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Balanced Functors Applied to Modules

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1. INTRODUCTION

Cartan and Eilenberg [1, p. 96] call a functor T of several variables right balanced if T becomes an exact functor of the remaining variables when any covariant variable is replaced by an injective module or when any contravariant variable is replaced by a projective module. An advantage of knowing that a functor of several variables is right balanced is that the right derived functors can be computed using resolutions of any of the variables. When the functor Hom(–, –) is used this gives the familiar result that we get the same global dimension of a ring using injective resolutions or projective resolutions.

The object of this paper is to modify and make a straightforward extension of this definition to the relative homological algebra situation formalized by Eilenberg and Moore in [2] (in fact, Eilenberg and Moore [2, p. 7] comment on a special case when Hom(–, –) is balanced without making the general definition). One application includes the interesting phenomenon of a difference in two between two definitions of the (relative) global dimension of a ring. This puts in perspective the fact, first noted in Bernecker [3], that certain full subcategories of the category of modules are reflective (or coreflective) only if the weak global dimension of the ring is at most two.

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2. Definitions and Terminology

Let **C**, **D** and **E** denote additive categories and let $\mathscr{F}$ and $\mathscr{G}$ be classes of objects of **C** and **D** respectively (or equivalently, full subcategories of each). Let $T: \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ be an additive functor, contravariant in the first and covariant in the second variable. Then we have:

Definition 2.1. T is right balanced relative to $(\mathscr{F}, \mathscr{G})$ if for each object A of **C** there is a complex $\cdots \to F_1 \to F_0 \to A \to 0$ with each $F_n \in \mathscr{F}$ such that the functor $T(-, G)$ applied to the complex gives an exact sequence whenever $G \in \mathscr{G}$, and if for each object **B** of **D** there is a complex $0 \to \mathbf{B} \to G^0 \to G^1 \to \cdots$ with each $G^n \in \mathscr{G}$ which becomes exact when $T(F, -)$ is applied for any $F \in \mathscr{F}$.

Notation. In case $\mathbf{C} = R$-mod (the category of left R-modules or Mod-R (right R-modules)), $\mathscr{F}\ell a t$ will denote the class of flat left R-modules or flat right R-modules. In the context it will be easy to deduce which is meant. Similarly $\mathscr{A}bs$ will denote the class of absolutely pure modules and $\mathscr{F}\ \mathscr{P}res$ the class of finitely presented modules.

The definition above is easily modified if, for example, T is covariant in both variables where we would postulate the existence of complexes $0 \to A \to F^0 \to F^1 \to \cdots$ and $0 \to B \to G^0 \to G^1 \to \cdots$ with obvious properties. Similar modifications give the definition of a left balanced functor.

In most applications $\mathscr{F}$ and $\mathscr{G}$ will be projective or injective classes of objects. We have (slightly modifying Eilenberg and Moore's terminology):

Definition 2.2. $\mathscr{F}$ is a projective class of objects for **C** if for each object A of **C** there is a complex $\cdots \to F_1 \to F_0 \to A \to 0$ with each $F_n \in \mathscr{F}$ such that $\mathrm{Hom}(F, -)$ makes the complex exact for each $F \in \mathscr{F}$.

Such a complex will be called a projective resolution of A for $\mathscr{F}$. An injective class and injective resolutions for the class are defined similarly. When F is a projective class we define the global dimension of **C** relative to $\mathscr{F}$ (as a projective class) to be the smallest n such that we always have a complex as above with $F_{n+1} = 0$, or to be infinite if there is no such n. Similarly we define the global dimension relative to an injective class.

Projective and injective classes are useful for defining derived functors. However, without assuming $\mathscr{F}$ projective or $\mathscr{G}$ injective we have, using the notation above:

Proposition 2.3. *If* **E** *is an abelian category and T is right balanced by $(\mathscr{F}, \mathscr{G})$ then the double complex $(T(F_n, G^m))$ and the complexes $(T(F_n, B))$ and $(T(A, G^n))$ have isomorphic homology.*

Proof. This is standard. The hypotheses guarantee that the spectral sequence of the double complex collapses using either of the standard filtrations.

COROLLARY 2.4. *If* $\cdots \to F_1' \to F_0' \to A \to 0$ *is another complex with* $F_n' \in \mathscr{F}$ *for all* n *which becomes exact when we apply* $T(-, G)$ *for any* $G \in \mathbf{G}$, *then the complexes* $(T(F_n', B))$ *and* $(T(F_n, B))$ *have isomorphic homology for any object* B *of* $\mathbf{D}$.

Proof. By the proposition, both have their homology isomorphic to that of $(T(A, G^n))$.

If either $\mathscr{F}$ is a projective class or $\mathscr{G}$ is an injective class, we can define the right functors R^nT as usual. We then get a natural transformation $T(A, B) \to R^0T(A, B)$ which can be computed using a resolution of A or of B. For left balanced functors we get, instead, a natural transformation $R_0T(A, B) \to T(A, B)$.

3. EXAMPLES

EXAMPLE 3.1. Let $T = \mathrm{Hom}(-,-)$ with $\mathbf{C} = \mathbf{D} = R$-Mod for some ring R. Then if $\mathscr{F} = \mathscr{G} = \mathscr{I}nj$ (the class of injective modules) and if R is left noetherian, T is left balanced by $(\mathscr{I}nj, \mathscr{I}nj)$. This follows from Proposition 2.2 or [4] which implies that $\mathscr{I}nj$ is a projective class for R-Mod. In this case the left derived functors will be denoted Ext_n. As usual, $\mathrm{Hom}(-,-)$ is right balanced by $(\mathscr{P}roj, \mathscr{I}nj)$ for any ring R with derived functors Ext^n.

EXAMPLE 3.2. Let $T = - \otimes -$ on Mod T is left balanced by $(\mathscr{P}roj, \mathscr{P}roj)$ and by $(\mathscr{F}lat, \mathscr{F}lat)$ with left derived functors Tor_n.

Absolutely pure modules were defined in [5] and in [6] it was shown $\mathscr{A}bs$ is an injective class in R-Mod. Würfel [6, Satz 1.6, p. 383] shows that if R is left coherent and G an absolutely pure left R-module then $G^+ = \mathrm{Hom}_Z(G, Q/Z)$ is a flat right R-module. In [4, Proposition 5.1, p. 201] the class $\mathscr{F}lat$ is shown to be injective in Mod-R if R is left coherent. This allows us to prove:

PROPOSITION 3.3. *If* R *is left coherent then* $T = - \otimes -$ *on* R-Mod $\times$ Mod-R *is right balanced by* $(\mathscr{F}lat, \mathscr{A}bs)$.

Proof. Given an injective resolution for $\mathscr{F}lat$ $0 \to A \to F^0 \to F^1 \to \cdots$ of A we need to show that if G is an absolutely pure left R-module then $0 \to A \otimes G \to F^0 \otimes G \to F^1 \otimes G \to \cdots$ is exact. This is so if the sequence we get when applying $\mathrm{Hom}_Z(-, Q/Z)$ is exact, or using a standard identity, when $0 \leftarrow \mathrm{Hom}(A, G^+) \leftarrow \mathrm{Hom}(F^0, G^+) \leftarrow \mathrm{Hom}(F^1, G^+) \leftarrow \cdots$ is exact. But since G^+ is flat this follows from the definition of an injective resolution for $\mathcal{F}lat$. If $0 \to B \to G^0 \to G^1 \to \cdots$ is an injective resolution for $\mathcal{A}bs$ then it's easy to argue that the sequence is exact, so if F is flat, $0 \to F \otimes B \to F \otimes G^0 \to F \otimes G^1 \to \cdots$ is exact. Hence T is right balanced by $(\mathcal{F}lat, \mathcal{A}bs)$. The right derived functors will be denoted by Tor^n.

EXAMPLE 3.4. If, again, R is left coherent, the class $\mathcal{F}\mathcal{G}\ \mathcal{P}roj$ (finitely generated projective right R-modules) is an injective class for $\mathcal{F}\ \mathcal{P}res$ (finitely presented right R-modules). For if M is a finitely presented right R-module we construct the desired resolution $0 \to M \to P^0 \to P^1 \to \cdots$ by steps. Since $\mathcal{F}lat$ is an injective class, there is a map $u: M \to F$ with F flat (but not necessarily finitely generated) such that $\mathrm{Hom}(F, \bar{F}) \to \mathrm{Hom}(M, \bar{F}) \to 0$ is exact for every flat $\bar{F}$. By Lazard [7], there is a finitely generated projective P and a factorization $M \to P \to F$ of u. Let $P^0 = P$. Then if $M' = \mathrm{Coker}(M \to P^0)$ we repeat the procedure with M' getting P'. Continuing we get the desired complex $0 \to M \to P^0 \to P^1 \to \cdots$. It's useful to note that such a sequence becomes exact if we apply $\mathrm{Hom}(-, P)$ with P finitely generated projective or $\mathrm{Hom}(-, F)$ with F flat.

Now we have $T = - \otimes -$ on $\mathcal{F}\ \mathcal{P}res \times R\text{-Mod}$ is right balanced by $(\mathcal{F}\mathcal{G}\ \mathcal{P}roj, \mathcal{A}bs)$ with the same derived functors (at least on the subcategory $\mathcal{F}\ \mathcal{P}roj \times R\text{-Mod}$) as in the previous example.

EXAMPLE 3.5. Using the above we see that if R is left coherent, $T = \mathrm{Hom}(-, -)$ is left balanced by $(\mathcal{F}\mathcal{G}\ \mathcal{P}roj, \mathcal{F}\mathcal{G}\ \mathcal{P}roj)$ on $\mathcal{F}\ \mathcal{G}en \times \mathcal{F}\ \mathcal{G}en$. Here the derived functors Ext_n differ from those of Example 3.1 with the same notation.

4. APPLICATIONS

We first need:

LEMMA 4.1. *If $M_1 \to M_2 \to M_3 \to M_4$ is an exact sequence of left R-modules such that for every finitely presented right R-module P, $P \otimes M_1 \to P \otimes M_2 \to P \otimes M_3 \to P \otimes M_4$ is exact, then $K =$ $\mathrm{Ker}(M_3 \to M_4)$ is a pure submodule of M_3.*

Proof. $P \otimes M_1 \to P \otimes M_2 \to P \otimes K \to 0$ is exact and $P \otimes K \to P \otimes M_3 \to P \otimes M_4$ is a complex. Thus $P \otimes M_1 \to P \otimes M_2 \to P \otimes M_3 \to P \otimes M_4$ exact means $0 \to P \otimes K \to P \otimes M_3$ is exact. This means K is a pure submodule of M_3.

PROPOSITION 4.2. *Let R be a left coherent ring. Then the right global dimension of R with respect to the injective class* $\mathscr{F}\ell a\ell$ *is exactly two less than the left global dimension of R with respect to the injective class* $\mathscr{A}bs$ *if this dimension is* $\geqslant 2$, *or is* 0 *if the latter is* $\leqslant 1$.

Proof. We argue the case $2 \leqslant n < \infty$ where n is the right global dimension with respect to $\mathscr{A}bs$. The proof of the other cases is similar. We know $T = - \otimes -$ is right balanced by $(\mathscr{F}\ell a\ell, \mathscr{A}bs)$. For a left R-module B let $0 \to B \to G^0 \to G^1 \to \cdots \to G^n \to 0$ be an injective resolution for $\mathscr{A}bs$. Since the sequence is exact and $A \otimes -$ is right exact for any right R-module A, we get that $\mathrm{Tor}^k(A, B) = 0$ when $k \geqslant n-1$. Then if $0 \to A \to F^0 \to F^1 \to \cdots$ is an injective resolution of A for $\mathscr{F}\ell a\ell$, we get $F^{n-2} \otimes B \to F^{n-1} \otimes B \to F^n \otimes B \to F^{n+1} \otimes B$ is exact for any B. Hence by Lemma 4.1, $K = \mathrm{Ker}(F^n \to F^{n+1})$ is pure in F^n so is flat. But $F^{n-2} \to F^{n-1} \to K \to 0$ is exact so $L = \mathrm{Ker}(F^{n-2} \to F^{n-1})$ is pure in F^{n-2} and so is flat. But then $0 \to A \to F^0 \to \cdots \to F^{n-3} \to L \to 0$ is an injective resolution of A for $\mathscr{F}\ell a\ell$.

This shows that the right global dimension of R for $\mathscr{F}\ell a\ell$ is at most $n-2$. If in fact it were less than $n-2$, then $\mathrm{Tor}^{n-2}(A, B) = \mathrm{Tor}^{n-1}(A, B) = 0$ for all A, B. Then arguing as above we see that if $0 \to B \to G^0 \to G^1 \to \cdots$ is an injective resolution for $\mathscr{A}bs$ of B then $C = \mathrm{Ker}(G^{n-1} \to G^n)$ is pure in G^{n-1} and so is absolutely pure. This gives an injective resolution $0 \to B \to G^0 \to \cdots \to G^{n-2} \to C \to 0$ for $\mathscr{A}bs$ of B. This contradicts the choice of n.

COROLLARY (Bernecker [3]). *The full subcategory* $\mathscr{F}\ell a\ell$ *in* Mod-*R is reflective if and only if the weak global dimension of R is at most* 2.

Proof. By Stenström [8], the weak global dimension of R is the same as the left global dimension of R for the injective class $\mathscr{A}bs$. By the proposition this means that when this dimension is $\leqslant 2$ every A has an injective resolution $0 \to A \to F \to 0$ for $\mathscr{F}\ell a\ell$ (i.e., $\mathrm{Hom}(F, \bar{F}) \to \mathrm{Hom}(A, \bar{F})$ is an isomorphism for every flat). This means $\mathscr{F}\ell a\ell$ is a reflective subcategory.

For the converse we simply reverse the steps above.

In a similar vein we have the following proposition (first noted in [4]) and its corollary:

PROPOSITION 4.3. *If R is left noetherian then the left global dimension of R for the projective class* $\mathscr{I}nj$ *is two less than the left global dimension of R with respect to the injective class* $\mathscr{I}nj$ (*i.e., the usual left global dimension*) *if the latter is* $\geqslant 2$, *otherwise it is* 0.

COROLLARY. *The full category of injective modules is a coreflective subcategory of R*-Mod *if and only if the usual left global dimension of R is at most* 2.

The proofs are similar to those for the preceding proposition and corollary.

THEOREM 4.4. *If R is a left coherent ring and $n \geqslant 0$ then the following are equivalent:*

(1) *For every flat left R-module F, there is an exact sequence $0 \to F \to A^0 \to \cdots \to A^n \to 0$ with each A^i absolutely pure.*

(2) *If $0 \to M \to P^0 \to P^1 \to \cdots$ is an injective resolution for $\mathcal{FG}\ \mathcal{P}roj$ of the finitely presented right R-module M, then the sequence is exact at P^k for $k \geqslant n-1$ (where $P^{-1} = M$ if $n = 0$).*

(3) *For every absolutely pure left R-module A there is an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to A \to 0$ with each F_i flat.*

(4) *There is an exact sequence $0 \to R \to A^0 \to \cdots \to A^n \to 0$ of left R-modules with each A^i absolutely pure.*

Proof. (1) $\Rightarrow$ (4) is immediate. To show that (4) $\Rightarrow$ (2) we note that $T = - \otimes -$ is right balanced on $\mathcal{F}\ \mathcal{P}roj \times R$-Mod by $(\mathcal{FG}\ \mathcal{P}res, \mathcal{A}bs)$ with right derived functors Tor^k.

If $n \geqslant 2$ then using the exact sequence $0 \to R \to A^0 \to \cdots \to A^n \to 0$ and the fact that $- \otimes M$ is right exact we get $\mathrm{Tor}^k(M, R) = 0$ for $k \geqslant n-1$. Computing using $0 \to M \to P^0 \to P^1 \to \cdots$ as in (2) we see that $\mathrm{Tor}^k(M, R)$ is just the homology of this complex, giving the desired result.

For $n = 1$, $0 \to R \to A^0 \to A^1 \to 0$ exact gives $\mathrm{Tor}^1(M, R) = 0$ so that, as above, $P^0 \to P^1 \to P^2$ is exact and $M \otimes R \to \mathrm{Tor}^0(M, R)$ is onto. Computing the latter morphism using $0 \to M \to P^0 \to P^1$ shows that $M \to P^0 \to P^1$ is exact.

If $n = 0$ then (4) means R is absolutely pure in a left R-module. By Proposition 3.3 this means $0 \to M \otimes R \to P^0 \otimes R \to P^1 \otimes R \to \cdots$ is exact, i.e., $0 \to M \to P^0 \to P^1 \to \cdots$ is exact.

We remark that (2) for $n = 0$ is equivalent to the requirement that every finitely presented right R-module be a submodule of a free R-module.

To prove (2) $\Rightarrow$ (1) assume (2) with $n \geqslant 2$. Let $0 \to F \to A^0 \to A^1 \to \cdots$ be exact with F flat and each A^i absolutely pure. Then by (2) we get $\mathrm{Tor}^k(M, F) = 0$ for $k \geqslant n-1$ since F is flat. Computing using $0 \to A^0 \to A^1 \to A^2 \to \cdots$ and using Lemma 4.1 we get $K = \mathrm{Ker}(A^n \to A^{n+1})$ is pure in A^n so is absolutely pure. Hence $0 \to F \to A^0 \to \cdots \to A^{n-1} \to K \to 0$ gives the desired exact sequence.

Now let $n = 1$. Then (2) says $M \to P^0 \to P^1 \to \cdots$ is exact, so $\mathrm{Tor}^k(M, F) = 0$ for $k > 0$ and $M \otimes F \to \mathrm{Tor}^0(M, F)$ is onto. Hence if $0 \to F \to A^0 \to A^1 \to \cdots$ is exact, $M \otimes F \to M \otimes A^0 \to M \otimes A^1 \to M \otimes A^2$ is exact for all finitely presented M. By Lemma 4.1 again we get the desired exact sequence $0 \to F \to A^0 \to K \to 0$ with $K = \mathrm{Ker}(A^1 \to A^2)$.

If $n=0$ then $0 \to M \to P^0 \to P^1 \to \cdots$ exact means $\mathrm{Tor}^k(M,F)=0$ for $k>0$ and $M \otimes F \to \mathrm{Tor}^0(M,F)$ is an isomorphism. This gives that $0 \to M \otimes F \to M \otimes A^0 \to M \otimes A^1$ is exact for all M which implies F is a pure submodule of A^0 so is absolutely pure.

The proofs $(2) \Rightarrow (3)$ and $(3) \Rightarrow (2)$ are similar but use the derived functors $\mathrm{Ext}_n(M,A)$ and the natural homomorphisms $\mathrm{Hom}(M,A) \to \mathrm{Ext}_0(M,A)$.

We note that the equivalence (1), (2), (3) and (4) for $n=0$ gives

COROLLARY 1. *If R is left coherent, then the following are equivalent*:

(1) *Every flat left R-module is absolutely pure.*

(2) *Every finitely presented right R-module is a submodule of a free module.*

(3) *Every absolutely pure left R-module is flat.*

(4) *R is absolutely pure as a left R-module.*

R. Colby in [9, Theorem 1, p. 245] proved that for any ring, coherent or not, every injective right R-module is flat (in his terminology R is right IF) if and only if (2) above holds. This implies that for a left coherent ring R, left absolutely pure modules are flat if and only if right injective modules are flat. Since injective modules are flat, this remark coupled with Corollary 1 gives:

COROLLARY 2. *If R is two-sided coherent then the following are equivalent*:

(1) *R is absolutely pure as a left R-module.*

(2) *R is absolutely pure as a right R-module.*

(3) *R is left IF.*

(4) *R is right IF.*

This result can be used to strengthen results of Würfel [6] and Sabbagh [10]. It stands in contradiction to Colby's example [9, Example 2, p. 249] which is claimed to be two-sided coherent and right IF but not left IF. In fact the ring is neither left nor right coherent. It suffices to note that neither the right nor the left annihilator of $1-e_1$ is finitely generated.

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