Topology and its Applications 43 (1992) 1-14
North-Holland

Countable dense homogeneity and the Baire property

Ben Fitzpatrick Jr

Division of Mathematics, Auburn University, AL 36849, USA

Zhou Hao-xuan

Department of Mathematics, Sichuan University, Chengdu, Sichuan, People's Rep. of China

Received 18 June 1987
Revised 5 April 1990 and 16 August 1990

Abstract

Fitzpatrick Jr, B. and Zhou H., Countable dense homogeneity and the Baire property, Topology and its Applications 43 (1992) 1-14.

Conditions are given that ensure that certain open subsets of countable dense homogeneous spaces are countable dense homogeneous. Also, results are given which pertain to the questions: Is every countable dense homogeneous metric space Baire? Is every one completely metrizable?

Keywords: Countable dense homogeneity, homogeneity, Menger curve, λ-set, Baire property, completely metrizable.

AMS (MOS) Subj. Class.: 54E52, 54E35, 54A35.

1. Introduction

All spaces under consideration are assumed to be T_1. If X is a space, then $H(X)$ denotes the group of all autohomeomorphisms on X.

Even before Sierpinski [17] had introduced the notion of a homogeneous space, Frechet [14] and Brouwer [7] had observed that Euclidean n-dimensional space has the property that if A and B are two countable dense sets in $\mathbb{R}^n$, then some h in $H(\mathbb{R}^n)$ takes A onto B; they needed this theorem in their development of dimension theory. In 1962, Fort [13] proved that the Hilbert cube has the same property. In 1972, Bennett [6] isolated this property. He called a space X countable dense homogeneous (CDH) provided that X is separable and that if A and B are two countable dense subsets of X, then there is an $h \in H(X)$ such that $h(A) = B$.

Theorem 1.1 (Bennett). *Every first countable, connected, CDH space is homogeneous.*

The condition of first countability was later removed independently by Cook and Ford. Their work was not published; a theorem from which theirs follows has recently appeared [10]. That the connectedness condition cannot be removed is readily seen by considering the disjoint union of a 1-sphere and a 2-sphere. Noting that each component of this example is CDH and, therefore, homogeneous led to the observation that every component of a CDH space is CDH and is, if non-degenerate, open in the space. The same arguments will work for quasi-components, but in fact in CDH spaces the quasi-components are precisely the components, so this gives nothing new. Section 2 of this paper discusses the hereditary character of countable dense homogeneity in open subsets.

Theorem 1.2 (Bennett). *Every locally compact, separable, metric, strongly locally homogeneous space is CDH.*

Recall that a space X is strongly locally homogeneous provided that there is a basis B for the topology of X such that if $\{p, q\} \subseteq U \in B$, then there is an $h \in H(X)$ such that $h(p) = q$ and $h(x) = x$ for all $x \in X \setminus U$. That local compactness can be replaced by completeness in this theorem was shown by Fletcher and McCoy [12] and by Anderson, Curtis and van Mill [3]. Van Mill [20] demonstrated that the condition cannot be further relaxed; he gave an example of a connected, locally connected, Baire subset of the plane that is strongly locally homogeneous but not CDH. We have provided [11] an example of a connected and locally connected Baire Hausdorff space which is CDH but not strongly locally homogeneous. In [21], Watson and Simon constructed such an example which is also regular. The question as to whether there is an example which is metrizable remains open. All previously known examples of CDH spaces seem to be Baire, which leads us to examine in Sections 3 and 4 the relation between CDH and the Baire property.

Section 4 is devoted to a technique for constructing CDH spaces, thereby providing pertinent examples for the questions mentioned.

2. Open subsets of CDH spaces and a decomposition theorem

In 1978, Ungar [19] established that a continuum other than S^1 is CDH if and only if it is strongly n-homogeneous for all positive integers n. It was stated as corollary to the main results that every open dense subset of a CDH continuum is itself CDH. There appears to be a gap in the argument for this corollary, and we regard the question as still open.[1] For 1-dimensional continua the statement does hold. The proof is a straightforward consequence of several known theorems; we include it here for the sake of completeness.

[1] We are indebted to J.M.S. White for this observation.

Theorem 2.1. *Every open subset of a 1-dimensional CDH continuum is CDH.*

Proof. If X is a CDH continuum, then X is homogeneous [6]; also, X is locally connected [9]. Anderson showed [2] that a 1-dimensional locally connected homogeneous continuum must be either the simple closed curve or the Menger universal curve. He also showed [1] that the Menger curve is strongly locally homogeneous. Clearly, S^1 is, also. It is trivial that open subsets of strongly locally homogeneous spaces are themselves strongly locally homogeneous. So, if U is an open subset of a 1-dimensional CDH continuum, then U is a locally compact, separable, strongly locally homogeneous metric space and is therefore by Bennett's Theorem 1.2 cited above, CDH. □

Remark. That not every open dense subset of a 1-dimensional homogeneous continuum is homogeneous can be easily seen by removing from the dyadic solenoid two points of the same composant. The resulting subspace has one arc component that is locally compact and others which are not.

Remark. Our example [11] mentioned in the introduction is a connected, locally connected, CDH, Baire, Hausdorff space with a dense, open, connected subspace that is not CDH, not even homogeneous. We do not know of such an example that is metrizable.

It is known [10] that components of CDH spaces are CDH and are, if nontrivial, open sets. It is easy to verify that the components of a CDH space are the quasi-components of the space. The following are further partial results about the hereditary nature of CDH.

Theorem 2.2. *If X is a CDH metric space and U is a locally compact subset of X that is both open and closed, then U is CDH.*

Proof. Let $X = A \cup B$, where A is the union of all nondegenerate components of X and $B = X \setminus A$. Since A is the union of sets that are open and closed and CDH, A is CDH. Since every member of $H(X)$ maps A onto itself, A must have empty boundary. Therefore, both A and B are open and closed and CDH. Let D be the union of all degenerate open components of X and E be the union of the rest of the degenerate components of X. Then D and E are open and closed and CDH. If U is open and closed in X, then $U = (U \cap A) \cup (U \cap D) \cup (U \cap E)$. Since U contains every component of X that it intersects, $U \cap A$ is CDH. Clearly, $U \cap D$ is CDH. The subspace $U \cap E$ is locally compact, 0-dimensional, and every point of $U \cap E$ is a limit point of $U \cap E$, and $U \cap E$ is the union of a countable discrete collection of Cantor sets and is therefore CDH. It follows that U is CDH. □

Corollary. *Let X be a locally compact, CDH, metric space. If every dense open set in X is CDH, then so is every open set.*

Proof. Let U be open in X. Let $Y = X \setminus \mathrm{bdry}(U)$. Then Y is open and dense in X, so Y is CDH. Then U, as an open and closed subset of the locally compact, CDH space Y, is CDH. □

Theorem 2.3. *If X is a complete, metric CDH, 0-dimensional space, then every open subset of X is CDH.*

Proof. Let A be the set of all isolated points of X; let B be the set of all points of $X \setminus A$ at which X is locally compact; and let $C = X \setminus (A \cup B)$.

It follows that each of A, B, and C is open and closed in X and CDH. As before, B is the union of a countable discrete (possibly empty) collection of Cantor sets, and so is every open subset of B. The set C is, if nonempty, a separable, 0-dimensional, nowhere locally compact, complete metric space and is therefore homeomorphic to the irrationals, and so is every open subset of C. It follows that every open subset of X is CDH. □

Corollary. *Let X be a complete, 0-dimensional separable metric space. Then X is CDH if and only if X is strongly locally homogeneous.*

Theorem 2.4. *If every CDH continuum is hereditarily CDH with respect to open subsets, then so is every CDH compact metric space.*

Proof. If X is a compact, CDH metric space, then $X = A \cup B \cup C$, where A is empty or the union of a finite number of nondegenerate continua, B is finite, and C is empty or the Cantor set, and each of A, B, and C is open and closed in X. Let U be open in X. By hypothesis, $U \cap A$ is CDH. Clearly, $U \cap B$ and $U \cap C$ are CDH. Thus, U is CDH. □

Remark. Every 1-dimensional CDH compact metric space is hereditarily CDH with respect to open subsets, by the above argument and by Theorem 2.1.

Remark. The results of this section are, admittedly, fragmentary, indicating that there is much that remains to be done.

We conclude this section with a decomposition theorem.

Theorem 2.5. *If X is CDH, densely homogeneous (= homogeneous with respect to σ-discrete dense subsets) or strongly locally homogeneous, then X is the union of a* *discrete collection of homogeneous subspaces, each of which is CDH, densely homogeneous, or strongly locally homogeneous, respectively.*

Proof. Partition the space into equivalence classes, where $x \sim y$ if and only if $y = h(x)$ for some $h \in H(X)$. We claim that each equivalence class $[x]$ is open. That this is so in case X is strongly locally homogeneous is immediate. Suppose that there exists a class $\mathscr{C}$ of dense subsets of X such that

(1) if $C \in \mathscr{C}$, $D \subseteq C$, and D is dense in X, then $D \in \mathscr{C}$,

(2) if $C \in \mathscr{C}$ and $x \in X$, then $C \cup \{x\} \in \mathscr{C}$, and

(3) if $C \in \mathscr{C}$ and $D \in \mathscr{C}$, then $h(C) = D$ for some $h \in H(X)$.

Such is the case provided that X is CDH or densely homogeneous.

Let E be the collection of all equivalence classes which are open. If $\bigcup E = X$, then the desired conclusion follows. Suppose, then, that $\bigcup E \neq X$. Let $X' = X \setminus (\bigcup E)$. Then X' is invariant under every autohomeomorphism on X, and if $x \in X'$ then $[x] \subseteq X'$, and X' is open and closed in X and has a class $\mathscr{C}'$ of dense subsets satisfying (1)–(3) above. Therefore, we may, without loss of generality, assume that $X = X'$, that is, that no equivalence class in X is open. It is easy to see that no $[x]$ can contain a nonempty open set. It follows that if $S \in \mathscr{C}$ and $p \in X$, then neither $S \cap [p]$ nor $S \cap (X \setminus [p])$ is dense in X.

The argument on [10, p. 21], repeated word for word, yields a contradiction, so that $\bigcup E = X$, and each equivalence class $[p]$ is open. Surely, each is homogeneous and, respectively, CDH, densely homogeneous, or strongly locally homogeneous. Since E is a covering of X by disjoint open sets, E is discrete, and the proof is complete. □

We are indebted to Mr. J. McGrath, who asked a question which led to the above observation.

3. The Baire property and λ-sets

In this section we derive a necessary condition for homogeneous, non-Baire metric spaces to be CDH. We also discuss applications of set-theoretic techniques of Miller [16] and of Baumgartner [5] to CDH spaces.

Lemma 3.1. *If X is a homogeneous, non-Baire space, then X is meager (= 1st category).*

Proof. There exists an open set U such that, for some countable collection $\mathscr{J}$ of dense open sets in X, $U \cap (\bigcap \mathscr{J}) = \emptyset$. By homogeneity, every point of X belongs to such an open set U. Indeed, the collection $\mathscr{U}$ of all such open sets forms a basis for the topology of X. Let $\mathscr{A}$ be a maximal disjoint collection of elements of $\mathscr{U}$. For each $A \in \mathscr{A}$, let $\{G_n(A)\colon n < \omega\}$ be a countable collection of dense open sets in X such that $(\bigcap_{n<\omega} G_n(A)) \cap A = \emptyset$. For each n, let $H_n = \bigcup_{A \in \mathscr{A}} (G_n(A) \cap A)$. Then each H_n is a dense open set in X, and $\bigcap_{n<\omega} H_n = \emptyset$. So $X = \bigcup_{n<\omega}(X \setminus H_n)$ is meager. □

Remark. An eariier version of this lemma had an additional, unnecessary hypothesis; this was kindly pointed out by Mr. R. Knight and by R.W. Heath. It should also be pointed out that Lemma 3.1, together with Theorem 2.5, shows that there is a CDH non-Baire metric space if and only if there is a CDH meager metric space.

Lemma 3.2. *If X is a separable, metric, meager space, then some countable dense subset of X is a G_δ-set in X.*

Proof. Suppose $X = \bigcup_{i=1}^{\infty} F_i$, where each F_i is closed and nowhere dense. Let $U = \{U_i : i \in Z^+\}$ be a countable basis for X. Choose $x_1 \in U_1$ and, for each $n > 1$, choose $x_n \in U_n \setminus (\bigcup_{i=1}^{n-1} F_i)$. Let $C = \{x_n : n \in Z^+\}$. Then C is a countable dense subset of X, and, for each $n \in Z^+$, C intersects F_n in at most n points. It follows that C is a G_δ-set. □

Lemma 3.3. *If X is a CDH space and some countable dense subset of X is a G_δ-set, then every countable subset of X is a G_δ-set.*

Proof. Firstly, every countable dense subset of X must be a G_δ-set, since X is CDH. Now, suppose A is a countable subset of X. Let B be a countable, dense subset of X; then $A \cup B$ is countable and dense in X. Since $A \cup B$ is a G_δ-set in X, and A is a G_δ-set in $A \cup B$, A is a G_δ-set in X. □

Definition. A space in which every countable set is a G_δ-set is called a λ*-set*. We have the following.

Theorem 3.4. *Every CDH, meager, metric space is a λ-set. Every CDH, homogeneous, non-Baire metric space is a λ-set.*

Remark. This can be used to show that certain spaces which seem likely candidates for CDH, non-Baire spaces are actually not CDH. For example, consider the subset of $\mathbb{R}^2$ consisting of all points with exactly one rational coordinate. This space, although strongly locally homogeneous and strongly n-homogeneous for all $n \in Z^+$, is not CDH, since it contains as a closed, nowhere dense set, a copy I of the irrationals. Then I contains a countable dense subset C, which cannot be a G_δ-set in the space, for if it were it would be a G_δ-set in I and, therefore, completely metrizable; but C is homeomorphic to $\mathbb{Q}$, the rationals. In a similar manner it follows that the set of all rational points in real Hilbert space is not CDH.

Remark. It is easy to see that if Y is a λ-set and $f: X \to Y$ is a continuous bijection, then X is a λ-set. Therefore, no continuous bijective image of the irrationals can be a λ-set.

Theorem 3.5. *If X is a meager, CDH metric space, then X does not contain a copy of an uncountable Borel set in $\mathbb{R}$.*

Proof. If $Y \subseteq X$ is homeomorphic to an uncountable Borel set in $\mathbb{R}$, then there is a continuous bijection f from a closed subset A of the irrationals onto Y. But A cannot be a λ-set. □

Corollary. *$\mathbb{Q}^{\omega}$ is not CDH, and if Z is a separable metric space which contains a copy of an uncountable Borel set in $\mathbb{R}$, then $\mathbb{Q} \times Z$ is not CDH.*

Theorem 3.6. *It is consistent with ZFC that every CDH metric space X with $|X| = \mathfrak{c}$ is nonmeager.*

Proof. By a result of Miller [15] there is a model M of ZFC such that the Cantor set contains no λ-sets of cardinality $\mathfrak{c}$. Since the Hilbert cube is a continuous image of the Cantor set, it follows that each separable metrizable space X is a continuous bijective image of a subspace Z of the Cantor set. By the above, if $|X| = |Z| = \mathfrak{c}$, then Z is not a λ-set. Thus, by the second remark after Theorem 3.4, X is not a λ-set. By Theorem 3.4, X is nonmeager. □

Corollary. *It is consistent with ZFC that every CDH metric space X with $|U| = \mathfrak{c}$ for every open set U in X is Baire.*

Proof. Assume X is CDH, metric, non-Baire, and $|U| = \mathfrak{c}$ for every open U in X. There exists an open set U in X such that, for some countable collection G of dense open sets in X, $(\bigcap G) \cap U = \emptyset$. Let X' be the union of all such open sets U. If $h \in H(X)$, then $h(X') = X'$. Suppose X' is not closed, and let β be the boundary of X'. Then β is nowhere dense, and $h(\beta) = \beta$ for every $h \in H(X)$. Let C be a countable dense subset of $X \backslash \beta$, and let D be a countable subset of β. There is no $h \in H(X)$ that takes C to $C \cup D$, unless $D = \emptyset$. Therefore, β is empty. We have that X' is open and closed in X and CDH. There is a countable collection $\{U_n : n \in Z^+\}$ of open sets in X such that, for each $n \in Z^+$, there exists a countable collection $\{g_{m,n} : m \in Z^+\}$ of dense open sets in X such that $(\bigcap_{m \in Z^+} g_{m,n}) \cap U_n = \emptyset$, and such that $X' = \bigcup_{n \in Z^+} U_n$. Let $F_{m,n} = X \backslash g_{m,n}$. Then $X' = \bigcup_{m,n} F_{m,n}$ is meager. □

Remark. We will see in Section 4 that the continuum hypothesis implies that there is a meager CDH $X \subseteq \mathbb{R}$; hence, it is independent with ZFC that there is a meager CDH metric space of size $\mathfrak{c}$.

Definition. A subset X of $\mathbb{R}$ is *ω_1-dense* if every open interval contains exactly ω_1-many point of X.

Theorem 3.7 (PFA). *There is a non-Baire CDH space of cardinality less than* $\mathfrak{c}$.

Indication of Proof. The proof is very similar to the proof of Theorem 6.9 of Baumgartner [5] and yields the stronger result that, under proper forcing, if X and Y are any two ω_1-dense subsets of $\mathbb{R}$ and if A and B are countable dense subsets of X and Y, respectively, then there exists an order-preserving homeomorphism $h: X \to Y$ such that $h(A) = B$. Therefore, every ω_1-dense set in $\mathbb{R}$ is CDH.

Modify Baumgartner's proof as follows. Replace the poset Q by $Q_1 = \{\langle f, g\rangle: f$ and g are order-preserving finite functions such that (1) $\operatorname{dom} f \subseteq A$, $\operatorname{ran} f \subseteq B$, $\operatorname{dom} g \subseteq X$, $\operatorname{ran} g \subseteq Y$, (2) $\operatorname{dom} f \cap \operatorname{dom} g = \emptyset$, and (3) $f \cup g$ is order-preserving$\}$. Then, we copy the proof of Theorem 6.9 except for the definition of $\bar{Q}$. Here $\bar{Q}_1$ will consist of all $q \in Q_1$ with $g = \pi_1(q)$ satisfying the conditions (6) and (7) on p. 943.

Note that, since PFA implies $\mathfrak{c} > \omega_1$, it is easy to construct an ω_1-dense set in $\mathbb{R}$. Also, PFA implies MA_{ω_1}, which in turn implies that every dense subset of $\mathbb{R}$ of cardinality ω_1 is not Baire. □

Remarks. Spaces indicated in Theorem 3.7 in fact exist when MA and $\omega_1 < \mathfrak{c}$ are true. It directly follows from the method used in [18], also see [22].

4. Constructing CDH spaces

The following lemma from set theory is useful in producing spaces that are CDH.

Lemma 4.1. *Suppose that F is a countable subset of $H(X)$, that A is an uncountable subset of X, that D and C are countable subsets of X, and that if $g \in G$, the subgroup of $H(X)$ generated by F, then $g(D) \cap C = \emptyset$. Then there exists a countable subset E of X such that $E \cap A \neq \emptyset$, $D \subseteq E$, $g(E) = E$ and $E \cap C = \emptyset$ for all $g \in G$.*

Proof. Assume the hypothesis. Let $C' = \{g(C): g \in G\}$, $D' = \{g(D): g \in D\}$. Then $C' \cap D' = \emptyset$ and C' and D' are countable. Pick $p \in A \backslash C'$, $D'' = D' \cup \{p\}$, and set $E = \bigcup \{g(D''): g \in G\}$. Then E has the desired properties. □

Remark. Let k and n be positive integers. The standard proof that $\mathbb{R}^n$ is CDH yields the following observation. If $A_1, \ldots, A_k$ are disjoint countable dense sets in $\mathbb{R}^n$ and $B_1, \ldots, B_k$ are disjoint countable dense sets in $\mathbb{R}^n$, then there exists $h \in H(\mathbb{R}^n)$ such that $h(A_i) = B_i$, $i = 1, \ldots, k$. Call such a space k-CDH.

Van Mill [20] gave an example of a connected and locally connected Baire subspace X of $\mathbb{R}^2$ such that X is strongly locally homogeneous but not countable dense homogeneous. On the other hand, it is known [12] that if X is a separable completely metrizable, strongly locally homogeneous space, then X is necessarily countable dense homogeneous. The following theorem shows that, under the continuum hypothesis, complete metrizability does not follow from the joint assumption of homogeneity, countable dense homogeneity, and strong local homogeneity, and being Baire.

Theorem 4.2 (CH). *Let n be a positive integer. There exists a Baire subspace X of $\mathbb{R}^n$ such that X is homogeneous, strongly locally homogeneous, and countable dense homogeneous, but X contains no uncountable Borel set; hence X is not completely metrizable. In case $n=2$ X is connected and locally connected.*

Proof. We will write $\mathbb{R}^n$ as the union of two disjoint sets X and Y such that X has the desired homogeneity properties and both X and Y intersect every uncountable Borel set in $\mathbb{R}^n$. Let $\mathbb{R}^n=\{r_\alpha: \alpha<\omega_1\}$. Let $\pi:\omega_1\times\omega_1\to\omega_1$ be a bijection such that $\pi(\beta,\gamma)\geq\beta$ and $\pi(0,0)=0$. $\pi(\beta,\gamma)=\beta$ happens iff $\gamma=0$ and β is in a fixed closed unbounded subset of ω_1. Let $\{B_\alpha: \alpha<\omega_1\}$ be the set of all uncountable Borel sets in $\mathbb{R}^n$. Let $\{U_k: k<\omega\}$ be a countable basis for $\mathbb{R}^n$, $U_0=\mathbb{R}^n$, and if $k>0$, U_k is an open ball. Note that if D is a countable dense subset of $\mathbb{R}^n$ and $\{x,y\}\subset D\cap U_k$, $k\geq 0$, then there is an $h\in H(\mathbb{R}^n)$ such that $h(x)=y$, $h(D)=D$, and h is the identity outside U_k. Choose D_0 to be a countable dense subset of $\mathbb{R}^n$, $D_0\cap B_0\neq\emptyset$, $r_0\in D_0$. $D_0=\{x_{0,i}: i<\omega\}$. For each $i,j,k<\omega$, let $h^0_{i,j,k}$ be an element h of $H(\mathbb{R}^n)$ such that if $\{x_{0,i},x_{0,j}\}\not\subset U_k$, then h is the identity and such that if $\{x_{0,i},x_{0,j}\}\subset U_k$, then $h(x_{0,i})=x_{0,j}$, $h(D_0)=D_0$, and h is the identity outside U_k.

Let H_0 be the subgroup of $H(\mathbb{R}^n)$ generated by $\{h^0_{i,j,k}: i,j,k<\omega\}$. There exists a countable dense subset C_0 of $\mathbb{R}^n\setminus D_0$ such that $C_0\cap B_0\neq\emptyset$, and $h(C_0)=C_0$ for all $h\in H_0$. Let $\{\langle A_{0,\gamma},B_{0,\gamma}\rangle: \gamma<\omega_1\}$ be the collection of all ordered pairs of countable dense subsets of D_0, chosen so that $D_0\setminus A_{0,0}$ and $D_0\setminus B_{0,0}$ are dense in D_0. By the observation above, there exists an $f_0\in H(\mathbb{R}^n)$ such that $f_0(C_0)=C_0$, $f_0(D_0)=D_0$, and $f_0(A_{0,0})=B_{0,0}$. Now suppose that $0<\alpha<\omega_1$, and that C_ξ, D_ξ, f_ξ have been defined for $0\leq\xi<\alpha$, and two listings $\langle A_{\xi\gamma},B_{\xi\gamma}\rangle_{\gamma<\omega_1}$ and $\langle x_{\xi,i}\rangle_{i<\omega}$, and a collection $\langle h^\xi_{i,j,k}\rangle_{i,j,k<\omega}$, have been defined such that

(1) C_ξ and D_ξ are countable dense subsets of $\mathbb{R}^n$, $C_\xi\cap D_\xi=\emptyset$,

(2) $\{r_\eta: \eta<\xi\}\subseteq D_\xi\cup C_\xi$,

(3) $D_\xi\cap B_\xi\neq\emptyset$ and $C_\xi\cap B_\xi\neq\emptyset$,

(4) $\bigcup_{\eta<\xi}D_\eta\subseteq D_\xi$ and $\bigcup_{\eta<\xi}C_\eta\subseteq C_\xi$,

(5) $f_\xi\in H(\mathbb{R}^n)$, and if $0\leq\eta\leq\xi$ then $f_\eta(C_\xi)=f_\eta^{-1}(C_\xi)=C_\xi$ and $f_\eta(D_\xi)=f_\eta^{-1}(D_\xi)=D_\xi$,

(6) $\{\langle A_{\xi,\gamma},B_{\xi,\gamma}\rangle: \gamma<\omega_1\}$ is the collection of all ordered pairs of countable dense subsets of D_ξ, such that if $\pi^{-1}(\xi)=\langle\xi,\gamma\rangle$ (in fact $\pi^{-1}(\xi)=\langle\xi,0\rangle$), then let $A_{\langle\xi,0\rangle}=B_{\langle\xi,0\rangle}=D_0$ and

(a) $D_\xi\setminus A_{\pi^{-1}(\xi)}$ and $D_\xi\setminus B_{\pi^{-1}(\xi)}$ are dense in D_ξ,

(b) $f_\xi(A_{\pi^{-1}(\xi)})=B_{\pi^{-1}(\xi)}$,

(c) each pair $\langle A_{\xi,\gamma},B_{\xi,\gamma}\rangle$ appears ω_1 times in $\{\langle A_{\xi,\gamma},B_{\xi,\gamma}\rangle: \xi,\gamma<\omega_1\}$,

(7) $D_\xi=\{x_{\xi,i}: i<\omega\}$, $h^{\xi}_{i,j,k}=h\in H(\mathbb{R}^n)$, and
   (a) if $\{x_{\xi,i}, x_{\xi,j}\}\not\subset U_k$, then h is the identity, and
   (b) if $\{x_{\xi,i}, x_{\xi,j}\}\subset U_k$, then $h(D_\xi)=D_\xi$, $h(x_{\xi,i})=x_{\xi,j}$, and h is the identity outside U_k,

(8) for all $\eta\leq\xi$, $h^{\eta}_{i,j,k}(D_\xi)=(h^{\eta}_{i,j,k})^{-1}(D_\xi)=D_\xi$ and $h^{\eta}_{i,j,k}(C_\xi)=(h^{\eta}_{i,j,k})^{-1}(C_\xi)=C_\xi$.

Let $(\beta,\gamma)=\pi^{-1}(\alpha)$. If $\beta=\alpha$ define $D_\alpha=\bigcup_{\xi<\alpha}D_\xi$, $C_\alpha=\bigcup_{\xi<\alpha}C_\xi$, $f_\alpha=$identity. If $\beta<\alpha$, define D_α, C_α, f_α as follows.

(a) D_α contains the first element of $\mathbb{R}^n$ not in $\bigcup_{\xi<\alpha}D_\xi\cup\bigcup_{\xi<\alpha}C_\xi$,
(b) $D_\alpha\cap B_\alpha\neq\emptyset$, D_α is countable, $\bigcup_{\xi<\alpha}D_\xi\subseteq D_\alpha$,
(c) $D_\alpha\cap(\bigcup_{\xi<\alpha}C_\xi)=\emptyset$,
(d) $f_\xi(D_\alpha)=D_\alpha$ for all $\xi<\alpha$, i.e., D_α is closed under f_ξ,
(e) D_α is closed under $h^{\xi}_{i,j,k}$, for all $\xi<\alpha$, for all $i,j,k<\omega$,
(f) $D_\alpha\backslash A_{\beta,\gamma}$ and $D_\alpha\backslash B_{\beta,\gamma}$ are dense in D_α.

Let $\{\langle A_{\alpha,\gamma}, B_{\alpha,\gamma}\rangle: \gamma<\omega_1\}$ be the collection of all ordered pairs of countable dense subsets of D_α. Let $D_\alpha=\{x_{\alpha,i}: i<\omega\}$. For each $i,j,k<\omega$, let $h^{\alpha}_{i,j,k}$ be an element h of $H(\mathbb{R}^n)$ such that if $\{x_{\alpha,i}, x_{\alpha,j}\}\not\subset U_k$, then h is the identity and if $\{x_{\alpha,i}, x_{\alpha,j}\}\subset U_k$, then $h(x_{\alpha,i})=x_{\alpha,j}$, $h(D_\alpha)=D_\alpha$, $h(\bigcup_{\xi<\alpha}C_\xi)=\bigcup_{\xi<\alpha}C_\xi$ and h is the identity outside U_k. Next, choose C_α so that

(a') C_α is countable, $C_\alpha\cap B_\alpha\neq\emptyset$, and $\bigcup_{\xi<\alpha}C_\xi\subseteq C_\alpha$,
(b') $C_\alpha\cap D_\alpha=\emptyset$,
(c') C_α is closed under the elements of the group generated by f_ξ, $0\leq\xi<\alpha$, and $h^{\eta}_{i,j,k}$, $0\leq\eta\leq\alpha$, $i,j,k<\omega$.

Next, define $f_\alpha\in H(\mathbb{R}^n)$ to be such that $f_\alpha(C_\alpha)=C_\alpha$, $f_\alpha(D_\alpha)=D_\alpha$, and $f_\alpha(A_{\beta,\gamma})=B_{\beta,\gamma}$. Finally, let $X=\bigcup_{\alpha<\omega_1}D_\alpha$ and $Y=\bigcup_{\alpha<\omega_1}C_\alpha$.

Clearly, X contains no uncountable Borel set, since $X\cap(\bigcup_{\alpha<\omega_1}C_\alpha)=\emptyset$ and $\bigcup_{\alpha<\omega_1}C_\alpha$ intersects every uncountable Borel set.

Let M and N be two countable dense subsets of X. There exists $\beta<\omega_1$ such that $M\cup N\subseteq D_\beta$. There exists $\gamma<\omega_1$ such that $M=A_{\beta,\gamma}$ and $N=B_{\beta,\gamma}$. Let $\alpha=\pi(\beta,\gamma)$. Then $\alpha\geq\beta$, so $M\cup N\subseteq D_\alpha$, and, moreover, $f_\alpha(M)=N$. Therefore, X is CDH.

Let x and y be two points of X. There exists α such that $\{x,y\}\subset D_\alpha$. There exist i,j such that $x=x_{\alpha,i}$ and $y=x_{\alpha,j}$. Then $h^{\alpha}_{i,j,0}$ is an automorphism on $\mathbb{R}^n$ that takes x to y and X to X. Therefore, X is homogeneous. Let $x\in X$ and let U be open in X, $x\in U$. There is a $k>0$ such that $x\in U_k\cap X\subseteq U$. Let $y\in U_k$. There exists $\alpha<\omega_1$ and there exist $i<\omega$ and $j<\omega$ such that $x=x_{\alpha,i}$ and $y=x_{\alpha,j}$. Then $h^{\alpha}_{i,j,k}$ is an autohomeomorphism on $\mathbb{R}^n$ that takes x to y, is fixed outside U and takes X onto X. Therefore, X is strongly locally homogeneous.

In case $n=2$ then since Y contains no uncountable closed set, $X=\mathbb{R}^2\backslash Y$ is connected and locally connected, by a theorem of Sierpinski (see Section 31 of Hausdorff's *Mengenlehre* 1931). □

Remark. Standard techniques developed by Bennett and others can actually be used to show that if X is a strongly locally homogeneous, complete separable metric space with no isolated points, then (1) X is k-CDH for all $k<\omega$, and (2) X has a countable basis G such that if D is a countable dense subset of X, $U\in G$, and $\{x, y\}\subseteq D\cap U$, then there is $h\in H(X)$ such that $h(x)=y$, $h(D)=D$, and h is the identity outside U. This, together with the proof of Theorem 4.2, yields the following.

Theorem 4.2′ (CH). *If Z is a complete separable metric space which is strongly locally homogeneous and has no isolated points, then Z contains a dense subspace X which is Baire, strongly locally homogeneous, CDH, homogeneous if Z is, and which is not completely metrizable.*

Remark. If we are not concerned with whether X is dense in Z or with whether X is connected and locally connected, it suffices to consider $Z=$ Cantor set, in which case Baldwin and Beaudoin [4] have shown, assuming Martin's axiom, that a CDH subset X exists which is not completely metrizable.

We conclude by showing that, assuming the continuum hypothesis, there exists a meager CDH subspace of $\mathbb{R}$. G. Gruenhage has kindly informed us that he also has a proof of this. His argument is apparently somewhat more complicated than ours; on the other hand it yields the stronger result that, under CH, there is a meager CDH subspace X of $\mathbb{R}$ such that X is of universal measure 0, that is, there is no nontrivial Borel measure on X.

Lemma 4.3. *If A_i and B_i $(i<\omega)$ are countable dense sets in $\mathbb{R}$ and $(A_i\cap A_j)=\emptyset$ and $B_i\cap B_j=\emptyset$ for $i\neq j$, then there exists $h\in H(\mathbb{R})$ such that $h(A_i)=B_i$, and h has a continuous, positive derivative on $\mathbb{R}$.*

Proof. This is a modification of [8, Lemma 1]. Proofs are essentially the same. □

Notation. Let $H'(\mathbb{R})$ denote the subgroup of $H(\mathbb{R})$ whose elements have continuous, positive derivatives. Let μ denote Lebesgue measure. If G is a subgroup of $H'(\mathbb{R})$ and $M\subseteq X$, then $\mathrm{Orb}(M, G)$ denotes $\{g(x): x\in M \text{ and } g\in G\}$.

Lemma 4.4. *Suppose $g\in H'(\mathbb{R})$, $\tilde{Y}\subseteq Z\subseteq\mathbb{R}$, $\mu(\tilde{Y})=0$, $g(\tilde{Y})=\tilde{Y}$, and Z is a G_δ-set in $\mathbb{R}$. Then there exists a G_δ-set Y in $\mathbb{R}$ such that $\tilde{Y}\subseteq Y\subseteq Z$, $\mu(Y)=0$, and $g(Y)=Y$.*

Proof. There exists a G_δ-set $T_0\subseteq Z$ such that $\mu(T_0)=0$ and $\tilde{Y}\subseteq T_0$. For each $n\geqslant 0$, let $T_{n+1}=T_0\cap g(T_n)\cap g^{-1}(T_n)$; let $Y=\bigcap_{n<\omega} T_n$. □

Lemma 4.5. *Suppose G is a countable subgroup of $H'(\mathbb{R})$, $\tilde{Y}\subseteq\mathbb{R}$, $\mu(\tilde{Y})=0$, and $g(\tilde{Y})=\tilde{Y}$ for every $g\in G$. Then there exists a G_δ-set Y in $\mathbb{R}$ such that $\tilde{Y}\subseteq Y$, $\mu(Y)=0$, and $g(Y)=Y$ for every $g\in G$.*

Proof. Enumerate $G=\{g_n: n<\omega\}$ in such a way that every g in G appears infinitely many times in the enumeration. There is a G_δ-set Y_0 such that $\tilde{Y}\subseteq Y_0$, $\mu(Y_0)=0$, and $g_0(Y_0)=Y_0$. For each $n\geq 0$, there is a G_δ-set Y_{n+1} such that $\tilde{Y}\subseteq Y_{n+1}\subseteq Y_n$, $\mu(Y_{n+1})=0$, and $g_{n+1}(Y_{n+1})=Y_{n+1}$. Let $Y=\bigcap_{n<\omega} Y_n$. □

Theorem 4.6 (CH). *There exists a meager CDH subspace X of* $\mathbb{R}$.

Proof. Our plan is to find a λ-set that is CDH and dense in $\mathbb{R}$.

Let $\pi:\omega_1\times\omega_1\to\omega_1$ be a bijection such that $\pi(0,0)=0$ and $\pi(\beta,\gamma)\geq\beta$. $\pi(\beta,\gamma)=\beta$ happens if and only if $\gamma=0$ and β is in a fixed closed unbounded subset of ω_1. For $\alpha<\omega_1$, inductively define countable dense subsets X_α of $\mathbb{R}$, G_δ-sets $Y_\alpha\subseteq\mathbb{R}$, and functions $f_\alpha\in H'(\mathbb{R})$. For each $\alpha<\omega_1$, $\{\langle A_{\alpha,\gamma}, B_{\alpha,\gamma}\rangle: \gamma<\omega_1\}$ is the collection of all ordered pairs of countable dense sets in X_α, with repetition such that each pair of countable dense sets of X_α appears ω_1 times in $\{\langle A_{\alpha,\gamma}, B_{\alpha,\gamma}\rangle: \gamma<\omega_1\}$ and $\forall\alpha$, if $\pi^{-1}(\alpha)=\langle\alpha,0\rangle$, then $A_{\alpha,0}=B_{\alpha,0}=A_{0,0}$. G_α is the subgroup of $H'(\mathbb{R})$ generated by $\{f_\beta: \beta\leq\alpha\}$.

Choose X_0 to be a countable dense subset of $\mathbb{R}$, and choose $A_{0,0}$ and $B_{0,0}$ so that $X_0\backslash A_{0,0}$ and $X_0\backslash B_{0,0}$ are dense in X_0. Choose $f_0\in H'(\mathbb{R})$ so that $f_0(A_{0,0})=B_{0,0}$ and $f_0(X_0)=X_0$. There exists a G_δ-set Y_0 in $\mathbb{R}$ such that $X_0\subset Y_0$, $\mu(Y_0)=0$, and $f_0(Y_0)=Y_0$.

Suppose $0<\alpha<\omega_1$, and suppose X_ξ, Y_ξ, f_ξ have been defined for $0<\xi<\alpha$, and that

(1) X_ξ is a countable dense set in $\mathbb{R}$, Y_ξ is a G_δ-set, $X_\xi\subseteq Y_\xi$, $\mu(Y_\xi)=0$, $f_\xi\in H'(\mathbb{R})$,
(2) $\forall\eta<\xi$, $\mathrm{Orb}(X_\eta, G_\eta)=X_\eta\subset X_\xi$, $Y_\eta\subseteq Y_\xi$,
(3) $\forall g\in G_\xi$, $g(Y_\xi)=Y_\xi$, $\mathrm{Orb}(Y_\eta, G_\eta)\subseteq Y_\xi$, for any $n<\xi$,
(4) $\forall\eta<\xi$, $(X_\xi\backslash X_\eta)\cap Y_\eta=\emptyset$, and
(5) $f_\xi(A_{\pi^{-1}(\xi)})=B_{\pi^{-1}(\xi)}$ and $f_{\xi'}(X_\xi)=X_\xi$ for all $\xi'\leq\xi$.

Case 1: *α is a successor ordinal*, $\alpha=\nu+1$. Take $x\in\mathbb{R}\backslash Y_\nu$. Then $(\mathrm{Orb}(\{x\}, G_\nu))\cap Y_\nu=\emptyset$. Let $(\beta,\gamma)=\pi^{-1}(\alpha)$. If $\beta=\alpha$, let $X_\alpha=X_\nu$, $Y_\alpha=Y_\nu$, and f_α be the identity. Assume $\beta<\alpha$, $A_{\beta,\gamma}$, $B_{\beta,\gamma}$ are countable dense sets of X_β. Of course they are also dense in $\mathbb{R}$. Take a countable dense subset $C\subseteq\mathbb{R}\backslash\mathrm{Orb}(Y_\nu, G_\nu)$ with $C=\mathrm{Orb}(C, G_\nu)$. Let $X_\alpha=X_\nu\cup C$. By Lemma 4.3, there is $f_\alpha\in H'(\mathbb{R})$ such that $f_\alpha(A_{\beta,\gamma})=B_{\beta,\gamma}$, $f_\alpha(C\cup X_\nu\backslash A_{\beta,\gamma})=C\cup X_\nu\backslash B_{\beta,\gamma}$. So $X_\alpha=\mathrm{Orb}(X_\alpha, G_\alpha)$ if G_α is the group generated by $G_\nu\cup\{f_\alpha\}$. Clearly, $X_\alpha\cap Y_\eta=X_\eta$ for $\eta<\alpha$. Let $\tilde{Y}_\alpha=\mathrm{Orb}(X_\alpha\cup Y_\nu, G_\alpha)$. Then $g(\tilde{Y}_\alpha)=\tilde{Y}_\alpha$ for all $g\in G_\alpha$. There exists a G_δ-set Y_α in $\mathbb{R}$ such that $\tilde{Y}_\alpha\subseteq Y_\alpha$, $\mu(Y_a)=0$, and $g(Y_\alpha)=Y_\alpha$ for all $g\in G_\alpha$.

Case 2: *α is a limit ordinal.* Let $X_\alpha=\bigcup_{\xi<\alpha}X_\xi$. Note that $X_\alpha\backslash A_{\pi^{-1}(\alpha)}$ and $X_\alpha\backslash B_{\pi^{-1}(\alpha)}$ are dense in X_α. Take $f_\alpha\in H'(\mathbb{R})$, $f_\alpha(A_{\pi^{-1}(\alpha)})=B_{\pi^{-1}(\alpha)}$, $f_\alpha(X_\alpha)=X_\alpha$. Let $Y=\mathrm{Orb}(\bigcup_{\xi<\alpha}Y_\xi, G_\alpha)$. Now, $\mu(Y)=0$, and $g(Y)=Y$ for all $g\in G_\alpha$. There exists a G_δ-set Y_α such that $Y\subseteq Y_\alpha$, $\mu(Y_\alpha)=0$, and $g(Y_\alpha)=Y_\alpha$ for all $g\in G_\alpha$.

It follows that (1)-(5) hold for $0\leq\xi\leq\alpha$.

Let $X=\bigcup_{\alpha<\omega_1}X_\alpha$. To show that X is CDH, it suffices to show that $f_\xi(X)=X$ for all $\xi<\omega_1$. Suppose $x\in X$. Then $x\in X_\alpha$ for some $\alpha>\xi$, so $f_\xi(x)\in X_{\alpha+1}$; so $f_\xi(x)\in X$.

Therefore, $f_\xi(X)\subseteq X$. Again, suppose $x\in X$; then $x\in X_\alpha$ for some $\alpha\geq\xi$, so $f_\xi^{-1}(x)\in X_{\alpha+1}$, so $f_\xi^{-1}(x)\in X$, so $x\in f_\xi(X)$. Therefore, $X\subseteq f_\xi(X)$, and we have $f_\xi(X)=X$.

We claim that X is a λ-set. To see this, suppose D is a countable subset of X. Then $D\subseteq X_\alpha$ for some $\alpha<\omega_1$. Now, $X\cap Y_\alpha$ is a G_δ-set in X, since Y_α is a G_δ-set in $\mathbb{R}$. But $(X\setminus X_\alpha)\cap Y_\alpha=\emptyset$, so X_α is a G_δ-set in X and is countable, so it follows that every subset of it is a G_δ-set in X. Therefore, D is a G_δ-set in X.

Finally, we observe that every separable λ-set with no isolated points is meager. For, let $D=\{x_n: n<\omega\}$ be a countable dense subset of the λ-set X. Since $D=\bigcap_{n<\omega} G_n$, where each G_n is open, we have $X=(\bigcup_{n<\omega}(X\setminus G_n))\cup\bigcup_{n<\omega}\{x_n\}$, so X is the countable union of closed, nowhere dense sets. This completes the proof. □

5. Questions

Question 1. Is there an absolute example of a CDH metric space of cardinality ω_1?

Question 2. If X is CDH and U is open in X and U is homogeneous, is U necessarily CDH?

Question 3. Is every open subset of a CDH continuum CDH?

References

[1] R.D. Anderson, A characterization of the universal curve and a proof of its homogeneity, Ann. of Math. 67 (1958) 313–324.
[2] R.D. Anderson, One dimensional continuous curves and a homogeneity theorem, Ann. of Math. 68 (1958) 1–16.
[3] R.D. Anderson, D.W. Curtis and J. van Mill, A fake topological Hilbert space, Trans. Amer. Math. Soc. 272 (1982) 311–321.
[4] S.L. Baldwin and R.A. Beaudoin, Countable dense homogeneous spaces under Martin's Axiom, Israel J. Math. 65 (1989) 153–164.
[5] J.E. Baumgartner, Applications of the proper forcing axiom, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 913–960.
[6] R.B. Bennett, Countable dense homogeneous spaces, Fund. Math. 74 (1972) 189–194.
[7] L.E.J. Brouwer, Some remarks on the coherence type η, Proc. Akad. Amsterdam 15 (1913) 1256–1263.
[8] F. Cater, Derivatives on countable dense subsets, Real Anal. Exchange 11 (1985–86) 159–167.
[9] B. Fitzpatrick, A note on countable dense homogeneity, Fund. Math. 75 (1972) 33–34.
[10] B. Fitzpatrick Jr and N.F. Lauer, Densely homogeneous spaces I, Houston J. Math. 13 (1987) 19–25.
[11] B. Fitzpatrick Jr and Zhou H., Densely homogeneous spaces II, Houston J. Math. 14 (1988) 57–68.
[12] P. Fletcher and R.A. McCoy, Conditions under which a connected representable space is locally connected, Pacific J. Math. 51 (1974) 433–437.
[13] M.K. Fort Jr, Homogeneity of infinite products of manifolds with boundary, Pacific J. Math. 12 (1962) 879–884.
[14] M. Frechet, Les dimension d'un ensemble abstrait, Math. Ann. 68 (1910) 145–168.
[15] A.W. Miller, Mapping a set of reals onto the reals, J. Symbolic Logic 48 (1983) 575–583.
[16] A.W. Miller, Special subsets of the real line, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 201–234.

[17] W. Sierpinski, Sur une propriété topologique des ensembles dénombrable dense en soi, Fund. Math. 1 (1920) 11–28.

[18] J. Steprans and S. Watson, Homeomorphisms of manifolds with prescribed behaviour on large dense sets, Bull. London Math. Soc. 19 (1987) 305–310.

[19] G.S. Ungar, Countable dense homogeneity and *n*-homogeneity, Fund. Math. 99 (1978) 155–160.

[20] J. van Mill, Strong local homogeneity does not imply countable dense homogeneity, Proc. Amer. Math. Soc. 84 (1982) 143–148.

[21] S. Watson and P. Simon, Open subsets of countable dense homogeneous spaces, Fund. Math., to appear.

[22] Zhou H., Two applications of set theory to homogeneity, Questions Answers Gen. Topology 6 (1988) 49–56.