Pergamon

Computers Math. Applic. Vol. 29, No. 6, pp. 67–73, 1995

0898-1221/95 \$9.50 + 0.00

0898-1221(95)00008-9

A Modification of Adomian's Solution for Nonlinear Oscillatory Systems

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(Received January 1994; accepted February 1994)

Abstract—In this paper, we consider nonlinear oscillatory equations of the form $\frac{d^2u}{dt^2} + \omega^2 u = \varepsilon f\left(u, \frac{du}{dt}\right)$, whose solutions can be obtained by the decomposition method. But these solutions do not exhibit periodicity, which is characteristic of oscillatory systems. We use an alternative technique by which the solution obtained by the decomposition method is made periodic. The method is described and illustrated with examples. There is no smallness assumption on the parameter ε occuring in the equation.

Keywords—Decomposition method, Nonlinearity, Oscillatory equations, Laplace transform, Padé approximant.

1. INTRODUCTION

The decomposition method of Adomian has been applied to a rather wide class of nonlinear differential and partial differential equations [1]. The general nonlinear deterministic operator in an equation $\mathcal{F} y = g$ is split into linear and nonlinear parts denoted by $\mathcal{L}$ and $\mathcal{N}$. $\mathcal{L}$ is written as $L + R$, where L is the linear part of $\mathcal{L}$ with the highest derivative, and R is the remaining part of the linear operator. The unknown function y is written as $y = \sum_{n=0}^{\infty} \lambda^n y_n$, where λ is a parameter. The nonlinear term $N y$ is represented as a series of Adomian polynominals $N y = \sum_{n=0}^{\infty} \lambda^n A_n$, where A_n is defined as

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} f(y(\lambda))\bigg|_{\lambda=0},$$

and $A_n = A_n(y_0, y_1, \ldots, y_n)$. Now,

$$y = y_0 - L^{-1} R y - L^{-1} N y.$$

If, for example, $L = \frac{d^2}{dx^2}$, then $y_0 = y(0) + y'(0) + L^{-1} g$. Each y_n is calculated from the preceeding term y_{n-1}:

$$y_{n+1} = -L^{-1}(R y_n + A_n).$$

Thus, the method results in a series solution.

In general, there exists no method which yields an exact solution of any nonlinear differential equation. For certain classes of differential equations, the only methods available are approximation procedures such as linearization, perturbative methods, etc. The advantage of the decomposition method over the other approximate methods, apart from computational simplicity, is

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX that the method is nonperturbative and does not involve any linearization or smallness assumptions. Hence, the solution obtained by this method is expected to be a better approximation. In this paper, we consider nonlinear oscillatory systems whose solutions can be obtained by the decomposition method. But these solutions do not exhibit periodicity, which is characteristic of oscillatory systems. We use an alternative technique to be applied to the series solution of the decomposition method that yields sine and cosine functions in the solution which are periodic. The procedure is described and illustrated with examples.

2. DESCRIPTION OF THE METHOD

We consider equations of the form

$$\frac{d^2u}{dt^2} + \omega^2 u = \varepsilon f\left(u, \frac{du}{dt}\right),$$

where ε is a parameter (not necessarily small). We solve this nonlinear equation using the decomposition method with $\frac{d^2}{dt^2}$ as the linear operator and obtain a series solution in t. This series solution does not exhibit the periodic behaviour which is characteristic of oscillator equations. Adomian, in his book [2], has suggested that if we choose $\frac{d^2}{dt^2} + \omega^2$ as the linear operator L, the solution converges faster than the previous case and one gets sine and cosine functions for solutions of the homogeneous equations which are used as an initial approximation. But inversion of the operator produces computational difficulties. Using a technique suggested in [3], we apply Laplace transformation to the series obtained by Adomian's decomposition method, then convert the transformed series into a meromorphic function by forming its Padé approximant, and then invert the approximant which yields a better solution that is also periodic.

In perturbative schemes, the frequency and the amplitude of the oscillator are considered as varying functions of time, and the frequency is also perturbed with respect to the small parameter ε. In conservative systems, steady oscillations occur for arbitrary amplitude, and hence the amplitude remains a constant. In nonconservative systems, stationary oscillations are possible only for special values, and so the amplitude is a varying function of time which tends to the fixed amplitude as $t \to \infty$. Here, in our case, the amplitude is not a varying function of time. So, for conservative systems, where amplitude is a constant, we apply the Laplace transformation to the series taken up to n terms and find the Padé approximant of the transformed series. But for nonconservative systems, to get the fixed amplitude of the steady oscillation, one can apply the Laplace transformation to the series and find the Padé approximant to the terms containing ε^0, ε^1, $\varepsilon^2, \dots$ separately, and obtain the solution. The value of the amplitude for which the higher order Padé approximant is reduced to the previous order Padé approximant gives the amplitude of the steady oscillation. The method is illustrated with the help of examples.

For comparison with the perturbative solutions, the parameter ε is taken to be small in these examples.

3. EXAMPLES

Example 1. The Duffing Equation

Consider the equation

$$\frac{d^2u}{dt^2} + u + \varepsilon u^3 = 0, \qquad u(0) = a, \quad u'(0) = 0.$$

By Adomian's method, we have

$$
\begin{aligned}
u_0 &= a, \\
u_1 &= -a\,(1+\varepsilon\,a^2)\,\frac{t^2}{2!}, \\
u_2 &= a\,(1+\varepsilon\,a^2)\,(1+3\varepsilon\,a^2)\,\frac{t^4}{4!}, \\
u_3 &= -a\,(1+\varepsilon\,a^2)\,(27\,\varepsilon^2\,a^4+24\,\varepsilon\,a^2+1)\,\frac{t^6}{6!}, \\
u_4 &= \left[a\,(1+4\varepsilon\,a^2+3\varepsilon^2\,a^4)\,(1+114\,\varepsilon\,a^2+117\,\varepsilon^2\,a^4)+90\,\varepsilon\,a^3\,(1+\varepsilon\,a^2)^3\right]\frac{t^8}{8!}.
\end{aligned}
$$

We take

$$\phi_5 = u_0 + u_1 + u_2 + u_3 + u_4. \tag{1}$$

Here, we apply the Laplace transform to ϕ_5 and find the [4/4] Padé approximant of the resulting series. The [4/4] approximant is

$$s\,L(u) = \frac{s^4 + (19\,\varepsilon\,a^2+9)\,s^2 + 16\,\varepsilon^2\,a^4}{s^4+(10+20\,\varepsilon\,a^2)\,s^2+(33\,\varepsilon^2\,a^4+26\,\varepsilon\,a^2+9)}.$$

Hence,

$$
\begin{aligned}
L(u) &= \frac{A}{s} + \frac{B\,s}{s^2+p} + \frac{C\,s}{s^2+q}, \qquad \text{where} \\
p &= 5\,(1+2\varepsilon\,a^2) + (67\,\varepsilon^2\,a^4+74\,\varepsilon\,a^2+16)^{1/2}, \\
q &= 5\,(1+2\varepsilon\,a^2) - (67\,\varepsilon^2\,a^4+74\,\varepsilon\,a^2+16)^{1/2}, \\
A &= \frac{16\,\varepsilon^2\,a^4}{33\,\varepsilon^2\,a^4+26\,\varepsilon\,a^2+9}, \\
C &= \frac{A\,(q-10-20\,\varepsilon\,a^2)+19\,\varepsilon\,a^2+9-q}{p-q}, \qquad \text{and} \\
B &= 1 - A - C.
\end{aligned}
$$

On inverting the approximant, we get periodic functions in the solution, for all values of the amplitude a, as p and q are positive for all a. The solution is

$$u = a\left\{A + B\cos(\sqrt{p}\,t) + C\cos(\sqrt{q}\,t)\right\},$$

which exhibits periodicity as is characteristic of the equation. The solution holds for arbitrary a, as the system is conservative. The solution is compared with that of the perturbative solution [4], for $a = 1$ and $\varepsilon = 0.1$, in Figure 1.

Example 2. The Linear Damping Oscillator Equation

Consider the linear damping oscillator equation:

$$\frac{d^2u}{dt^2} + u = -2\varepsilon\,\frac{du}{dt}, \qquad u(0) = a, \quad u'(0) = 0.$$

By the decomposition method, we have

$$
\begin{aligned}
u_0 &= a, \\
u_1 &= -a\,\frac{t^2}{2!}, \\
u_2 &= 2a\,\varepsilon\,\frac{t^3}{3!} + a\,\frac{t^4}{4!}, \\
u_3 &= -4a\,\varepsilon^2\,\frac{t^4}{4!} - 4a\,\varepsilon\,\frac{t^5}{5!} - a\,\frac{t^6}{6!}, \\
\phi_4 &= u_0 + u_1 + u_2 + u_3.
\end{aligned} \tag{2}
$$

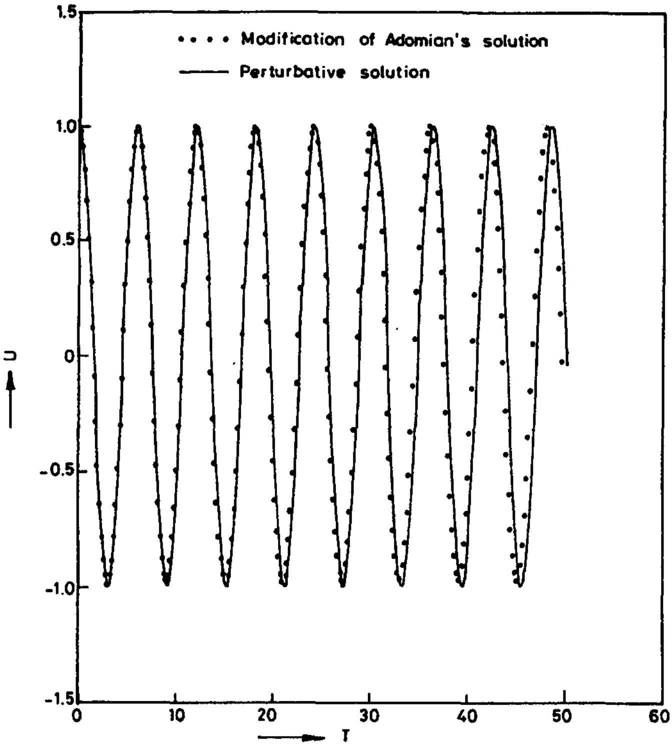

Figure 1. Solution of the Duffing equation.

The [2/2] Padé approximant of the Laplace transform of the series (2) is

$$L(u) = \frac{a\,(s+2\varepsilon)}{s^2+2\varepsilon+1}.$$

On inverting, we have

$$u = a\,e^{-\varepsilon t}\left\{\cos(b\,t) + \frac{\varepsilon}{b}\,\sin(b\,t)\right\},$$

where $b = (1-\varepsilon^2)^{1/2}$. Though the equation is nonconservative, it is linearly dissipative and, hence, the amplitude is a constant. This solution is compared with the exact solution for $a = 1$ and $\varepsilon = 0.1$, in Figure 2. Here, the solution exhibits oscillatory behaviour only when $\varepsilon < 1$.

Example 3. The Vanderpol Equation

The problem to be solved is

$$\frac{d^2u}{dt^2} + u = \varepsilon\,(1-u^2)\,\frac{du}{dt}, \qquad u(0) = a, \quad u'(0) = 0.$$

Using the decomposition method, we have

$$\begin{aligned}
u_0 &= a,\\
u_1 &= -a\,\frac{t^2}{2!},\\
u_2 &= -a\,\varepsilon\,(1-a^2)\,\frac{t^3}{3!} + a\,\frac{t^4}{4!},\\
u_3 &= -a\,\varepsilon^2\,(1-a^2)^2\,\frac{t^4}{4!} + 2a\,\varepsilon\,(1-4a^2)\,\frac{t^5}{5!} - a\,\frac{t^6}{6!},\\
u_4 &= -a\,\varepsilon^3\,(1-a^2)^3\,\frac{t^5}{5!} + a\,\varepsilon^2\,(1-32\,a^2+29\,a^4)\,\frac{t^6}{6!} + a\,\varepsilon\,(69\,a^2-3)\,\frac{t^7}{7!} + a\,\frac{t^8}{8!},\\
\phi_5 &= u_0+u_1+u_2+u_3+u_4.
\end{aligned} \tag{3}$$

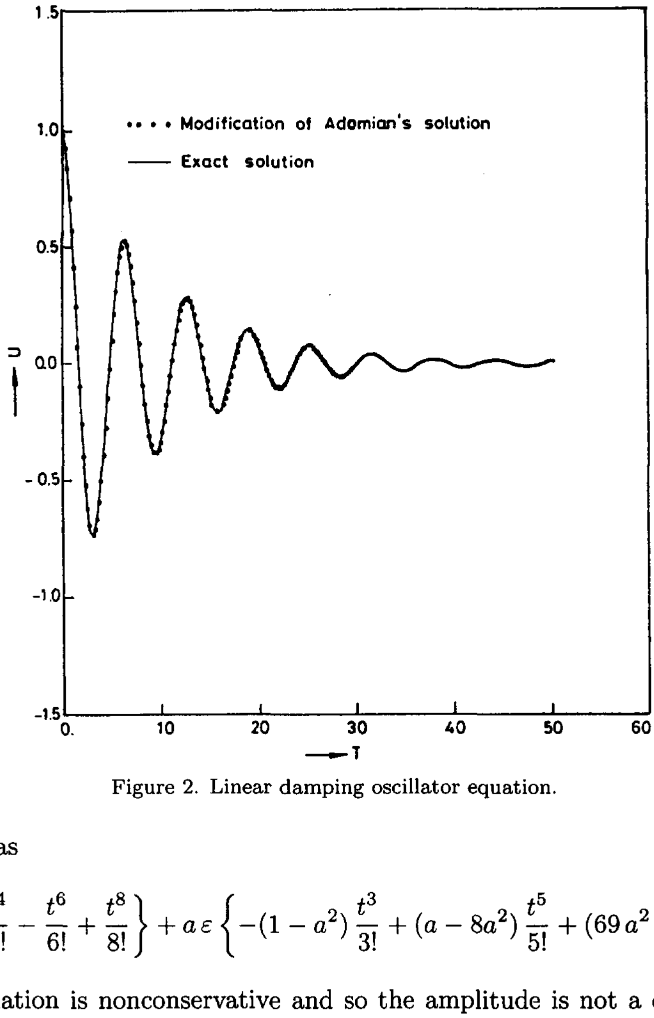

Figure 2. Linear damping oscillator equation.

Here, we write (3) as

$$u = a\left\{1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!}\right\} + a\,\varepsilon\left\{-(1-a^2)\,\frac{t^3}{3!} + (a - 8a^2)\,\frac{t^5}{5!} + (69\,a^2 - 3)\,\frac{t^7}{7!}\right\} + \cdots. \quad (4)$$

The Vanderpol equation is nonconservative and so the amplitude is not a constant. We apply the Laplace transform to the series [3] and find the Padé approximant for the terms containing ε^0, ε^1, $\varepsilon^2, \dots$ separately, and on inverting the [4/4] approximant, we have

$$u = a\cos(t) + a\,\varepsilon(a^2-1)\left\{\frac{b}{\alpha}\,\sin(\alpha t) - \frac{b}{\beta}\,\sin(\beta\,t)\right\}, \qquad \text{where}$$

$$b = \frac{1-a^2}{2a\,(21\,a^2 - 48)},$$

$$\alpha = \left(\frac{4a^2 - 1 + a\,(21\,a^2 - 48)^{1/2}}{a^2-1}\right)^{1/2},$$

$$\beta = \left(\frac{4a^2 - 1 - a\,(21\,a^2 - 48)^{1/2}}{a^2-1}\right)^{1/2}.$$

The [4/6] approximant reduces to the [4/4] approximant for $a = 2$. From perturbative techniques, it is known that steady oscillations occur when $a = 2$. The graph of the steady oscillation is sketched in Figure 3 and is compared with the perturbative solution [4].

Example 4. The Rayleigh Equation

The Rayleigh Equation is

$$\frac{d^2u}{dt^2} + u = \varepsilon\left[\frac{du}{dt} - \frac{1}{3}\left(\frac{du}{dt}\right)^3\right], \qquad u(0) = 0, \quad u'(0) = a.$$

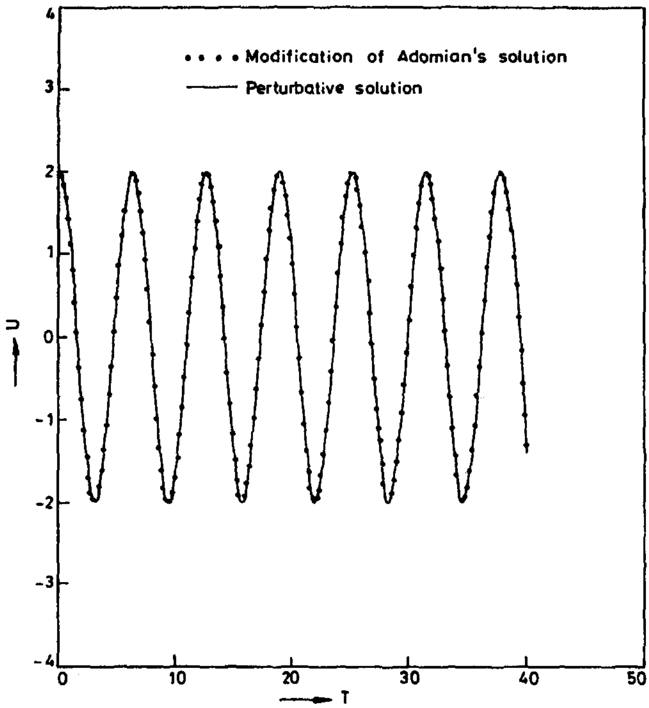

Figure 3. Solution of the Vanderpol equation for $a = 2$.

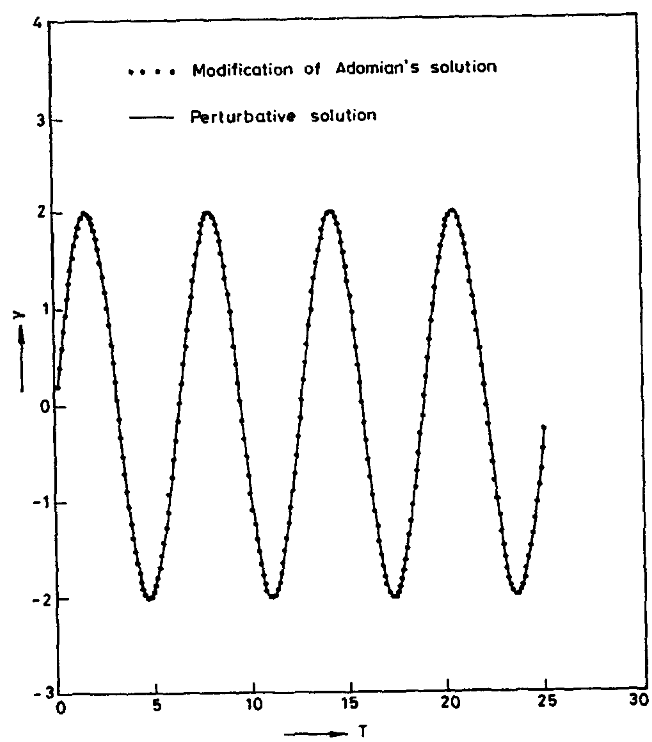

Figure 4. Solution of the Rayleigh equation for $a = 2.058171$.

By the decomposition method, the solution is

$$u_0 = a\,t,$$
$$u_1 = a\,\varepsilon\left(1 - \frac{1}{3}a^2\right)\frac{t^2}{2!} - a\,\frac{t^3}{3!},$$

$$u_2 = a\,\varepsilon^2\left(1 - \frac{4}{3}a^2 + \frac{1}{3}a^4\right)\frac{t^3}{3!} - a\,\varepsilon\left(2 - \frac{4}{3}a^2\right)\frac{t^4}{4!} + a\,\frac{t^5}{5!}.$$

By separating terms containing ε^0, $\varepsilon^1, \dots$ we have

$$\phi_3 = a\left(t - \frac{t^3}{3!} + \frac{t^5}{5!}\right) + a\,\varepsilon\left\{\left(1 - \frac{1}{3}a^2\right)\frac{t^2}{2!} - \left(2 - \frac{4}{3}a^2\right)\frac{t^4}{4!}\right\}. \tag{5}$$

The inversion of the [3/2] Padé approximant to the Laplace transform of (5) gives the solution as

$$u = a\sin(t) + \frac{a\,\varepsilon}{3}\,(3 - a^2)\,b\cos(t/\sqrt{b}),$$

where $b = (3 - a^2)/(6 - 4a^2)$. The [3/4] approximant reduces to the [3/2] approximant for $a = 2.058171$. From the solution obtained by perturbative techniques [4], the stable oscillation is attained at $a = 2$. Figure 4 represents the graph of the steady oscillation.

4. CONCLUSION

The examples considered show that the method illustrated above yields a more convenient form of the solution compared to the series solution of the decomposition method for a class of nonlinear oscillatory problems. The solution exhibits periodic behaviour and is compared with the perturbative solutions.

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