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Chaos in the one-dimensional wave equation

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Abstract

This paper deals with the chaotic behavior of the solutions of a mixed problem for the one-dimensional wave equation with a quadratic boundary condition. This behavior is studied through the connection between the energy function and quadratic discrete dynamical systems.

Keywords: Wave equation; Quadratic maps; Chaos

1. Introduction

In the literature there are plenty of examples of dynamical systems that exhibit chaotic behavior [2,3]. In spite of the complexity of these behaviors, sometimes chaos occurs in systems that are relatively simple [4,5]. Recently, in the paper [2], the appearance of chaos in the one-dimensional wave equation with a cubic boundary condition was shown. The authors studied the dynamics of the solution at the nonlinear boundary by generating a discrete dynamical system and suggesting that chaos is due to the changes of sign of the derivative of the energy function, the so-called 'self-excited oscillations'.

The aim of this paper is to contribute to the explanation of the appearance of chaotic behavior in the one-dimensional wave equation. We consider a mixed problem for this equation with a special quadratic boundary condition that generates families of general type quadratic discrete dynamical systems.

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We show that by choosing a family of discrete dynamical systems the change of sign of the derivative of the energy function is not a necessary condition for the appearance of chaos. Also, we show the numerical difficulties that arise due to the presence of the nonlinear boundary condition.

This paper is organized as follows. In Section 2 we present the mixed problem concerning the one-dimensional wave equation and derive from it a family of discrete dynamical systems, which describe the behavior of the solution at the nonlinear boundary domain. We also study the relationship between the energy function and parametric coefficients of the introduced discrete systems. In Section 3 we draw consequences about the discrete systems regarding periodicity and chaotic behavior. Finite difference methods are also used to solve the wave equation to see what the effect of the discretization errors on the chaotic behavior is. Finally, in Section 4 the general conclusions of the work are summarized.

2. Wave equation

Consider the one-dimensional wave equation

$$u_{tt}(x,t) = c\triangle u(x,t)$$

with initial conditions

$$u(x,0) = f(x) \qquad \text{and} \qquad u_t(x,0) = g(x) \tag{1}$$

and boundary conditions $u(t,0) = 0$ and

$$(u_x(1,t) - u_t(1,t))^2 - 2b(\lambda)(u_x(1,t) - u_t(1,t)) - 2(u_x(1,t) + u_t(1,t)) = 0. \tag{2}$$

We are interested in the solution to the above system in the strip $0 < x < 1$ and $t > 0$. It is well known that the solution can be obtained by using characteristics: let $\psi = x + ct$ and $\eta = x - ct$.

Let

$$w(x,t) = \frac{u_x(x,t) + u_t(x,t)}{2} \tag{3}$$

and

$$z(x,t) = \frac{u_x(x,t) - u_t(x,t)}{2}, \tag{4}$$

so with this condition w and z are constant along $\psi = \xi_1$ and along $\eta = \xi_2$ respectively, where ξ_1 and ξ_2 are constants. Thus, w and z satisfy the equations $w_t - cw_x = 0$ and $z_t + cz_x = 0$ with initial conditions given by

$$w(x,0) = \frac{f'(x) + g(x)}{2}$$

$$z(x,0) = \frac{f'(x) - g(x)}{2}$$

whereas using (3) and (4) the boundary conditions become

$$w(0,t) = z(0,t)$$

and

$$z^2(1,t) - b(\lambda)(z(1,t)) = w(1,t). \tag{5}$$

The boundary condition (5) becomes

$$z(1,t) = H(w(1,t)) \quad (6)$$

where

$$H(w) = \frac{b(\lambda) - \sqrt{b^2(\lambda) - 4w}}{2}.$$

Let us study the behavior of a sequence of values of the solution reflected in the nonlinear boundary at $x = 1$ along the characteristic curves starting from $(x_o, 0)$. The characteristic curve $x - ct = x_o$ intersects the boundary $x = 1$ at the point $(1, \tau_o)$ where $\tau_o = \frac{1-x_o}{c}$. Here we have that $z(1, \tau_o) = z(x_o, 0)$ and $w(1, \tau_o) = H^{-1}(z(x_o, 0))$, where

$$H^{-1}(z) = z^2 - b(\lambda)z.$$

Departing from the point $(1, \tau_o)$ along the characteristic $x + ct = 2 - x_o$ intersect the boundary $x = 0$ at the point $(0, \tau_1)$ where $\tau_1 = \frac{2-x_o}{c}$ here we have that $z(0, \tau_1) = w(0, \tau_1) = H^{-1}(z(x_o, 0))$. Notice that, along the characteristic $x - ct = x_o - 2$, z is constant, therefore its value at $x = 1$ and $\tau_2 = \frac{3-x_o}{c}$ is given by $z(1, \tau_2) = H^{-1}(z(x_o, 0))$ and

$$w(1, \tau_2) = H^{-1}(z(1, \tau_2)) = H^{-1} \circ H^{-1}(z(x_o, 0))$$

or

$$w(1, \tau_2) = H^{-1}(w(1, \tau_o)).$$

By induction we can write

$$w(1, \tau_{2n}) = H^{-1}(w(1, \tau_{2(n-1)})) \quad (7)$$

where $\tau_{2n} = \frac{2n+1-x_o}{c}$. That is, the solution can be obtained by iterates of H^{-1}.

If we define $q_n = w(1, \tau_{2n})$ then Eq. (7) takes the form

$$q_{n+1} = q_n^2 - b(\lambda)q_n \quad (8)$$

which is a quadratic unimodal map.

Notice that the discrete dynamical system (8) turns out from the general quadratic dynamical system of the form

$$x_{n+1} = \alpha(\lambda)x_n^2 + \beta(\lambda)x_n + \gamma(\lambda)$$

taking the substitution $x_n = Aq_n + B$ with $A = \alpha(\lambda)^{-1}$ and $B = (2\alpha)^{-1}(\beta - \sqrt{\beta^2 - 4\alpha\gamma})$.

Let us study the relationship between the derivative of the energy function and the coefficient $b(\lambda)$ arising in (8). The energy of the system is given by

$$E(t) \equiv \frac{1}{2}\int_0^1 (u_x^2 + u_t^2)\,\mathrm{d}x$$

so

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = u_t(1,t)u_x(1,t)$$

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = u_t(1,t)\left(u_t(1,t) + b(\lambda) + 1 - \sqrt{(b(\lambda)+1)^2 + 4u_t(1,t)}\right).$$

For values of $w_t(1,t)$ small enough, we have that

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = u_t(1,t)\left(\frac{b(\lambda)-1}{b(\lambda)+1}\right) + o(u_t^2(1,t)).$$

Thus the sign of the derivative of the energy depends only on the value $b(\lambda)$ with respect to one. That is, if $0 < b(\lambda) < 1$ then the sign of $\frac{\mathrm{d}E(t)}{\mathrm{d}t}$ is equal to the sign of $-u_t(1,t)$ while if $b(\lambda) > 1$ then the sign of $\frac{\mathrm{d}E(t)}{\mathrm{d}t}$ coincides with the sign of $u_t(1,t)$. Thus the sign of the derivative of the energy depends locally on $b(\lambda)$.

In the next section we study the discrete system (8) for different functions $b(\lambda)$.

3. Chaos and the role of $b(\lambda)$

Let us choose $b(\lambda)$ as an unimodal function, given by

$$b(\lambda) = \frac{4\alpha}{\beta^2}\lambda(\beta - \lambda), \tag{9}$$

where $\beta > 0$ and $0 < \alpha$. Here α and β are the maximum value and a zero of the function $b(\lambda)$ respectively. For this example it will be interesting to show the different scenarios obtained for different values of α, since the behavior of the system $q_{n+1} = q_n^2 - b(\lambda)q_n$ depends strongly on the value of α. If $0 < \alpha < 1$ then $\lim_{n\to\infty} q_n = 0$ for every value of λ. If $1 < \alpha < \sqrt{6} - 1$ then we obtain a bifurcation diagram for q_n that consists of a straight line with an asymmetric close loop homeomorphic to a circle; see Fig. 1(a). For values of $\sqrt{6} - 1 < \alpha < 1.57$ then we have bifurcation diagrams consisting of a straight line with a collection of nested loops. It is in this range where periodic behavior takes place. If $\sqrt{6} - 1 < \alpha < 1.5441$ then there are two loops, as is shown in Fig. 1(b). If $1.5441 < \alpha < 1.564$ then there are four loops, see Fig. 1(c), and so on. However, if $\alpha > 1.57$ we have a chaotic behavior, as is shown in the bifurcation diagram of Fig. 1(d).

3.1. *Finite differences methods*

There are not many hyperbolic partial differential equations that can be solved exactly and that have chaotic solutions. So this one-dimensional wave equation with the nonlinear right boundary condition gives a good case study to see what is the effect of the discretization on the behavior of the solutions. It is well known that for the one-dimensional wave equation with Dirichlet boundary conditions, the standard explicit centered differences scheme with equal uniform time and space steps produces the exact solution except for round-off errors [1]. For a time step, Δt, different from the space step, Δx, the truncation error is $\mathcal{O}((\Delta t)^2 + (\Delta x)^2)$. To have the same degree of accuracy in the approximation of the right boundary condition, we approximate both the time and spatial first derivatives using centered differences. Since the x-derivative is evaluated at $x = 1$, it is necessary to add a fictitious line of nodes at $x = 1 + \Delta x$. From evaluating the approximation to the wave equation at $x = 1$ and from the approximation to the right boundary condition, the values at the fictitious nodes can be eliminated. Centered time differences require initial values at $t = 0$ and $t = \Delta t$. Ref. [1] shows how to use a Maclaurin series to get second order approximations at $t = \Delta t$. In our case, we could also use the exact value at $t = \Delta t$, and in some runs we did.

Calculations were done using $\Delta x = 1/100$, $\Delta x = 1/200$ and $\Delta x = 1/400$ with $\Delta t = \Delta x$, $\Delta t = \Delta x/10$ and $\Delta t = \Delta x/100$. Values of α, β and λ were chosen to see if we could reproduce the bifurcation diagrams of Fig. 1. An initial condition $f(x) = \sin(\pi x)$ and initial derivative $g(x) = 0$ were chosen. Some subharmonics were also used. In all the calculations, the values at the right boundary $x = 1$

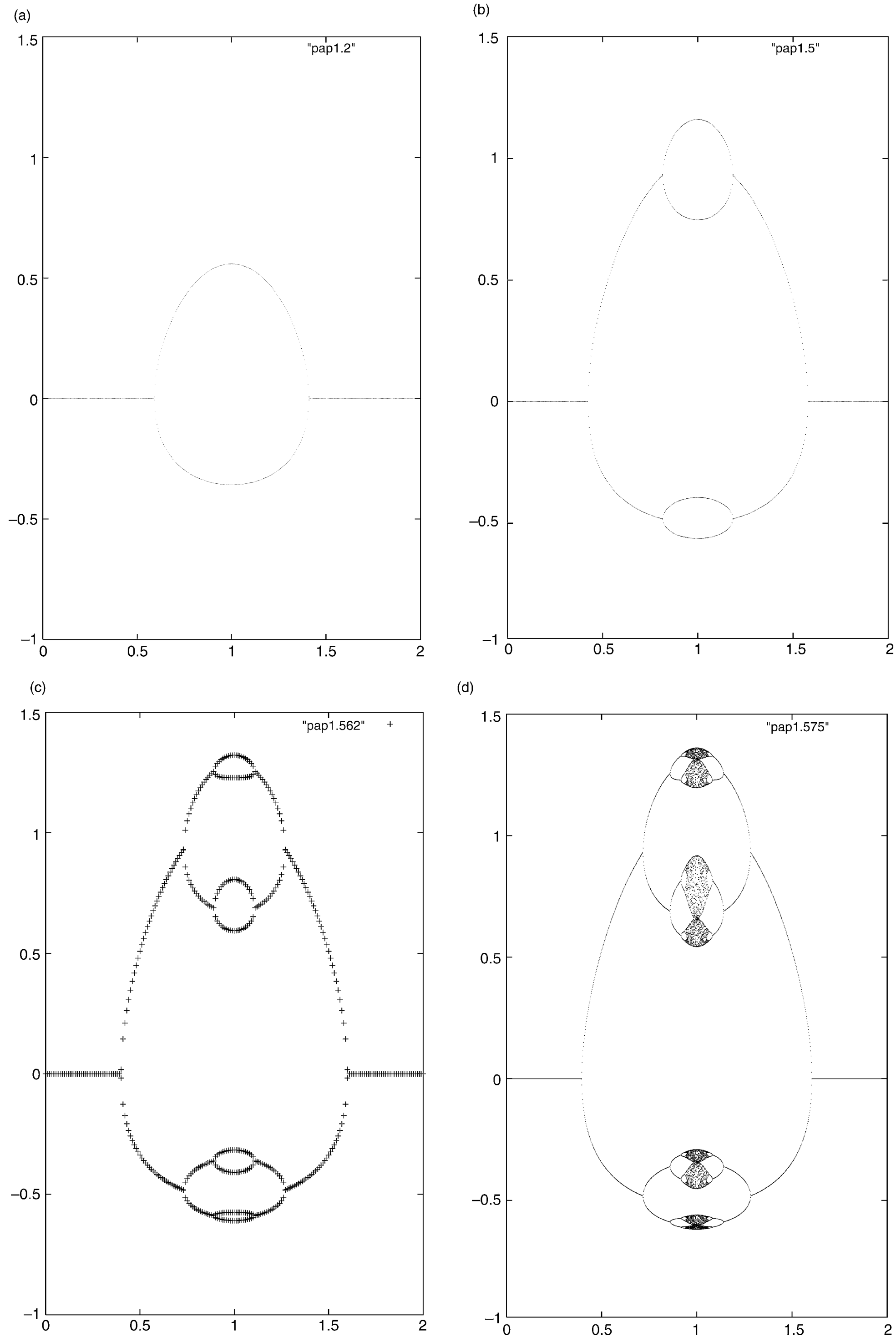

Fig. 1. Regular reversal maps for $\alpha = 1.2$, 1.5, 1.562 and 1.575 respectively.

started to grow and propagate into the domain, and eventually the program tried to take the square root of a negative number. The numerical approximation of that boundary condition creates an instability. Since the numerical scheme for the wave equation is stable for all $\Delta t/\Delta x \leq 1$, we expect that any instabilities are due to the boundary condition. There are no general methods to study nonlinear instabilities such as the one we are getting. Reducing the time step does not help control the instability.

4. Conclusions

We have shown that the appearance of periodic/chaotic behavior in the one-dimensional wave equation depends on the behavior of the solution at the nonlinear boundary condition. For bounded functions $b(\lambda)$ it is possible to obtain only periodic solutions and for unbounded functions there is always periodic and chaotic behavior.

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