Interpolation by universal, hypercyclic functions

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Abstract

In the present article we study an interpolation problem for classes of analytic functions, in a systematic manner. More precisely, we provide sufficient conditions so that proper and “big”, in the Baire category sense, subclasses of analytic functions have an interpolation property at an infinite set of points. We then apply our main theorems to several classes of universal, hypercyclic functions.

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1. Introduction

In [23], Maclane proved the existence of a universal entire function $f$ with respect to differentiation, namely the sequence of its derivatives $\{f^{(n)} : n \in \mathbb{N}\}$ is dense in the space $H(\mathbb{C})$ of the entire functions endowed with the topology of uniform convergence on compacta. Later in [13], Duyos Ruiz proved that the above notion of universality is a generic property, that is the set of universal entire functions is $G_δ$ and dense in $H(\mathbb{C})$ and Gethner and Shapiro in [15] and Grosse-Erdmann in [16] extended the previously mentioned results for functions, analytic on any simply connected domain. These properties are particular instances of a general phenomenon in analysis, so called hypercyclicity. Let us give the precise definition of this notion as it appeared in the very influential survey article of Grosse-Erdmann [17].

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Definition 1.1. Let $X$ be a topological vector space. A sequence of linear and continuous operators $T_n : X \to X, n = 1, 2, \ldots$ is said to be hypercyclic if there exists a vector $x \in X$ so that the sequence
\[ \{T_1 x, T_2 x, \ldots\}, \]
is dense in $X$. In this case the vector $x$ will be called hypercyclic for $\{T_n\}$ and the symbol $HC(\{T_n\})$ stands for the set of hypercyclic vectors for $\{T_n\}$. If the sequence $\{T_n\}$ comes from the iterates of a single operator $T$, i.e. $T_n = T^n, n = 1, 2, \ldots$ then $T$ is called hypercyclic and the set of hypercyclic vectors for $T$ is denoted by $HC(T)$.

There are many examples of such operators and several deep results, in this area, have been established during the last decade, see, for instance, the survey articles [7,17,18,26,32] and the recent books [3,19]. It is well known that under some mild assumptions on $X$, if $T : X \to X$ is hypercyclic then the set $HC(T)$ is $G_\delta$ and dense in $X$, i.e. a countable intersection of open and dense sets (see [17]).

Let $\Omega \subset \mathbb{C}$ be an open set and let $H(\Omega)$ be the space of analytic functions in $\Omega$ endowed with the topology of uniform convergence on compacta. It is well known that for every sequence $\{a_n\}_{n \in \mathbb{N}}$ of points in $\Omega$, having no accumulation point in $\Omega$ and for every sequence $\{b_n\}_{n \in \mathbb{N}}$ of complex numbers, there exists a function $f \in H(\Omega)$ such that $f(a_n) = b_n, n = 1, 2, \ldots$. The general question we would like to address in the present work is the following: if we consider a “large” subset $\mathcal{A}$ of $H(\Omega)$, say $\mathcal{A}$ is $G_\delta$ and dense in $H(\Omega)$, can we retain this interpolation property? In particular, for which classes of universal-hypercyclic functions does such an interpolation property hold? Roughly speaking, we aim to show that interpolation properties are compatible with universality.

Let us mention that in [10], an analogous interpolation problem was studied within the class of universal Taylor series on simply connected domains. The proof of the later result involves the following lemma due to Herzog [20].

Lemma 1.2. Let $(X, d_X)$ and $(Y, d_Y)$ be a metric Polish space and a separable metric space, respectively. Let $\{L_n\}$ be a sequence of continuous mappings from $X$ to $Y$. Assume that the set $U(\{L_n\})$ of all universal elements (i.e. of all points $x$ with $\{L_n x : n = 1, 2, \ldots\}$ dense in $Y$) is residual in $X$. For a sequence $\{A_k\}$ of open subsets of $X$, put $A = \bigcap_{k=1}^\infty A_k \neq \emptyset$ and let $L_n|_A$ denote the restriction of $L_n$ on $A$. If
\[ \lim_{k \to \infty} \sup_{n \in \mathbb{N}} \inf_{z \in A} (d_X(a_k, z) + d_Y(L_n a_k, L_n z)) = 0 \]
for every sequence $\{a_k\}$ with $a_k \in A_k$ for $k \in \mathbb{N}$, then $U(\{L_n|_A\})$ is residual in $A$.

At a first glance, Herzog’s lemma seems a little bit technical. However, a very nice discussion in [18] clarifies many issues concerning the meaning of the above lemma and its consequences. So far all the existing results establishing interpolation type properties of universal-hypercyclic functions rely on Lemma 1.2., see [4,5,10,29,30]. Our approach, which avoids Herzog’s lemma, is more elementary.

Before stating our first main theorem it is convenient to introduce the following definitions.

Definition 1.3. We say that a subset $\mathcal{A}$ of $H(\Omega)$ has the interpolation property with respect to a sequence $\{a_n\}_{n \in \mathbb{N}}$ of complex numbers in $\Omega$, if for every sequence $\{b_n\}_{n \in \mathbb{N}}$ of complex numbers there exists a function $f \in \mathcal{A}$ such that $f(a_n) = b_n, n = 1, 2, \ldots$. A sequence $\{a_n\}_{n \in \mathbb{N}}$ of points in $\Omega$ is called admissible if it has no accumulation point in $\Omega$. Moreover, if a subset $\mathcal{A}$ of $H(\Omega)$
has the interpolation property with respect to any admissible sequence \( \{a_n\}_{n \in \mathbb{N}} \) of points in \( \Omega \) then we simply say that \( \mathcal{A} \) has the interpolation property.

**Definition 1.4.** Let \( \Omega \) be an open subset of the complex plane. We say that an open subset \( \mathcal{D} \) of \( H(\Omega) \) has the property \((\ast)\) if there exists a fixed compact set \( L_\mathcal{D} \subset \Omega \) such that every function \( g \in \mathcal{D} \) has an open neighborhood \( V_g \subset \mathcal{D} \) of the form

\[
V_g = \left\{ f \in H(\Omega) : \sup_{z \in L_\mathcal{D}} |f(z) - g(z)| < \varepsilon \right\}
\]

for some \( \varepsilon = \varepsilon(g) > 0 \).

We are now in a position to state our first main result.

**Theorem 1.5.** Let \( \Omega \) be an open subset of \( \mathbb{C} \) and let \( \mathcal{A} \) be a \( G_\delta \) and dense subset of \( H(\Omega) \). Then \( \mathcal{A} \) has the interpolation property if the following two conditions are satisfied:

(i) The set \( \mathcal{A} \) can be written as a denumerable intersection of open sets \( A_j \), \( j = 1, 2, \ldots \) so that each \( A_j \) has the property \((\ast)\).

(ii) If \( f \in \mathcal{A} \) and \( p \) is a polynomial then \( f + p \in \mathcal{A} \).

The proof of Theorem 1.5 is given in Section 2. As we shall see in the next section, Theorem 1.5 can be applied to several classes of universal functions, e.g. universal functions with respect to differentiation and universal Taylor series of various types.

In order to state our second main result we introduce the following

**Definition 1.6.** Let \( \Omega \) be an open set of \( \mathbb{C} \) and let \( \{a_n\}_{n \in \mathbb{N}} \) be an admissible sequence in \( \Omega \). We say that an open subset \( \mathcal{D} \) of \( H(\Omega) \) has the property \((\ast\ast)\) with respect to \( \{a_n\}_{n \in \mathbb{N}} \) if for every compact set \( L \subset \Omega \) there exists a function \( g \in \mathcal{D} \) such that \( g \) has an open neighborhood of the form

\[
V_g = \left\{ f \in H(\Omega) : \sup_{z \in L_g} |f(z) - g(z)| < \varepsilon \right\}
\]

for some \( \varepsilon > 0 \) and \( L_g \subset \Omega \) compact and disjoint from \( L \cup \{a_n : n \in \mathbb{N}\} \).

We are now ready to state our second main theorem.

**Theorem 1.7.** Let \( \Omega \) be an open subset of \( \mathbb{C} \), \( \mathcal{A} \) be a \( G_\delta \) and dense subset of \( H(\Omega) \) and \( \{a_n\}_{n \in \mathbb{N}} \) be an admissible sequence of points in \( \Omega \). Suppose that \( \mathcal{A} \) can be written as a countable intersection of open sets \( A_j \), \( j = 1, 2, \ldots \) such that each \( A_j \) has the property \((\ast\ast)\) with respect to \( \{a_n\}_{n \in \mathbb{N}} \). Then \( \mathcal{A} \) has the interpolation property with respect to the sequence \( \{a_n\}_{n \in \mathbb{N}} \).

The proof of Theorem 1.7, which shares certain similarities with the proof of Theorem 1.5, is given is Section 3. Using Theorem 1.7 we will deal with the problem of interpolation in classes of universal functions with respect to translations and with respect to translations–dilations. In this case, some natural restrictions for the sequence \( \{a_n\}_{n \in \mathbb{N}} \) appear, due to the nature of the universality under consideration. For details see Section 3.

2. **Proof of Theorem 1.5**

Before starting the proof of Theorem 1.5, let us state a well known lemma which will be used frequently throughout this paper; we give its proof for completeness.
**Remark 1.** Let \( \tilde{f} \) defined on \( L \) such that

\[
\sup_{z \in L} |\tilde{f}(z) - g(z)| \leq \sup_{z \in L} |f(z) - g(z)|(1 + M)
\]

and

\[
\tilde{f}(a_{\ell}) = g(a_{\ell}), \quad \ell = 1, 2, \ldots, k,
\]

where \( M \) is a positive number depending only on \( L \) and the points \( a_1, a_2, \ldots, a_k \).

**Proof.** Let \( p(z) = (z - a_1) \cdots (z - a_k) \) and set \( M = \sum_{\ell=1}^{k} \sup_{z \in L} \frac{|p(z)|}{|z-a_{\ell}| |p'(a_{\ell})|} \). Then

\[
\tilde{f}(z) = \sum_{\ell=1}^{k} \left[ g(a_{\ell}) - f(a_{\ell}) \right] \frac{p(z)}{(z-a_{\ell}) p'(a_{\ell})} + f(z)
\]

has the desired properties. \( \Box \)

**Remark 1.** Note that \( \tilde{f} \) equals \( f \) plus a polynomial.

**Proof of Theorem 1.5.** Let \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence of points in \( \Omega \) having no accumulation point in \( \Omega \) and let \( \{b_n\}_{n \in \mathbb{N}} \) be any sequence of complex numbers. Set

\[
\Gamma = \{ f \in H(\Omega) : f(a_n) = b_n, \ n = 1, 2, \ldots \}.
\]

Then the set \( \Gamma \) is a closed subset of \( H(\Omega) \), thus endowed with the topology of uniform convergence on compacta, it is a complete space. In view of Baire’s category theorem it suffices to prove that the set \( \mathcal{A} \cap \Gamma \) is equal to a denumerable intersection of open and dense sets in \( \Gamma \). \( \Box \)

Now by the assumptions, \( \mathcal{A} = \bigcap_{j=1}^{\infty} \mathcal{A}_j \), where each \( \mathcal{A}_j \) is an open and dense subset of \( H(\Omega) \) and in addition each \( \mathcal{A}_j \) has the property (\( \ast \)). Obviously \( \mathcal{A} \cap \Gamma = \bigcap_{j=1}^{\infty} (\mathcal{A}_j \cap \Gamma) \) and \( \mathcal{A}_j \cap \Gamma \) is open in \( \Gamma \) for every \( j = 1, 2, \ldots \). We will prove that for every \( j \), \( \mathcal{A}_j \cap \Gamma \) is dense in \( \Gamma \) and the result will follow.

For this reason, we fix an index \( j_0 \), a function \( h \in \Gamma \), a compact set \( L \subset \Omega \) and a positive number \( \varepsilon < 1 \). Our aim is to find a function \( g \in \mathcal{A}_{j_0} \cap \Gamma \) with the property:

\[
\sup_{z \in L} |g(z) - h(z)| < \varepsilon.
\]

We consider an exhausting sequence \( \{L_N\}_{N \geq 1} \) of compact subsets of \( \Omega \) (see [31]). Since \( \mathcal{A}_{j_0} \) has the property (\( \ast \)), we may also fix a compact set \( L_{A_{j_0}} \) as mentioned in the property. We assume, without loss of generality, that the compact set \( L \cup L_{A_{j_0}} \) is contained in \( L_1 \).

Because the sequence \( \{a_n\}_{n \in \mathbb{N}} \) has no accumulation point in \( \Omega \), the set \( \{a_n : n \in \mathbb{N}\} \cap L_k \) is finite for every \( k \in \mathbb{N} \). Therefore, without loss of generality (changing the enumeration of \( \{a_n\}_{n \in \mathbb{N}} \), if necessary), we may assume that for every \( k \in \mathbb{N} \):

\[
\{a_1, \ldots, a_{m_k}\} \subset L_k
\]

and

\[
\{a_{m_k+1}, \ldots\} \cap L_k = \emptyset,
\]

where \( \{m_k\}_{k \in \mathbb{N}} \) is an increasing sequence (not necessary strictly increasing) of natural numbers.

We will achieve our goal by constructing a sequence of functions \( \{g_k\}_{k \in \mathbb{N}} \) with the following properties:

1. \( \sup_{z \in L_k} |h(z) - g_k(z)| < \varepsilon/2, k = 1, 2, \ldots \),
2. \( \sup_{z \in L_k} |g_{k+1}(z) - g_k(z)| < 1/2^k, \ k = 1, 2, \ldots \),
3. $g_k \in V$, for a closed set $V \subset \mathcal{A}_{j_0}$ to be chosen later,
4. $g_k(a_\ell) = b_\ell$, $\ell = 1, 2, \ldots, m_k$.

Because of property 2, the sequence $\{g_k\}_{k \in \mathbb{N}}$ will be a Cauchy sequence in $H(\Omega)$, thus it will converge to a function $g \in H(\Omega)$. It is then easy to see that this function, $g$, has the desired properties.

CONSTRUCTION OF $\{g_k\}_{k \in \mathbb{N}}$. We proceed by induction.

**Step 1.** We construct the function $g_1$. Since $\mathcal{A}$ is dense in $H(\Omega)$, there exists a function $\tilde{g}_1 \in \mathcal{A}$, such that:

$$\sup_{z \in L_1} |h(z) - \tilde{g}_1(z)| < \frac{\varepsilon}{2^2(1 + M_1)},$$

where $M_1$ is the positive number of Lemma 2.1 for the points $a_1, a_2, \ldots, a_{m_1}$ and the compact set $L_1$. We now apply Lemma 2.1 and we obtain a function $g_1 \in H(\Omega)$ such that:

$$\sup_{z \in L_1} |h(z) - g_1(z)| \leq \sup_{z \in L_1} |h(z) - \tilde{g}_1(z)|(1 + M_1) < \frac{\varepsilon}{2^2}$$

and

$$g_1(a_\ell) = h(a_\ell) = b_\ell, \quad \ell = 1, 2, \ldots, m_1.$$
and
\[ g_{k+1}(a_\ell) = b_\ell, \quad \ell = 1, 2, \ldots, m_{k+1}. \]

It is easy to see that all requirements are satisfied, since
\[ \sup_{z \in L_k} |g_{k+1}(z) - g_1(z)| < \frac{a \epsilon}{2^2}, \quad a < 1, \quad \epsilon < 1, \quad L \cup L_{A_j_0} \subset L_1. \]

2.1. Applications of Theorem 1.5

We would like to present two classes of universal functions for which the Theorem 1.5 holds. We start with the class of universal functions with respect to differentiation. As we already mentioned in the Introduction, for any simply connected domain \( \Omega \) the operator \( D_\Omega : H(\Omega) \to H(\Omega) \) defined by \( D_\Omega f = f' \), \( f \in H(\Omega) \) is hypercyclic. The latter result implies that the set
\[ HC(D_\Omega) = \{ f \in H(\Omega) : \{ f^{(n)} \}_{n \in \mathbb{N}} \text{ is dense in } H(\Omega) \} \]
is a \( G_\delta \) and dense subset of \( H(\Omega) \). We prove the following

**Theorem 2.2.** Let \( \Omega \subset \mathbb{C} \) be a simply connected domain. Then the class \( HC(D_\Omega) \) has the interpolation property.

**Proof.** As we already mentioned, if \( \Omega \subset \mathbb{C} \) is a simply connected domain, then the class \( HC(D_\Omega) \) is a \( G_\delta \) and dense subset of \( H(\Omega) \). Moreover, if \( f \in HC(D_\Omega) \) and \( p \) is a polynomial then \( f + p \in HC(D_\Omega) \). So, in order to finish the proof and in view of Theorem 1.5, it suffices to prove that the class \( HC(D_\Omega) \) is equal to a denumerable intersection of open sets so that each open set has the property \( (\ast) \).

Let \( \{ f_j \}_{j \in \mathbb{N}} \) be an enumeration of all polynomials with coefficients in \( \mathbb{Q} + i \mathbb{Q} \). Let, in addition, \( \{ L_N \}_{N \in \mathbb{N}} \) be an exhausting sequence of compact subsets of \( \Omega \) with connected complement (see Theorem 13.3 in [31]). Then the following equality holds (see [8]):
\[ HC(D_\Omega) = \bigcap_{j=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcap_{s=0}^{\infty} \bigcup_{n=0}^{\infty} O(j, N, s, n), \]
where
\[ O(j, N, s, n) = \left\{ f \in H(\Omega) : \sup_{z \in L_N} |f^{(n)}(z) - f_j(z)| < \frac{1}{3} \right\}. \]

For every \( j, s, N \) and \( n \) the set \( \bigcup_{n=0}^{\infty} O(j, N, s, n) \) is open (see, for instance, [8]). Moreover, it has the property \( (\ast) \) since it can be easily seen that an appropriate compact is \( L_{N+1} \).

**Remark 2.** Nieß in [29], following a method developed in [9], gave a different proof of Theorem 2.2. Subsequently, Bernal extended the above result for hypercyclic differential operators \( \phi(D_C) \) where \( \phi \) is an entire function of subexponential type, see [4]. For an even more general result see [5].

It is time to proceed to the second class of universal functions for which Theorem 1.5 applies, the class of universal Taylor series.

**Definition 2.3.** Let \( \Omega \subset \mathbb{C} \) be an open set and \( \zeta \in \Omega \). Let, in addition, \( f \) be a holomorphic function in \( \Omega \), and consider the Taylor development of \( f \) with center \( \zeta \in \Omega \), i.e.
\[ f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\zeta)}{n!} (z - \zeta)^n. \]
\[ \sum_{n=0}^{\infty} \alpha_n (z - \zeta)^n, \] inside the disk of convergence. By \( S_n(f, \zeta)(z), \ n = 1, 2, \ldots \) we denote the partial sums of the Taylor development of \( f \) with center \( \zeta \). The function \( f \) is called universal Taylor series with center \( \zeta \) (or belongs to the class \( U(\Omega, \zeta) \)), if for every compact set \( K \subset \mathbb{C} \setminus \Omega \), with \( K^c \) connected and for every continuous \( h : K \to \mathbb{C} \), which is holomorphic in \( K^\alpha \), there exists a sequence of natural numbers \( \{\lambda_n\}_{n \in \mathbb{N}} \), such that
\[
\sup_{z \in K} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \to 0, \text{ as } n \to \infty.
\]

The above definition was introduced in [27,28]. Roughly speaking, the partial sums of a universal Taylor series have very strong approximation properties on or outside their circle of convergence. For a background and details on this subject we refer to [2,25] and the references therein.

**Theorem 2.4.** Let \( \Omega \subset \mathbb{C} \) be a simply connected domain. Then \( U(\Omega, \zeta) \) has the interpolation property.

**Proof.** It is known that the class \( U(\Omega, \zeta) \) is a \( G_\delta \) and dense subset of \( H(\Omega) \) (see [27,28]). Obviously if \( f \in U(\Omega, \zeta) \) and \( p \) is a polynomial then \( f + p \in U(\Omega, \zeta) \). So, as before, it suffices to prove that the class is equal to a denumerable intersection of open sets having the property \((*)\). Let \( \{f_j\}_{j \in \mathbb{N}} \) be an enumeration of all polynomials with coefficients in \( \mathbb{Q} + i\mathbb{Q} \). Let, in addition, \( \{K_m\}_{m \in \mathbb{N}} \) be a sequence of compact sets in \( \Omega^c \) with connected complement such that for every \( K \subset \Omega^c \) with connected complement, \( K \subset K_{m_0} \) for some \( m_0 \), see [27,28]. Then the following equality holds (see, for instance, [27] or [25]):
\[
U(\Omega, \zeta) = \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} E(m, j, s, n),
\]
where \( E(m, j, s, n) = \left\{ f \in H(\Omega) : \sup_{z \in K_m} |S_n(f, \zeta)(z) - f_j(z)| < \frac{1}{s} \right\} \).

For every \( m, j, s, n \), the set \( \bigcup_{n=1}^{\infty} E(m, j, s, n) \) is open (see [28]) and it has the property \((*)\); an appropriate compact set is any closed disk \( B \) centered at \( \zeta \) with \( B \subset \Omega \). This covers the case of simply connected domains done in [10].

**Remark 3.** We turn our attention to universal Taylor series on non-simply connected domains. Contrary to the case of a simply connected domain, the existence of universal Taylor series on non-simply connected domains is a quite delicate question and remains a subtle problem despite the recent progress has been done, see [1,9,12,14,24,33–35]. Our last application of Theorem 1.5 deals with universal Taylor series on non-simply connected domains, in particular doubly connected domains. To recall briefly, fix a compact and connected set \( K \subset \mathbb{C} \) such that \( \Omega := \mathbb{C} \setminus K \) is also connected. In [24] Melas showed that for every \( \zeta \in \Omega \) the class \( U(\Omega, \zeta) \) is non-empty, in fact \( G_\delta \) and dense in \( H(\Omega) \). Later on Bayart, answering a question of the first author, established that the class \( \bigcap_{\zeta \in \Omega} U(\Omega, \zeta) \) is residual in \( H(\Omega) \), see [1]. We stress that for non-simply connected domains \( W \) having more than one “holes” the question whether the class \( U(W, \zeta) \) is non-empty, \( \zeta \in W \), is still open in its full generality. There is however some recent progress due to Gardiner and Tsirivas [14], where, through the use of potential theory, they prove very interesting results in the negative direction. Namely, they show that for certain non-simply connected domains \( W \) having two “holes” and certain \( \zeta \in W \) the class \( U(W, \zeta) \) is void. We now continue with the following result. Fix a compact and connected set \( K \subset \mathbb{C} \) such that \( \Omega := \mathbb{C} \setminus K \)
is also connected. Then the class \( \bigcap_{\zeta \in \Omega} U(\Omega, \zeta) \) has the interpolation property. In order to prove this, it suffices to use the fact that the class \( \bigcap_{\zeta \in \Omega} U(\Omega, \zeta) \) contains a \( G_\delta \) and dense subset in \( H(\Omega) \), see [1], and then argue as above. We leave the details to the reader.

3. Proof of Theorem 1.7

Let \( \Omega \subset \mathbb{C} \) be an open set. Let, in addition, \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence of points in \( \Omega \) having no accumulation point in \( \Omega \) and \( \{b_n\}_{n \in \mathbb{N}} \) be any sequence of complex numbers. Set

\[
\Gamma = \{ f \in H(\Omega) : f(a_n) = b_n, \ n = 1, 2, \ldots \}.
\]

Note that \( \Gamma \) is a closed subset of \( H(\Omega) \), thus endowed with the topology of uniform convergence on compacta, it is a Baire space.

Now let \( \mathcal{A} = \bigcap_{j=1}^{\infty} \mathcal{A}_j \), where every \( \mathcal{A}_j \) is open in \( H(\Omega) \) having the property (**)) with respect to \( \{a_n\}_{n \in \mathbb{N}} \). It suffices to prove that for every \( j \), \( \mathcal{A}_j \cap \Gamma \) is dense in \( \Gamma \).

For this reason, we fix a function \( h \in \Gamma \), a compact set \( L \subset \Omega \), a positive integer \( j \) and a positive number \( \varepsilon < 1 \). Our aim is to find a function \( g \in \mathcal{A}_j \cap \Gamma \) with the property:

\[
\sup_{z \in L} |g(z) - h(z)| < \varepsilon.
\]

We consider an exhausting sequence \( \{L_N\}_{N \geq 1} \) of compact subsets of \( \Omega \) (see [31]). Since \( \mathcal{A}_j \) has the property (**)) with respect to \( \{a_n\}_{n \in \mathbb{N}} \), we may fix a function \( f \in \mathcal{A}_j \) and an open neighborhood

\[
V_f = \left\{ g \in H(\Omega) : \sup_{z \in L_f} |f(z) - g(z)| < a \right\} \subset \overline{V_f} \subset \mathcal{A}_j,
\]

with \( L_f \) disjoint from \( L \cup L_1 \cup \{a_n : n \in \mathbb{N}\} \).

Since the set \( \{a_1, a_2, \ldots \} \cap (L \cup L_k) \) is finite for every \( k \in \mathbb{N} \) (our sequence has no accumulation points in \( \Omega \)), without loss of generality (changing the enumeration of \( \{a_n\}_{n \in \mathbb{N}} \), if necessary), we may assume that for every \( k \in \mathbb{N} \):

\[
\{a_1, \ldots, a_{m_k}\} \subset (L \cup L_k)
\]

and

\[
\{a_{m_k+1}, \ldots \} \cap (L \cup L_k) = \emptyset,
\]

where \( \{m_k\}_{k \in \mathbb{N}} \) is an increasing sequence (not necessary strictly increasing) of natural numbers.

We will achieve our goal by constructing a sequence of functions \( \{g_k\}_{k \in \mathbb{N}} \) with the following properties:

1. \( \sup_{z \in L} |h(z) - g_k(z)| < \varepsilon/2, \ k = 1, 2, \ldots, \)
2. \( \sup_{z \in L_k} |g_{k+1}(z) - g_k(z)| < 1/2^k, \ k = 1, 2, \ldots, \)
3. \( g_k \in V_f, \) with \( \overline{V_f} \subset \mathcal{A}_j, \)
4. \( g_k(a_{\ell}) = b_{\ell}, \ \ell = 1, 2, \ldots, m_k. \)

Note that the sequence \( \{g_k\}_{k \in \mathbb{N}} \) will be a Cauchy sequence in \( H(\Omega) \) (see property 2), thus it will converge to a function \( g \in H(\Omega) \). It is easy to see that this function is suitable for our purposes.
Construction of \( \{g_k\}_{k \in \mathbb{N}} \).

**Step 1.** We first construct the function \( g_1 \). We apply Runge’s theorem to find a function \( \tilde{g}_1 \in H(\Omega) \), such that:

\[
\sup_{z \in L_1 \cup L} |h(z) - \tilde{g}_1(z)| < \frac{\varepsilon}{2^2(1 + M_1)}
\]

and

\[
\sup_{z \in L_f} |f(z) - \tilde{g}_1(z)| < \frac{a\varepsilon}{2^2(1 + M_1)}
\]

where \( M_1 \) is the positive number of Lemma 2.1 for the points \( a_1, a_2, \ldots, a_{m_1} \) and the compact set \( L_1 \cup L \cup L_0 \). We now apply Lemma 2.1 and we obtain a function \( g_1 \in H(\Omega) \) such that:

\[
\sup_{z \in L} |h(z) - g_1(z)| < \frac{\varepsilon}{2^2},
\]

\[
\sup_{z \in L_f} |f(z) - g_1(z)| < \frac{a}{2^2}
\]

and

\[
g_1(a_\ell) = h(a_\ell) = b_\ell, \quad \ell = 1, 2, \ldots, m_1.
\]

Since \( g_1 \) is near \( f \) on \( L_f \), it belongs to \( V_f \).

**Step 2.** Let \( k \geq 1 \) and suppose that \( g_1, \ldots, g_k \) have been constructed. We will construct \( g_{k+1} \). First, we apply Runge’s theorem to find a function \( \tilde{g}_{k+1} \in H(\Omega) \) such that:

\[
\sup_{z \in L_k \cup L \cup L_f} |\tilde{g}_{k+1}(z) - g_k(z)| < \frac{a\varepsilon}{2^{k+1}(1 + M_{k+1})}
\]

and

\[
|\tilde{g}_{k+1}(a_\ell) - h(a_\ell)| < \frac{a\varepsilon}{2^{k+1}(1 + M_{k+1})}, \quad \ell = m_k + 1, \ldots, m_{k+1},
\]

where \( M_{k+1} \) is the positive number of Lemma 2.1 for the compact set

\[
L_f \cup L_k \cup L \cup \{a_1, \ldots, a_{m_{k+1}}\}
\]

and the points

\[
a_1, a_2, \ldots, a_{m_{k+1}}.
\]

We now apply Lemma 2.1 once more and we obtain a function \( g_{k+1} \in H(\Omega) \) such that:

\[
\sup_{z \in L_k \cup L \cup L_f} |g_{k+1}(z) - g_k(z)| < \frac{a\varepsilon}{2^{k+1}}
\]

and

\[
g_{k+1}(a_\ell) = b_\ell, \quad \ell = 1, 2, \ldots, m_{k+1}.
\]

It is easy to see that all requirements are satisfied.
3.1. Applications of Theorem 1.7

In 1929, Birkhoff was the first to give an example of a universal (entire) function with respect to translations (see [6]). Let us recall the definition of this class of functions using modern terminology. Consider $T_1$ to be the operator on $H(\mathbb{C})$ defined by $T_1(f(z)) = f(z + 1)$ for every $f \in H(\mathbb{C})$. It is well known that $T_1$ is hypercyclic, i.e. there exist entire functions $g$ such that the set \{g(z+n) : n = 0, 1, 2, \ldots\} is dense in $H(\mathbb{C})$, see [21,16]. Such functions $g$ are called universal functions with respect to (integer) translations.

**Theorem 3.1.** Let \{a_n\}_{n \in \mathbb{N}} be a sequence of distinct points in \mathbb{C}, having no accumulation point with the following property:

For every $N \in \mathbb{N}$, there exist infinitely many natural numbers $n \in \mathbb{N}$ such that:

\[ \{a_1, a_2, \ldots\} \cap \overline{D(n, N)} = \emptyset. \]

Then $HC(T_1)$ has the interpolation property with respect to $\{a_n\}_{n \in \mathbb{N}}$.

**Remark.** For example, it is sufficient to require that the points \{a_n\}_{n \in \mathbb{N}} avoid an angle with bisector part of the positive real axis.

**Proof.** The following equality holds (see, for instance, [11]):

\[ HC(T_1) = \bigcap_{j=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} O(j, N, s, n), \]

where $O(j, N, s, n) = \left\{ f \in H(\mathbb{C}) : \sup_{z \in \overline{D(n, N)}} |f(z + n) - f_j(z)| < \frac{1}{s} \right\}$ and $\{f_j\}$ is an enumeration of all the polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. One can easily see that $O(j, N, s, n)$ is an open subset of $H(\mathbb{C})$. It suffices to prove that $\bigcup_{n=0}^{\infty} O(j, N, s, n)$ has the property (**)). To this end, suppose $L$ is a given closed ball. We choose sufficiently large $n$ so that $\overline{D(n, N)} \cap (L \cup \{a_1, a_2, \ldots\}) = \emptyset$ and then consider the function $g \in H(\mathbb{C})$ defined by $g(z) = f_j(z - n)$, $z \in \mathbb{C}$. Clearly $g \in \bigcup_{l=0}^{\infty} O(j, N, s, l)$ and then it is easy to show that for $\epsilon > 0$ sufficiently small the set

\[ V_g := \left\{ f \in H(\mathbb{C}) : \sup_{z \in \overline{D(n, N)}} |f(z) - g(z)| < \epsilon \right\} \]

is an open neighborhood of $g$ contained in $\bigcup_{l=0}^{\infty} O(j, N, s, l)$. This completes the proof of the theorem. \hfill \Box

For our last example, we work on any simply connected domain $\Omega \subset \mathbb{C}$ and we are interested in a slightly different class of functions than before. Namely, our “universal” functions realize approximations using dilations and translations. To be more specific, let us consider two sequences of complex numbers \{c_n\}_{n \in \mathbb{N}} and \{d_n\}_{n \in \mathbb{N}} such that the following hold:

- $c_n \neq 0$, for every $n \in \mathbb{N}$
- $d_n \to d$ a boundary point of $\Omega$
- For every natural number $N$, there exists a natural number $n_0$, such that $c_n z + d_n \in \Omega$, $n \geq n_0$ and $z \in \overline{D(0, N)}$. 
Then the class we are working on is the following:

**Definition 3.2.** A function $f \in H(\Omega)$ belongs to the class $U_T(\Omega, \{c_n\}_{n \in \mathbb{N}}, \{d_n\}_{n \in \mathbb{N}})$, if for every compact set $L \subset \mathbb{C}$ with connected complement and every continuous function $h : L \to \mathbb{C}$, which holomorphic in $L^0$, there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ of natural numbers such that

$$\sup_{z \in L} |f(c_{\lambda_n}z + d_{\lambda_n}) - h(z)| \to 0, \quad \text{as } n \to +\infty.$$ 

The existence of functions satisfying the properties of the above definition has been established by Luh, see for example [21,22] for even stronger universality properties. If we would like to interpret the above definition in terms of “hypercyclicity”, we just define for every positive integer $n$ the operator $T_n : H(\Omega) \to H(\Omega)$ by

$$T_n(f)(z) = f(c_{\lambda_n}z + d_{\lambda_n}), \quad f \in H(\Omega)$$

(whenever the definition makes sense). Then we have

$$HC((T_n)) = U_T(\Omega, \{c_n\}_{n \in \mathbb{N}}, \{d_n\}_{n \in \mathbb{N}}).$$

**Theorem 3.3.** Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of points in $\Omega$ having no accumulation point in $\Omega \cup \{d\}$. Then the set $U_T(\Omega, \{c_n\}_{n \in \mathbb{N}}, \{d_n\}_{n \in \mathbb{N}}) \cap \Gamma$ has the interpolation property with respect to $\{a_n\}_{n \in \mathbb{N}}$.

**Proof.** Let $\{f_j\}_{j \in \mathbb{N}}$ be an enumeration of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. Then the following equality holds (see for example [11]):

$$U_T(\Omega, \{c_n\}_{n \in \mathbb{N}}, \{d_n\}_{n \in \mathbb{N}}) = \bigcap_{j=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} O(j, N, s, n),$$

where $O(j, N, s, n) = \left\{ f \in H(\Omega) : \sup_{z \in D(0,N)} |f(c_nz + d_n) - f_j(z)| < \frac{1}{s} \right\}$. □

In order to finish the proof of **Theorem 3.3** we use the above set theoretic description of the class $U_T(\Omega, \{c_n\}_{n \in \mathbb{N}}, \{d_n\}_{n \in \mathbb{N}})$ and then we follow a similar line of reasoning as in the proof of **Theorem 3.1**. The details are left to the interested reader.

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**References**