,0} \otimes A & \xrightarrow{\bar{\partial} + \mathbf{I}_x} & \mathcal{A}_U^{0,1} \otimes A \\ e^a \downarrow & & e^a \downarrow & & e^a \downarrow \\ \mathcal{O}_{\tilde{U}} & \longrightarrow & \mathcal{A}_U^{0,0} \otimes A & \xrightarrow{\bar{\partial}} & \mathcal{A}_U^{0,1} \otimes A \end{array}$$

is commutative, where $\tilde{U} = U \times \text{Spec } A$. In particular, $e^a : \mathcal{S}_x \xrightarrow{\sim} \mathcal{O}_{\tilde{U}}$ is an isomorphism of sheaf of algebras over A . Since $\mathcal{O}_{\tilde{U}}$ is flat over A , \mathcal{S}_x is also flat over A .

Let $y = s^*x = t^*x \in A_R^{0,1}(\Theta) \otimes m_A$. We have an analogous diagram on R :

$$\begin{array}{ccccc} s^{-1}\mathcal{S}_x = \text{Ker}(\bar{\partial} + \mathbf{I}_y) & \longrightarrow & \mathcal{A}_R^{0,0} \otimes A & \xrightarrow{\bar{\partial} + \mathbf{I}_y} & \mathcal{A}_R^{0,1} \otimes A \\ e^{s^*a} \downarrow & & e^{s^*a} \downarrow & & e^{s^*a} \downarrow \\ s^{-1}\mathcal{O}_{\tilde{U}} \cong \mathcal{O}_{\tilde{R}} & \longrightarrow & \mathcal{A}_R^{0,0} \otimes A & \xrightarrow{\bar{\partial}} & \mathcal{A}_R^{0,1} \otimes A, \end{array}$$

where $\tilde{R} = R \times \text{Spec } A$, and the same diagram also for $t^{-1}\mathcal{S}_x$. Hence we have $s^{-1}\mathcal{S}_x = t^{-1}\mathcal{S}_x$ as sub-sheaves of $\mathcal{A}_R^{0,0} \otimes A$. This means that the descent data for $\mathcal{A}_R^{0,0} \otimes A$ induces a descent data for \mathcal{S}_x . Therefore, \mathcal{S}_x descends to a sheaf of algebras on \mathcal{X} flat over A . $\mathcal{S}_x \otimes_A \mathbb{C} \cong \mathcal{O}_{\mathcal{X}}$ is obvious. \square

The proposition says that we have a morphism $\hat{\Gamma} : MC_{KS_{\mathcal{X}}} \rightarrow \text{Def}_{\mathcal{X}}$. The following proposition concludes the proof of Theorem 4.1.

Proposition 4.3. $\hat{\Gamma}$ descends to an injective morphism

$$\Gamma : \text{Def}_{KS_{\mathcal{X}}} \rightarrow \text{Def}_{\mathcal{X}} .$$

Proof. Assume $x, y \in MC_{KS_{\mathcal{X}}}(A)$ are gauge equivalent, i.e., there exists $r \in KS_{\mathcal{X}}^0 \otimes m_A$ such that $e^r * x = y$. By [Ia, Lemma II.5.5], we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{S}_x = \text{Ker}(\bar{\partial} + \mathbf{I}_x) & \longrightarrow & \mathcal{A}_U^{0,0} \otimes A & \xrightarrow{\bar{\partial} + \mathbf{I}_x} & \mathcal{A}_U^{0,1} \otimes A \\ e^r \downarrow & & e^r \downarrow & & e^r \downarrow \\ \mathcal{S}_y = \text{Ker}(\bar{\partial} + \mathbf{I}_y) & \longrightarrow & \mathcal{A}_U^{0,0} \otimes A & \xrightarrow{\bar{\partial} + \mathbf{I}_y} & \mathcal{A}_U^{0,1} \otimes A \end{array}$$

whose columns are isomorphisms. $r \in KS_{\mathcal{X}}^0 \otimes m_A$ implies $e^{s^*r} = e^{t^*r}$ so that e^r descends to an isomorphism between \mathcal{S}_x and \mathcal{S}_y on \mathcal{X} . This means that $\hat{\Gamma}$ factors through Γ .

To prove the injectivity of Γ , we show that given an isomorphism $\psi : \mathcal{S}_x \rightarrow \mathcal{S}_y$ satisfying $s^*\psi = t^*\psi$, there exists $r \in KS_{\mathcal{X}}^0 \otimes m_A$ such that $\psi = e^r$. It is enough to show this for small extensions by inductive argument. In other words, we can assume that there is an extension

$$0 \rightarrow I \rightarrow A \rightarrow \tilde{A} \rightarrow 0$$

with $I \cdot m_A = 0$ and $p = x - y \in A_U^{0,1}(\mathcal{O}_U) \otimes I$. Under this assumption, we have

$$0 = \bar{\partial}(y + p) + \frac{1}{2}[y + p, y + p] = \bar{\partial}y + \bar{\partial}p + \frac{1}{2}[y, y] = \bar{\partial}p.$$

Since we assumed that U is Stein, $H^1(U, \mathcal{O}_U)$ vanishes. By the (usual) Dolbeault theorem, this means that the Dolbeault complex $(A_U^{0,\bullet}(\mathcal{O}_U), \bar{\partial})$ is exact at the degree 1 place. Therefore, we have $u \in KS_{\mathcal{X}}^0 \otimes I$ such that $\bar{\partial}u = p$. Then, we have

$$e^u * x = x + \sum_{n=0}^{\infty} \frac{[u, -]^n}{(n+1)!}([u, x] - \bar{\partial}u) = x - \bar{\partial}u = x - p = y.$$

Take $a \in A_U^{0,0}(\mathcal{O}_U) \otimes m_A$ which induces an isomorphism $e^a : \mathcal{S}_x \rightarrow \mathcal{O}_{\tilde{U}}$ and φ be an automorphism of $\mathcal{O}_{\tilde{U}}$ making the diagram

$$\begin{CD} \mathcal{S}_x @>\psi>> \mathcal{S}_y @>e^{-u}>> \mathcal{S}_x \\ @VVe^aVV @. @VVe^aV \\ \mathcal{O}_{\tilde{U}} @>\varphi>> \mathcal{O}_{\tilde{U}} \end{CD}$$

commutative. Because $e^{-u} \circ \psi = \text{id} \pmod I$, $\varphi = \text{id} \pmod I$. Therefore, there exists $q \in H^0(U, \mathcal{O}_U) \otimes I$ such that $\varphi = e^q$. Note that e^q commutes with e^a since the coefficients of q are in I . This implies $\psi = e^u \circ e^{-a} \circ e^q \circ e^a = e^{u+q}$. In other words, we have $\psi = e^r$ for $r = u + q \in KS_{\mathcal{X}}^0 \otimes m_A$. \square

Remark. In the argument above, we can construct the lifting of the groupoid representation associated to $U \rightarrow \mathcal{X}$ without appealing to Corollary 2.4. Let $\tilde{R} \rightrightarrows \tilde{U}$ be the trivial lifting to A of the groupoid $R \rightrightarrows U$ representing \mathcal{X} and $q_0 = (s_0, t_0, e_0, m_0, i_0)$ the structural morphisms of $\tilde{R} \rightrightarrows \tilde{U}$. As we assumed U Stein, every lifting of $R \rightrightarrows U$ to A is given only by twisting q_0 . The isomorphism $e^a : \mathcal{S}_x \xrightarrow{\sim} \mathcal{O}_{\tilde{U}}$ appeared in the proof of Proposition 4.2 induces an automorphism

$$\eta_a : (\tilde{R}, \mathcal{O}_{\tilde{R}}) \xrightarrow{s^*(e^a)} (\tilde{R}, s^{-1}\mathcal{S}_x) = (\tilde{R}, t^{-1}\mathcal{S}_x) \xrightarrow{t^*(e^{-a})} (\tilde{R}, \mathcal{O}_{\tilde{R}}).$$

If we define $q_x = (s_x, t_x, e_x, m_x, i_x)$ by

$$s_x = s_0, \quad t_x = t_0 \circ \eta_a, \quad e_x = e_0, \quad m_x = m_0 \circ (p_2^* \eta_a^{-1}), \quad i_x = i_0 \circ \eta_a,$$

where p_2 is the projection $\tilde{R} \times_{\tilde{U}} \tilde{R} \rightarrow \tilde{R}$ to the second factor, it is straightforward to check that q_x satisfies the axioms of a groupoid space and the lifting q_x of the groupoid space actually corresponds to $\mathcal{S}_x \in \text{Def}_{\mathcal{O}_{\mathcal{X}}}(A)$. In this way, we can prove Corollary 2.4 by hand for a smooth separated DM stack \mathcal{X} .

Theorem 4.4. *Let \mathcal{X} be a smooth separated analytic Deligne–Mumford stack and $KS_{\mathcal{X}}$ be the associated Kodaira–Spencer algebra. Then, the morphism of functors $\Gamma : \text{Def}_{KS_{\mathcal{X}}} \rightarrow \text{Def}_{\mathcal{X}}$ in Theorem 4.1 is actually an isomorphism in the analytic category.*

Proof. To prove this theorem, we show that Γ is étale. According to Proposition 1.4, this is equivalent to show that

- (i) The ‘Dolbeault isomorphism’ in Theorem 3.3 on H^1 is actually the map induced by the morphism of functors $\Gamma : \text{Def}_{KS_{\mathcal{X}}} \rightarrow \text{Def}_{\mathcal{X}}$.
- (ii) The ‘Dolbeault isomorphism’ in Theorem 3.3 on H^2 satisfies the condition (ii) of Proposition 1.4.

For (i), the diagram chasing in (3) shows that $x \in Z^1(KS_{\mathcal{X}}^*, \bar{\partial})$ corresponds to an element $g \in \text{Ker}(\Gamma(U_1, \Theta) \xrightarrow{d} \Gamma(U_2, \Theta))$. As we consider over the ring of dual numbers $A = \mathbb{C}[\varepsilon]$, the condition $g \in \text{Ker} d$ is equivalent to say that $(\text{id} + g) : s^* \mathcal{O}_U \otimes A \rightarrow t^* \mathcal{O}_U \otimes A$ satisfies the cocycle condition of descent and the sheaf on \mathcal{X} given by descent with this twist is isomorphic to \mathcal{S}_x appeared in the proof of Theorem 4.1.

Now we prove (ii). Let $0 \rightarrow I \rightarrow A \rightarrow \bar{A} \rightarrow 0$ be a small extension. Let $x \in MC_{KS_{\mathcal{X}}(\bar{A})}$ be a Maurer–Cartan solution on \bar{A} and $\tilde{x} \in KS_{\mathcal{X}}^1 \otimes m_A$ an arbitrary lifting to A . The obstruction to the existence of a lifting of x in $MC_{KS_{\mathcal{X}}}(A)$ is

$$h = \bar{\partial} \tilde{x} + \frac{1}{2} [\tilde{x}, \tilde{x}] \in \text{Ker}(KS_{\mathcal{X}}^2 \otimes I \xrightarrow{\bar{\partial}} KS_{\mathcal{X}}^3 \otimes I),$$

which does not depend on the choice of a lifting \tilde{x} . By a diagram chasing in (3) gives $\tau \in \Gamma(U_0, \mathcal{A}^{0,1}(\Theta)) \otimes I$, $\rho \in \Gamma(U_1, \mathcal{A}^{0,0}(\Theta)) \otimes I$, and $\omega \in \text{Ker}(\Gamma(U_2, \Theta) \otimes I \rightarrow \Gamma(U_3, \Theta) \otimes I)$ satisfying

$$\bar{\partial} \tau = h, \quad \bar{\partial} \rho = t^* \tau - s^* \tau, \quad \omega = p_2^* \rho - m^* \rho + p_1^* \rho,$$

where p_1 , m , and p_2 are the projections pr_{12} , pr_{13} and pr_{23} on $U_2 = U \times_{\mathcal{X}} U \times_{\mathcal{X}} U$, respectively. If we put $\hat{x} = \tilde{x} - \tau$, we can check $\hat{x} \in MC_{KS_U}(A)$ (using the extension $A \rightarrow \bar{A}$ is small). This means that there is a sheaf $\mathcal{S}_{\hat{x}}$ on U , which is isomorphic to $\mathcal{O}_{\bar{U}}$. $\bar{\partial} \rho = t^* \tau - s^* \tau$ implies that $e^{\rho} : s^* \mathcal{S}_{\hat{x}} \rightarrow t^* \mathcal{S}_{\hat{x}}$ is an isomorphism (again using the smallness of the extension). The cocycle condition for descent is equivalent to the vanishing of the class $[\omega] = [p_2^* \rho - m^* \rho + p_1^* \rho]$ in the cohomology group $H^2(\mathcal{X}, \Theta_{\mathcal{X}})$. This implies that ω gives the obstruction to lifting $\mathcal{S}_{\hat{x}}$ to A , thus we checked the condition (ii) of Proposition 1.4. \square

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