Vogan Diagrams of Real Forms of Affine Kac–Moody Lie Algebras

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A Vogan diagram is actually a Dynkin diagram with some additional structure. This paper develops theory of Vogan diagrams for “almost compact” real forms of indecomposable nontwisted affine Kac–Moody Lie algebras. Here, the equivalence classes of Vogan diagrams are in one–one correspondence with the isomorphism classes of almost compact real forms.

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1. INTRODUCTION

Classification of finite-dimensional real semisimple Lie algebras has been a classical problem. In [9], Knapp has given classification of finite-dimensional real semisimple Lie algebras by using Vogan diagrams. A corresponding theory of Vogan diagrams for “almost compact” real forms of “indecomposable nontwisted affine Kac–Moody Lie algebras” has been developed in [1].

In [1], we have introduced Vogan diagrams for the almost compact real forms of an indecomposable nontwisted affine Kac–Moody Lie algebra. A Vogan diagram is actually a Dynkin diagram with a certain overlay of additional data. Namely one fixes a Cartan involution \( \theta \), chooses a maximally compact \( \theta \)-stable Cartan subalgebra, and introduces an ordering that takes the compact part before the noncompact part. The Cartan involution permutes the simple roots in orbits of one or two roots, and the two-element orbits are marked on the Dynkin diagram of the complexation. The one-element orbits consist of compact or noncompact imaginary roots, and the vertices of the Dynkin diagram corresponding to the noncompact roots are...
shaded. We have introduced the notion of equivalence of Vogan diagrams. Two Vogan diagrams are equivalent if they correspond to different orderings for the same real form.

This paper deals with the Vogan diagrams for the almost compact real forms of an indecomposable nontwisted affine Kac–Moody Lie algebra. The main theorem of this paper (Theorem 5.2) is that any diagram that looks like a Vogan diagram comes from some real form of an indecomposable nontwisted affine Kac–Moody Lie algebra. This theorem and Theorem 5.2 in [1] together provide a one–one correspondence between suitably defined equivalence classes of Vogan diagrams and isomorphism classes of almost compact real forms.

In [1], the Appendix gives representatives of equivalence classes of Vogan diagrams for some nontwisted affine Kac–Moody Lie algebras. A count of the number of these diagrams yields a match with the number of almost compact real forms. The theory of Vogan diagrams allows one to identify, for a given Lie algebra, which standard Lie algebra is isomorphic to it, without tracking down isomorphisms whose existence is known but for which there are no formulas.

This paper is organized as follows: Section 2 recites known facts about (complex) indecomposable nontwisted affine Kac–Moody Lie algebras. Section 3 records aspects of the classification of the real forms and then introduces Vogan diagrams, making the necessary justifications. Section 4 discusses some examples, showing explicitly what almost compact real forms are and what their Vogan diagrams look like. Section 5 states and proves the main theorem.

2. PRELIMINARIES

2.1. Affine Kac–Moody Lie Algebras

Let \( I = [1, n + 1], n \in \mathbb{N} \), be an interval in \( \mathbb{N} \). Let \( A = (a_{ij}), i, j \in I \) be a matrix with integer coefficients. \( A \) is called a generalized Cartan matrix if it satisfies the following conditions:

- \( a_{ii} = 2 \) for \( i = 1, \ldots, n + 1 \);
- \( a_{ij} \) are nonpositive integers for \( i \neq j \);
- \( a_{ij} = 0 \) implies \( a_{ji} = 0 \).

Let \( \mathfrak{g} = \mathfrak{g}(A) \) be the free complex Lie algebra generated by a basis of the standard Cartan subalgebra \( \mathfrak{h} \) and the elements \( e_i, f_i \) for \( i \in I \) with the following defining relations [5]:

\[
[e_i, f_j] = \delta_{ij} \alpha_i^\vee, \quad i, j \in I
\]

\[
[h, h'] = 0, \quad h, h' \in \mathfrak{h}
\]
The algebra \( \mathfrak{g} \) is called a Kac–Moody Lie algebra. The algebra \( \mathfrak{g} \) is an affine Kac–Moody Lie algebra if

- \( A \) is an indecomposable matrix; i.e., after the indices are reordered \( A \) cannot be written in the form
  \[
  \begin{pmatrix}
  A_1 & 0 \\
  0 & A_2
  \end{pmatrix}.
  \]
- there exists a vector \( (a_i)_{i=1}^{n+1} \), with \( a_i \) all positive such that
  \[
  A(a_i)_{i=1}^{n+1} = 0.
  \]

Then \( A \) is called a Cartan matrix of affine type. The affine algebra associated to a generalized Cartan matrix of type \( X_n^{(1)} \) is called a nontwisted affine Kac–Moody Lie algebra.

2.2. Dynkin Diagram Associated with a Generalized Cartan Matrix

We associate with a generalized Cartan matrix \( A \) a graph \( S(A) \) called the Dynkin diagram of \( A \) as follows. If \( a_{ij}a_{ji} \leq 4 \) and \( |a_{ij}| \geq |a_{ji}| \), the vertices \( i \) and \( j \) are connected by \( |a_{ij}| \) lines, and these lines are equipped with an arrow pointing toward \( i \) if \( |a_{ij}| > 1 \). If \( a_{ij}a_{ji} > 4 \), the vertices \( i \) and \( j \) are connected by a boldfaced line equipped with an ordered pair of integers \( |a_{ij}|, |a_{ji}| \).

It is clear that \( A \) is indecomposable if and only if \( S(A) \) is a connected graph. \( A \) is determined by the Dynkin diagram \( S(A) \) and an enumeration of its vertices.

2.3. A Realization of Nontwisted Affine Kac–Moody Lie Algebras

Let \( L = \mathbb{C}[t, t^{-1}] \) be the algebra of Laurent polynomials in \( t \). The residue of a Laurent polynomial \( P = \sum_{k \in \mathbb{Z}} c_k t^k \) (where all but a finite number of \( c_k \) are 0) is defined as \( \text{Res} P = c_{-1} \).

Let \( \mathfrak{g} \) be a finite-dimensional simple Lie algebra over \( \mathbb{C} \) of type \( X_n \). Then \( L(\mathfrak{g}) = L \otimes_{\mathbb{C}} \mathfrak{g} \) is an infinite-dimensional complex Lie algebra with the bracket

\[
[P \otimes x, Q \otimes y] = PQ \otimes [x, y], \quad P, Q \in L; x, y \in \mathfrak{g}.
\]

Fix a nondegenerate, invariant, symmetric bilinear form \( \langle \cdot, \cdot \rangle \) in \( \mathfrak{g} \) and extend this form to an \( L \)-valued form \( \langle \cdot, \cdot \rangle \), on \( L(\mathfrak{g}) \) by

\[
(P \otimes x, Q \otimes y) = PQ(x, y), \quad P, Q \in L; x, y \in \mathfrak{g}.
\]
The derivation \( t^l \frac{d}{dt} \) of \( L \) extends to \( L(\hat{\mathfrak{g}}) \) by

\[
t^l \frac{d}{dt} (P \otimes x) = t^l \frac{dP}{dt} \otimes x \quad P \in L; x \in \hat{\mathfrak{g}}.
\]

Therefore \( \psi(a, b) = \text{Res}(\frac{dP}{dt}; a, b; a, b \in L(\hat{\mathfrak{g}})) \) defines a two-cocycle on \( L(\hat{\mathfrak{g}}) \) (Co 1), (Co 2) [5, p. 97].

We denote by \( \hat{L}(\hat{\mathfrak{g}}) \) the central extension of the Lie algebra \( L(\hat{\mathfrak{g}}) \) associated to the cocycle \( \psi \). Explicitly \( \hat{L}(\hat{\mathfrak{g}}) = L(\hat{\mathfrak{g}}) \oplus \mathbb{C}c \) with the bracket

\[
[a + \lambda c, b + \mu c] = [a, b] + \psi(a, b)c \quad a, b \in L(\hat{\mathfrak{g}}), \lambda, \mu \in \mathbb{C}.
\]

Finally, we denote by \( \hat{\hat{L}}(\hat{\mathfrak{g}}) \) the Lie algebra which is obtained by adjoining to \( \hat{L}(\hat{\mathfrak{g}}) \) a derivation which acts on \( L(\hat{\mathfrak{g}}) \) as \( t^l \frac{d}{dt} \) and kills \( c \). Explicitly \( \hat{\hat{L}}(\hat{\mathfrak{g}}) = L(\hat{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d \) with the bracket defined by

\[
[t^k \otimes x + \lambda c + \mu d, t^l \otimes y + \lambda_1 c + \mu_1 d] = t^{l+k} \otimes [x, y] + \mu_1 t^l \otimes y - \mu_2 k t^k \otimes x + k \delta_{j,-k}(x, y)c,
\]

where \((x, y \in \hat{\mathfrak{g}}; \lambda, \mu, \lambda_1, \mu_1 \in \mathbb{C}; k, j \in \mathbb{Z})\) [5, Sect. 7.2.2]. \( \hat{\hat{L}}(\hat{\mathfrak{g}}) \) is a non-twisted affine Kac–Moody Lie algebra associated to the affine matrix \( A \) of type \( X_n^{(1)} \) [5, Theorem 7.4]. Here \( \mathbb{C}c \) is the center of \( \hat{\hat{L}}(\hat{\mathfrak{g}}) \) and \( L(\hat{\mathfrak{g}}) \oplus \mathbb{C}c \) is the derived algebra of \( L(\hat{\mathfrak{g}}) \).

**Notation.** From now on \( \hat{\hat{L}}(\hat{\mathfrak{g}}) = \mathfrak{g} \) and \( t^k \otimes x \) will be written as \( t^k x \).

### 2.4. Invariant Bilinear Form of \( \mathfrak{g} \)

The normalized invariant form of \( \mathfrak{g} \) [5, Sect. 6.2] can be described as follows. Take the normalized invariant form \((.,.)\) on \( \hat{\mathfrak{g}} \) and extend \((.,.)\) to all of \( \hat{\mathfrak{g}} \) by

\[
(P \otimes x, Q \otimes y) = (\text{Res} t^{-1} PQ)(x, y) \quad x, y \in \hat{\mathfrak{g}}; P, Q \in L;
\]

\((\mathbb{C}c \oplus \mathbb{C}d, L(\hat{\mathfrak{g}})) = 0; (c, c) = (d, d) = 0; (c, d) = 1\). It is a nondegenerate, symmetric, invariant, bilinear form of \( \mathfrak{g} \) [5, Sect. 7.5].

### 2.5. Roots of \( \mathfrak{g} \)

Let \( \Delta \subset \hat{\mathfrak{h}}^* \) be the root system of the finite-dimensional Lie algebra \( \hat{\mathfrak{g}} \), let \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) be the root basis, let \( \{H_1, H_2, \ldots, H_n\} \) be the coroot basis, and let \( E_{1,i}, F_{1,i}(i = 1, 2, \ldots, n) \) be the Chevalley generators. Let \( \theta \) be the highest root of the finite root system \( \Delta \). Let

\[
\hat{\mathfrak{g}} = \bigoplus_{\alpha \in \Delta \setminus 0} \hat{\mathfrak{g}}_\alpha
\]
be the root space decomposition of $\hat{\mathfrak{g}}$. Let $\omega$ be the Cartan involution of $\hat{\mathfrak{g}}$. We choose $F_0 \in \hat{\mathfrak{g}}_\theta$ such that $(F_0, \omega(F_0)) = -\frac{2}{(\theta, \theta)}$ and set $E_0 = -\omega(F_0)$. Then we have due to [5, Theorem 2.2e]

$$[E_0, F_0] = -\theta'.'$$

Now we come to the Lie algebra $\mathfrak{g} = \tilde{L}(\hat{\mathfrak{g}})$. Here $\mathfrak{h} = \mathfrak{h} + \mathbb{C}c + \mathbb{C}d$ is an $(n+2)$-dimensional commutative subalgebra in $\mathfrak{g}$. We extend $\lambda \in \mathfrak{h}^*$ to a linear functional on $\mathfrak{h}$ by setting $\langle \lambda, c \rangle = \langle \lambda, d \rangle = 0$, so that $\mathfrak{h}^*$ is identified with a subspace in $\mathfrak{h}^*$. We denote by $\delta$ the linear functional on $\mathfrak{h}$ defined by $\delta|_{\mathfrak{h} + \mathbb{C}c} = 0$, $(\delta, d) = 1$. Set

$$e_{n+1} = tE_0, \quad f_{n+1} = t^{-1}F_0$$
$$e_i = E_i, \quad f_i = F_i \quad (i = 1, 2, \ldots, n).$$

**Remark 1.** In [5], Kac uses $e_0$ for $e_{n+1}$ and $f_0$ for $f_{n+1}$. We see from [5, Sect. 7.4.1] that

$$[e_{n+1}, f_{n+1}] = \frac{2}{(\theta, \theta)}c - \theta'.$$

Now we describe the root system and the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$;

$$\Delta = \{j\delta + \gamma, \text{ where } j \in \mathbb{Z}, \gamma \in \Delta \} \cup \{j\delta, \text{ where } j \in \mathbb{Z} - 0\}.$$ 
$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} L(\hat{\mathfrak{g}})_{\alpha},$$

where $L(\hat{\mathfrak{g}})_{j\delta + \gamma} = t^j q_{\gamma}, L(\hat{\mathfrak{g}})_{j\delta} = t^j \hat{\mathfrak{h}}.$

We set

$$\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1} = \delta - \theta\},$$
$$\pi' = \{\alpha'_1, \alpha'_2, \ldots, \alpha'_n, \alpha'_{n+1} = \frac{2}{(\theta, \theta)}c - \theta'\}.$$ 

Then by [5, Proposition 6.4a]

$$A = \langle \alpha'_i, \alpha_j \rangle_{i, j = 1}^{n+1}.$$ 

In other words, $(\mathfrak{h}, \pi, \pi')$ is a realization of the affine matrix $A$.

### 2.6. Cartan Involution of $\mathfrak{g}$

The map $e_i \mapsto -f_i, f_i \mapsto -e_i, i \in I; h \mapsto -h, h \in \mathfrak{h}$ can be uniquely extended to an involution $\omega$ of $\mathfrak{g}$. $\omega$ is called the Cartan involution of $\mathfrak{g}$.
2.7. Weyl Group

For each \( i \in I \), we define a fundamental reflection \( r_i \) of the space \( \mathfrak{h}^* \) by

\[
r_i(\alpha) = \alpha - \langle \alpha, \alpha_i \rangle \alpha_i \quad \forall \alpha \in \mathfrak{h}^*.
\]

The subgroup \( W = GL(\mathfrak{h}^*) \) generated by \( r_i, i \in I \) is called the Weyl group of \( \mathfrak{g} \).

3. CLASSIFICATION OF REAL FORMS OF NONTWISTED AFFINE KAC–MOODY LIE ALGEBRAS

3.1. Cartan and Borel Subalgebras of \( \mathfrak{g} \)

We define a group \( G \) acting on the Lie algebra \( \mathfrak{g} \) via the adjoint representation \( \text{Ad}: G \to \text{Aut}(\mathfrak{g}) \). It is generated by the subgroups \( U_\alpha \) for \( \alpha \in \pm \pi \) and \( \text{Ad}(U_\alpha) = \exp(\text{ad}(U_\alpha)) \).

A maximal \( \text{ad}_\mathfrak{g} \)-diagonalizable subalgebra of \( \mathfrak{g} \) is called a Cartan subalgebra. Every Cartan subalgebra of \( \mathfrak{g} \) is \( \text{Ad}(G) \)-conjugate to the standard Cartan subalgebra \( \mathfrak{h} \) [6]. A Borel subalgebra of \( \mathfrak{g} \) is a maximal completely solvable subalgebra. It is always conjugated by \( \text{Ad}(G) \) to \( b^+ \) or \( b^- \), where

\[
b^+ = \mathfrak{h} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha \quad \text{and} \quad b^- = \mathfrak{h} \oplus \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha.
\]

But \( b^+ \) and \( b^- \) are not conjugated under \( \text{Ad}(G) \). So there are exactly two conjugacy classes of Borel subalgebra: the positive and negative subalgebras [6].

3.2. Real Forms of \( \mathfrak{g} \)

A real form of \( \mathfrak{g} \) is a Lie algebra \( \mathfrak{g}_\mathbb{R} \) over \( \mathbb{R} \) such that there exists an isomorphism from \( \mathfrak{g} \) to \( \mathfrak{g}_\mathbb{R} \otimes \mathbb{C} \) [2]. If we replace \( \mathbb{C} \) by \( \mathbb{R} \) in the definition of \( \mathfrak{g} \), we obtain a real form \( \mathfrak{g}_\mathbb{R} \), which is called \( \text{split} \). A real form of \( \mathfrak{g} \) corresponds to a semilinear involution of \( \mathfrak{g} \). This means to an automorphism \( \tau \) of \( \mathfrak{g} \) such that \( \tau^2 = \text{Id} \) and \( \tau(\lambda x) = \bar{\lambda} \tau(x) \) for \( \lambda \in \mathbb{C} \) (semilinear).

A linear or semilinear automorphism of \( \mathfrak{g} \) is said to be of the \textit{first kind} if it transforms a Borel subalgebra into a Borel subalgebra of the same sign. A linear or semilinear automorphism of \( \mathfrak{g} \) is said to be of the \textit{second kind} if it transforms a Borel subalgebra into a Borel subalgebra of the opposite sign.

Let \( \mathfrak{g}_\mathbb{R} \) be a real form of \( \mathfrak{g} \), and fix an isomorphism from \( \mathfrak{g} \) to \( \mathfrak{g}_\mathbb{R} \otimes \mathbb{C} \). Then the Galois group \( \Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) \) acts on \( \mathfrak{g} \) and the corresponding group \( G \). We identify \( \mathfrak{g}_\mathbb{R} \) with the fixed point set \( \mathfrak{g}^\Gamma \).
DEFINITION 3.1. If $\Gamma$ consists of the first kind automorphism we say that $\mathfrak{g}_R$ is *almost split*. Otherwise if the nontrivial element of $\Gamma$ is the second kind of automorphism, we say that $\mathfrak{g}_R$ is *almost compact* (in other words, a real form $\mathfrak{g}_R$ of $\mathfrak{g}$ is said to be almost split if for each $\gamma$ in $\Gamma$ the action of $\gamma$ on $\mathfrak{g}$ is of the first kind; otherwise $\mathfrak{g}_R$ is said to be almost compact [13]).

3.3. Automorphisms of $\mathfrak{g}$

Let $\mathfrak{h}$ be the standard Cartan subalgebra of $\mathfrak{g}$. Reference [6] defines a group $\tilde{H}$ which acts on $G$ and $\mathfrak{g}$. In fact $\tilde{H} = \text{Ad}(\tilde{H})$ is isomorphic to $(\mathbb{C}^*)^I$ and if the element $h$ of $\tilde{H}$ corresponds to $(h_i)_{i \in I}$, it acts on $\mathfrak{g}_\alpha$ by multiplication by the scalar

$$\prod_i (h_i)^{n_i} \text{ if } \alpha = \sum_i n_i \alpha_i.$$  

The group $\text{Int}(\mathfrak{g}) = \text{Ad}(\tilde{H} \ltimes G)$ of interior automorphisms of $\mathfrak{g}$ is the image of the semidirect product of $\tilde{H}$ and $G$. Its derived group is the adjoint group $\text{Ad}(G)$ (denoted by $\text{Int}(\mathfrak{g})$). As $G$ is transitive on the Cartan subalgebra, the group $\text{Int}(\mathfrak{g})$ does not depend on the choice of $\mathfrak{h}$.

We consider the group $\text{Aut}(A)$ of permutations $\rho$ of $I$ such that $a_{\rho ij} = a_{ij}$ for $i, j \in I$. We regard $\text{Aut}(A)$ as a subgroup of $\text{Aut}(\mathfrak{g}')$ by requiring $\rho(e_i) = e_{\rho i}$, $\rho(f_i) = f_{\rho i}$. The Cartan involution $\omega$ commutes with $\text{Aut}(A)$. We define the group of outer automorphisms of $\mathfrak{g}$ to be

$$\text{Ext}(\mathfrak{g}) = \{\text{Id}, \omega\} \times \text{Aut}(A).$$

Transvections. Following [2], we denote by $\text{Tr}$ the set of transvections of $\mathfrak{g}$, which means linear maps of $\mathfrak{g}$ in $\mathfrak{g}$ of the form $\phi = \exp \psi$, where $\psi$ is a linear map of $\mathfrak{g}$ into $c$, zero on $\mathfrak{g}'$. So we have $\phi(x) = x + \psi(x)$. A description of transvections of $\mathfrak{g}$ is also given in [7, Sect. 4.20].

Then according to [2, p. 187], we have the following decomposition of the group of automorphisms of $\mathfrak{g}$:

$$\text{Aut}(\mathfrak{g}) = [\text{Ext}(\mathfrak{g}) \ltimes \text{Int}(\mathfrak{g})] \times \text{Tr}. \quad (3.1)$$

If $\sigma$ is an automorphism of the first kind of $\mathfrak{g}$, then $\omega \circ \sigma$ is of the second kind of automorphism and conversely [2, p. 17]. From (3.1) it follows that any automorphism of $\mathfrak{g}$ is either an automorphism of the first kind or an automorphism of the second kind.

3.4. Semilinear Automorphisms

DEFINITION 3.2. Following [2, p. 190], we define by $\text{Aut}_{\mathbb{C}}(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$, which are $\mathbb{C}$-linear or semilinear. The group $\text{Aut}(\mathfrak{g})$ is normal in $\text{Aut}_{\mathbb{C}}(\mathfrak{g})$ and of index 2.
A semilinear automorphism of order 2 of \( \mathfrak{g} \) is called a *semiinvolution* of \( \mathfrak{g} \). For any semiinvolution \( \sigma' \) we have the decomposition \( \text{Aut}_\mathbb{R}(\mathfrak{g}) = \{1, \sigma'\} \rtimes \text{Aut}(\mathfrak{g}) \). We denote by \( \sigma'_\mathfrak{g} \) the conjugation of \( \mathfrak{g} \) with respect to the standard split real form. We call \( \sigma'_\mathfrak{g} \) the *standard normal semiinvolution* of \( \mathfrak{g} \). This commutes with the standard Cartan involution \( \omega \).

Let \( \omega' = \sigma'_\mathfrak{g} \omega = \omega \sigma'_\mathfrak{g} \). Then \( \omega' \) is called the *standard Cartan semiinvolution* of \( \mathfrak{g} \). Its algebra of fixed points is the *standard compact* real form of \( \mathfrak{g} \). A Cartan semiinvolution of \( \mathfrak{g} \) is a semiinvolution \( \omega' \) conjugate to \( \omega' \) by an element of \( \text{Aut}_\mathbb{R}(\mathfrak{g}) \). Then \( \omega' \) is a semiinvolution of the second kind and the associated real form is called compact real form \( \mathfrak{g}_1 \) of \( \mathfrak{g} \).

Let \( \sigma' \) be a semiinvolution and \( \omega' \) be a Cartan semiinvolution. Then using [2, Proposition 4.5] and after having supposed that \( \sigma' \) and \( \omega' \) stabilize the same Cartan subalgebra \( \mathfrak{h} \), one may suppose by conjugating by \( G \) that \( \omega' \) commutes with \( \sigma' \).

**Definition 3.3.** Let \( \sigma' \) be a semiinvolution of the second kind of \( \mathfrak{g} \) and let \( \mathfrak{g}_\mathbb{R} = \mathfrak{g}^\sigma' \) be the corresponding almost compact real form. A Cartan semiinvolution \( \omega' \), which commutes with \( \sigma' \) is called a Cartan semiinvolution for \( \sigma' \) of \( \mathfrak{g}_\mathbb{R} \). The involution \( \sigma = \sigma' \omega' \) (resp. its restriction \( \omega'_\mathfrak{g} \) to \( \mathfrak{g}_\mathbb{R} \)) is called a Cartan involution of \( \sigma' \) (resp. of \( \mathfrak{g}_\mathbb{R} \)). The algebra of fixed points \( \mathfrak{g}_0 = \mathfrak{g}^\sigma \) is called a *maximal compact subalgebra* of \( \mathfrak{g}_\mathbb{R} \). We have the Cartan decomposition \( \mathfrak{g}_\mathbb{R} = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \), with \( \mathfrak{t}_0 \subseteq \mathfrak{k}_\mathbb{R} \) and \( \mathfrak{a}_0 \subseteq \mathfrak{p}_\mathbb{R} \) [1, Proposition 3.5].

3.5. Maximally Compact Cartan Subalgebra of an Almost Compact Form

Let \( \mathfrak{t}_0 \) be a maximal abelian subspace of \( \mathfrak{t}_0 \). Then \( \mathfrak{t}_0 = Z_{\mathfrak{g}^\sigma}(\mathfrak{t}_0) \) is a \( \sigma \)-stable Cartan subalgebra of the almost compact real form \( \mathfrak{g}_\mathbb{R} \) of the form \( \mathfrak{t}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \), with \( \mathfrak{a}_0 \subseteq \mathfrak{p}_0 \) [1, Proposition 3.5]. We say that a \( \sigma \)-stable Cartan subalgebra of an almost compact real form \( \mathfrak{g}_\mathbb{R} \) of the form \( \mathfrak{t}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \), with \( \mathfrak{t}_0 \subseteq \mathfrak{t}_0 \) and \( \mathfrak{a}_0 \subseteq \mathfrak{p}_0 \), is *maximally compact* if \( \dim \mathfrak{t}_0 \) is as large as possible. A maximally compact Cartan subalgebra \( \mathfrak{b}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \) of an almost compact real form \( \mathfrak{g}_\mathbb{R} \) has the property that all of the roots are real on \( \mathfrak{a}_0 \) and imaginary on \( \mathfrak{t}_0 \) [1, Proposition 3.6].

3.6. Classification of Real Forms

Following [3, Theorem 4.4], we obtain under \( \text{Aut}(\mathfrak{g}) \) a one–one correspondence between the conjugacy classes of (linear) involutions of the second kind of \( \mathfrak{g} \) and the conjugacy classes of almost split real forms of \( \mathfrak{g} \).

**Theorem 3.4** [12]. We consider

1. *The semiinvolutions \( \sigma' \), of the second kind of \( \mathfrak{g} \).*
The involutions $\theta$, of the first kind of $\mathfrak{g}$.

The relation $\sigma' \approx \theta$ if and only if

(a) $\omega' = \theta \sigma' = \sigma' \theta$ is a Cartan semiinvolution.
(b) $\theta$ and $\sigma'$ stabilize the same Cartan subalgebra $\mathfrak{h}$.
(c) $\mathfrak{h}$ is contained in a minimal $\sigma'$-stable positive parabolic subalgebra.

Then this relation induces a bijection between the conjugacy classes under $\text{Aut}(\mathfrak{g})$ of semiinvolutions of the second kind and conjugacy classes of involutions of the first kind.

Thus we obtain under $\text{Aut}(\mathfrak{g})$ a one-one correspondence between the conjugacy classes of (linear) involutions of the first kind of $\mathfrak{g}$ (including identity) and the conjugacy classes of almost compact real forms of $\mathfrak{g}$. The compact real form is unique, it corresponds to identity. A classification of involutions of an affine Kac–Moody Lie algebra is given by Levtstein [11]. A classification of automorphisms of finite order of affine Kac–Moody Lie algebra of type $A_n^{(1)}$ is given by Kobayashi [10]. A description of automorphisms of Kac–Moody algebras is also given in [7].

3.7. Vogan Diagrams

Let $\mathfrak{g}_R$ be an almost compact real form of $\mathfrak{g}$ corresponding to the semiinvolution $\sigma'$ of the second kind of $\mathfrak{g}$. Let $\sigma$ be the Cartan involution of $\mathfrak{g}_R$, and let $\mathfrak{g}_R = \mathfrak{t}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. We write $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ for the complexation of the Cartan decomposition.

Let $\mathfrak{t}_0$ be a maximal abelian subspace of $\mathfrak{t}_0$. Then $\mathfrak{b}_0 = Z_{\mathfrak{g}_R}(\mathfrak{t}_0)$ is a $\sigma$-stable Cartan subalgebra of $\mathfrak{g}_R$ of the form $\mathfrak{b}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, with $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$ by [1, Proposition 3.5]. This $\mathfrak{b}_0$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_R$ because $\mathfrak{t}_0$ is as large as possible. By [1, Proposition 3.6], we note that the roots of $(\mathfrak{g}, \mathfrak{b})$ are imaginary on $\mathfrak{t}_0$ and real on $\mathfrak{a}_0$. A root is real if it takes real values on $\mathfrak{b}_0$ (i.e., vanishes on $\mathfrak{t}_0$), imaginary if it takes purely imaginary values on $\mathfrak{b}_0$ (i.e., vanishes on $\mathfrak{a}_0$), and complex otherwise.

For any root $\alpha$, $\sigma \alpha$ is the root $\sigma \alpha(H) = \alpha(\sigma^{-1}H)$. If $\alpha$ is imaginary, then $\sigma \alpha = \alpha$. Thus $\mathfrak{g}_\alpha$ is $\sigma$-stable, and we have $\mathfrak{g}_\alpha = (\mathfrak{g}_\alpha \cap \mathfrak{t}) \oplus (\mathfrak{g}_\alpha \cap \mathfrak{p})$. Since $\mathfrak{g}_\alpha$ is one-dimensional, $\mathfrak{g}_\alpha \subseteq \mathfrak{t}$ or $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$. We call an imaginary root $\alpha$ compact if $\mathfrak{g}_\alpha \subseteq \mathfrak{t}$, noncompact if $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$.

Let $\mathfrak{b}_0$ be a $\sigma$-stable Cartan subalgebra of $\mathfrak{g}_R$. Then there are no real roots if and only if $\mathfrak{b}_0$ is maximally compact [1, Proposition 3.8].

Let $\mathfrak{g}_R$ be an almost compact real form of $\mathfrak{g}$. Let $\sigma$ be the Cartan involution and $\mathfrak{g}_R = \mathfrak{t}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. Let $\mathfrak{b}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be a maximally compact $\sigma$-stable Cartan subalgebra of $\mathfrak{g}_R$, with complexation $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{a}$. We let $\Delta = \Delta(\mathfrak{g}, \mathfrak{b})$ be the set of roots. By [1, Proposition 3.8], there are no real roots, i.e., no roots that vanish on $\mathfrak{t}$. 

We choose a positive system $\Delta^+$ for $\Delta$ that takes $it_0$ before $a_0$ [8, p. 339]. Since $\sigma$ is $+1$ on $t_0$ and $-1$ on $a_0$ and since there are no real roots, $\sigma(\Delta^+) = \Delta^+$. Therefore $\sigma$ permutes the simple roots. It must fix the simple roots that are imaginary and permute in two-cycles the simple roots that are complex.

By the Vogan diagram of the triple $(\\mathfrak{g}, \phi, \Delta^+)$, we mean the Dynkin diagram of $\Delta^+$ with the two-element orbits under $\sigma$ labeled by an arrow and with the one-element orbits painted or not, according to the corresponding imaginary simple root, noncompact or compact.

**Equivalence of Vogan diagrams.** We define equivalence of Vogan diagrams to be the equivalence relations generated by the following two operations:

2. Change in the positive system by reflection in a simple, noncompact root, i.e., by a vertex which is colored in the Vogan diagram.

As a consequence of reflection by a simple, noncompact root $\alpha$, the rule for single and triple lines is that we leave $\alpha$ colored and its immediate neighbor changed to the opposite color. The rule for double lines is that if $\alpha$ is the smaller root, then there is no change in the color of its immediate neighbor, but we leave $\alpha$ colored. If $\alpha$ is the larger root, then we leave $\alpha$ colored and its immediate neighbor is changed to the opposite color.

If two Vogan diagrams are not equivalent to each other, we call them nonequivalent.

### 4. EXAMPLES

**4.1. Real Forms of $A_2^{(1)} = \mathfrak{sl}(3, \mathbb{C})[t, t^{-1}] \oplus \mathbb{C}e \oplus \mathbb{C}d$**

Here, the generalized Cartan matrix and corresponding Dynkin diagram are shown below.

\[
\begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{pmatrix}
\]

The generators are

\[
e_1 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\quad e_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\quad e_3 = t \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix},
\]

\[
f_1 = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\quad f_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
\quad f_3 = t^{-1} \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
The Cartan subalgebra of $A_2^{(1)}$ is 
\[(h_1 + h_2) \oplus \mathbb{C}c \oplus \mathbb{C}d,\]
where
\[h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad h_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

The Cartan semiinvolution $\omega'_s$ of $A_2^{(1)}$ is the following.
\[\omega'_s : \begin{array}{cccc}
    \phi & e_1 & \mapsto & -t^{-n}f_1 \\
    \phi & e_2 & \mapsto & -t^{-n}f_2 \\
    \phi & e_3 & \mapsto & -t^{-n}f_3 \\
    \phi & f_1 & \mapsto & -t^{-n}e_1 \\
    \phi & f_2 & \mapsto & -t^{-n}e_2 \\
    \phi & f_3 & \mapsto & -t^{-n}e_3 \\
    \phi & h_1 & \mapsto & -t^{-n}h_1 \\
    \phi & h_2 & \mapsto & -t^{-n}h_2 \\
    \phi & h_3 & \mapsto & -t^{-n}h_3 \\
    if & e_1 & \mapsto & ic \\
    if & e_2 & \mapsto & ic \\
    if & e_3 & \mapsto & ic \\
    if & f_1 & \mapsto & ic \\
    if & f_2 & \mapsto & ic \\
    if & f_3 & \mapsto & ic \\
    if & h_1 & \mapsto & ic \\
    if & h_2 & \mapsto & ic \\
    if & h_3 & \mapsto & ic \\
    c & \mapsto & -c \\
d & \mapsto & -d \\
ic & \mapsto & ic \\
id & \mapsto & id.
\end{array}\]

4.2. Almost Compact Real Forms of $\mathfrak{sl}(3, \mathbb{C})[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$

Using the involutions given by Kobayashi [10], we get the following five almost compact real forms of $\mathfrak{sl}(3, \mathbb{C})[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$.

(1) Involution of the first kind $\phi : e_1 \mapsto e_1, f_1 \mapsto f_1, e_2 \mapsto -e_2, f_2 \mapsto -f_2, e_3 \mapsto -e_3, f_3 \mapsto -f_3, c \mapsto c, d \mapsto d$. The corresponding semiinvolution of the second kind is $\omega'_s \circ \phi$. The corresponding almost compact real form is generated by \{\(e_1 t^n - f_1 t^{-n}, i(e_1 t^n + f_1 t^{-n}), f_1 t^n - e_1 t^{-n}, i(f_1 t^n + e_1 t^{-n}), e_2 t^n + f_2 t^{-n}, i(e_2 t^n - f_2 t^{-n}), f_2 t^n + e_2 t^{-n}, i(f_2 t^n - e_2 t^{-n}), e_3 t^n + f_3 t^{-n}, i(e_3 t^n - f_3 t^{-n})\); \(n \in \mathbb{Z}\) \(\oplus \mathbb{R}ic \oplus \mathbb{R}id\).
(2) Involution of the first kind \( \phi : e_1 \mapsto e_1, f_1 \mapsto f_1, e_2 \mapsto e_2, f_2 \mapsto f_2, e_3 \mapsto -e_3, f_3 \mapsto -f_3, c \mapsto c, d \mapsto d \). The corresponding semiinvolution of the second kind is \( \omega'_1 \circ \phi \). The corresponding almost compact real form is generated by \( \{ e_1 t^n + (-1)^{n+1} f_1 t^{-n}, i(e_1 t^n - (-1)^{n+1} f_1 t^{-n}), e_2 t^n + (-1)^{n+1} f_2 t^{-n}, i(e_2 t^n - (-1)^{n+1} f_2 t^{-n}), e_3 t^n + (-1)^{n+1} f_3 t^{-n}, i(e_3 t^n + (-1)^{n+1} f_3 t^{-n}); n \in \mathbb{Z} \} \oplus \mathfrak{Ric} \oplus \mathbb{R} d \).

(3) The compact real form, corresponding to the Cartan semiinvolution \( \omega'_2 \) is given by \( \{ e_1 t^n - f_1 t^{-n}, i(e_1 t^n + f_1 t^{-n}), h_1 t^n - h_1 t^{-n}, i(h_1 t^n + h_1 t^{-n}), e_2 t^n - f_2 t^{-n}, i(e_2 t^n + f_2 t^{-n}), h_2 t^n - h_2 t^{-n}, i(h_2 t^n + h_2 t^{-n}), e_3 t^n - f_3 t^{-n}, i(e_3 t^n + f_3 t^{-n}), h_3 t^n - h_3 t^{-n}, i(h_3 t^n + h_3 t^{-n}); n \in \mathbb{Z} \} \oplus \mathfrak{Ric} \oplus \mathbb{R} d \).

(4) Involution of the first kind \( \phi : e_1 \mapsto -f_1, f_1 \mapsto -e_1, e_2 \mapsto -f_2, f_2 \mapsto -e_3, e_3 \mapsto -t^2 f_3, f_3 \mapsto -t^{-2} e_3, c \mapsto c, d \mapsto d \). The corresponding semiinvolution of the second kind is \( \omega'_1 \circ \phi \). The corresponding almost compact real form is generated by \( \{ e_1 t^n + e_1 t^{-n}, i(e_1 t^n - e_1 t^{-n}), f_1 t^n + f_1 t^{-n}, i(f_1 t^n - f_1 t^{-n}), e_2 t^n + e_2 t^{-n}, i(e_2 t^n - e_2 t^{-n}), f_3 t^n + f_3 t^{-n}, i(f_3 t^n - f_3 t^{-n}); n \in \mathbb{Z} \} \oplus \mathfrak{Ric} \oplus \mathbb{R} d \).

(5) Involution of the first kind \( \phi : e_1 \mapsto -f_1, f_1 \mapsto -e_1, e_2 \mapsto -f_2, f_2 \mapsto -e_3, e_3 \mapsto t^2 f_3, f_3 \mapsto t^{-2} e_3, c \mapsto c, d \mapsto d \). The corresponding semiinvolution of the second kind is \( \omega'_1 \circ \phi \). The corresponding almost compact real form is generated by \( \{ e_1 t^n - (-1)^{n+1} e_1 t^{-n}, i(e_1 t^n + (-1)^{n+1} e_1 t^{-n}), f_1 t^n - (-1)^{n+1} f_1 t^{-n}, i(f_1 t^n + (-1)^{n+1} f_1 t^{-n}), e_2 t^n - (-1)^{n+1} e_2 t^{-n}, i(e_2 t^n + (-1)^{n+1} e_2 t^{-n}), f_3 t^n - (-1)^{n+1} f_3 t^{-n}, i(f_3 t^n + (-1)^{n+1} f_3 t^{-n}); n \in \mathbb{Z} \} \oplus \mathfrak{Ric} \oplus \mathbb{R} d \).

4.3. Vogan Diagrams Corresponding to Almost Compact Real Forms

(1)-(3) of \( \mathfrak{sl}(3, \mathbb{C})[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \)

Here, \( \phi \) is the complexified Cartan involution for the real form. So its fixed point set will give us \( \mathfrak{t}_0 \). The maximally compact Cartan subalgebra \( \mathfrak{h}_0 = i\mathfrak{t}_1 + i\mathfrak{t}_2 + ic + id = \mathfrak{t}_0 \).

We define simple roots as follows:

\( \alpha_1(h_2) = 2, \alpha_1(h_2) = -1, \alpha_1(c) = 0, \alpha_1(d) = 0 \), with corresponding root vector \( e_1 \).

\( \alpha_2(h_1) = -1, \alpha_2(h_2) = 2, \alpha_2(c) = 0, \alpha_2(d) = 0 \), with corresponding root vector \( e_2 \).

\( \alpha_3(h_1) = -1, \alpha_3(h_2) = -1, \alpha_3(c) = 0, \alpha_3(d) = 1 \), with corresponding root vector \( e_3 \).
So the Vogan diagram corresponding to (1) is the following, because $e_1$ is in $\mathfrak{f}$ and $e_2, e_3$ are in $\mathfrak{p}$.

The Vogan diagram corresponding to (2) is the following, because $e_1, e_2$ are in $\mathfrak{f}$ and $e_3$ is in $\mathfrak{p}$.

The Vogan diagram corresponding to (3) is the following, because $e_1, e_2, e_3$ are in $\mathfrak{f}$.

4.4. Vogan Diagrams Corresponding to Almost Compact Real Forms

(4) and (5) of $\mathfrak{sl}(3, \mathbb{C})[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$

The maximally compact Cartan subalgebra is

$$b_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + ic + id.$$  

We see from $\phi$'s that

$$t_0 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + ic + id, \quad \text{and} \quad a_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$
We define simple roots as follows:

\[ \alpha_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = 3, \quad \alpha_1 \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1, \quad \alpha_1(c) = 0, \quad \alpha_1(d) = 0. \]

The corresponding root vector is \( itf_3 + e_2 \).

\[ \alpha_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = -3, \quad \alpha_2 \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1, \quad \alpha_2(c) = 0, \quad \alpha_2(d) = 0. \]

The corresponding root vector is \( it^{-1}e_3 + f_2 \).

\[ \alpha_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = 0, \quad \alpha_3 \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -2, \quad \alpha_3(c) = 0, \quad \alpha_3(d) = 1. \]

The corresponding root vector is \( e_1t + f_1t - ih_1t \).

So the Vogan diagram for (4) is the following, because \( e_1t + f_1t - ih_1t \) is in \( p \).

![Vogan Diagram for (4)](image)

The Vogan diagram for (5) is the following, because \( e_1t + f_1t - ih_1t \) is in \( f \).

![Vogan Diagram for (5)](image)

5. MAIN RESULTS

**Definition 5.1.** An abstract Vogan diagram is an irreducible abstract Dynkin diagram of a nontwisted affine Kac–Moody Lie algebra with two pieces of additional structure indicated: One is an automorphism of order 1 or 2 of the diagram, which is indicated by labeling the two-element orbits.
The other is a subset of the one-element orbits, which is to be indicated by painting the vertices corresponding to the members of the subset. Every Vogan diagram is of course an abstract Vogan diagram.

**Theorem 5.2.** If an abstract Vogan diagram for a nontwisted affine Kac–Moody Lie algebra is given, then there exists an almost compact real form of a nontwisted affine Kac–Moody Lie algebra such that the given diagram is the Vogan diagram of this almost compact real form.

**Proof.** We can associate a generalized Cartan matrix $A$ with the given Dynkin diagram. According to Proposition 1.1 in [5], there exists a unique up-to-isomorphism realization for every such $n \times n$ matrix $A$. Following [5, Sect. 1.3], we can associate with $A$, a Kac–Moody Lie algebra $g = g(A)$. So $g$ is a nontwisted affine Kac–Moody Lie algebra with generators $(e_i, f_i, i \in I)$. Let $b$ be the Cartan subalgebra of $g$ as in the above realization. Let $\Delta = \Delta(g, h)$. Let $\Delta^+$ be the positive root system. Then by the conjugacy of compact real forms [1, Corollary 5.1], we can write an explicit formula for the compact real form $u_0$.

$$u_0 = \sum_{n, \alpha} \mathbb{R}(t^n e_a - f_a t^{-n}) + \sum_{n, \alpha} \mathbb{R}i(t^n e_a + f_a t^{-n})$$
$$+ \sum_{n, \alpha} \mathbb{R}i(t^n h_a + h_a t^{-n}) + \sum_{n, \alpha} \mathbb{R}(t^n h_a - h_a t^{-n}) + \mathbb{R}ic + \mathbb{R}id. \quad (5.1)$$

We define $Y_{n, \alpha} = t^n e_a$ and $Y_{-n, -\alpha} = t^{-n} f_a$. The given data determine an automorphism $\theta$ of the Dynkin diagram, which extends linearly to $h^*$ and is isometric. Let us see that $\theta(\Delta) = \Delta$. It is enough to see that $\theta(\Delta^+) \subset \Delta$.

We prove that $\theta(\Delta^+) \subset \Delta$ by induction on the level $\sum n_i$ of a positive root $\alpha = \sum n_i \alpha_i$. If the level is 1, then the root $\alpha$ is simple and we are given that $\theta \alpha$ is a simple root. Let $n > 1$, and assume inductively that $\theta \alpha$ is in $\Delta$ if $\alpha \in \Delta^+$ has level less than $n$. Let $\alpha$ be a positive root with level $n$. If we choose $\alpha_i$ simple with $(\alpha, \alpha_i) > 0$, then $r_{\alpha_i}(\alpha)$ is a positive root $\beta$ with smaller level than $\alpha$ [5, Lemma 3.7]. By inductive hypothesis $\theta \beta$ and $\theta \alpha_i$ are in $\Delta$. Since $\theta$ is isometric, it is easy to see that $\theta \alpha = r_\theta(\theta \beta)$ and therefore $\theta \alpha$ is in $\Delta$ by [5, Proposition 3.7b]. This completes the induction. Thus $\theta(\Delta) = \Delta$.

We can then transfer $\theta$ to $h$, retaining the same name $\theta$. We define $\theta$ on $Y_{n, \alpha}$ for simple roots $\alpha$ by

$$\theta(Y_{n, \alpha}) = Y_{n, \alpha}, \text{ if } \alpha \text{ is unpainted and forms a one-element orbit.} \quad (5.2)$$
$$\theta(Y_{n, \alpha}) = -Y_{n, \alpha}, \text{ if } \alpha \text{ is painted and forms a one-element orbit.} \quad (5.3)$$
$$\theta(Y_{n, \alpha}) = Y_{\theta(n, \alpha)}, \text{ if } \alpha \text{ is in a two-element orbit.} \quad (5.4)$$

By the isomorphism theorem [6], $\theta$ extends to an automorphism of $g$ consistently of these definitions on $h$ and on $Y_{n, \alpha}$’s for simple $\alpha$. It also follows by the isomorphism theorem [6] that $\theta^2 = 1$. 


The next step is to show that \( \theta(u_0) = u_0 \). For \( \alpha \in \Delta \), define a constant \( a_{n, \alpha} \) by \( \theta(Y_{n, \alpha}) = a_{n, \alpha} Y_{\theta(n, \alpha)} \). Then by [3, Proposition 3.4, Part R2], we get

\[
a_{n, \alpha} a_{-n, -\alpha} = (a_{n, \alpha} Y_{\theta(n, \alpha)}) (a_{-n, -\alpha} Y_{\theta(-n, -\alpha)}) \]

\[
= (\theta Y_{n, \alpha}, \theta Y_{-n, -\alpha})
\]

\[
= (Y_{n, \alpha}, Y_{-n, -\alpha})
\]

\[
= 1. \quad (5.5)
\]

We shall prove that

\[
a_{n, \alpha} = \pm 1 \quad \text{for all } \alpha \in \Delta \text{ and for all } n. \quad (5.6)
\]

To prove (5.6) it is enough to prove the result for \( \alpha \in \Delta^+ \) because of (5.5). We do so by induction on the level of \( \alpha \). If the level is 1, then (5.6) holds by (5.2)–(5.4). Thus it is enough to prove that if (5.6) holds for positive roots \( \alpha \) and \( \beta \) and if \( \alpha + \beta \) is a root, then it holds for \( \alpha + \beta \). In the notation of [4, Theorem 4.12], we have

\[
[Y_{n, \alpha}, Y_{m, \beta}] = N_{\alpha, \beta} Y_{n+m, \alpha+\beta}
\]

\[
\theta Y_{m+n, \alpha+\beta} = N_{\alpha, \beta}^{-1} \theta[Y_{n, \alpha}, Y_{m, \beta}]
\]

\[
a_{m+n, \alpha+\beta} Y_{\theta(m+n, \alpha+\beta)} = N_{\alpha, \beta}^{-1} [\theta Y_{n, \alpha}, \theta Y_{m, \beta}]
\]

\[
a_{m+n, \alpha+\beta} Y_{\theta(m+n, \alpha+\beta)} = N_{\alpha, \beta}^{-1} a_{n, \alpha} a_{m, \beta} [Y_{\theta(n, \alpha)}, Y_{\theta(m, \beta)}]
\]

\[
a_{m+n, \alpha+\beta} Y_{\theta(m+n, \alpha+\beta)} = N_{\alpha, \beta}^{-1} a_{n, \alpha} a_{m, \beta} N_{\theta\alpha, \theta\beta} Y_{\theta(m+n, \alpha+\beta)}
\]

\[
a_{m+n, \alpha+\beta} = N_{\alpha, \beta}^{-1} a_{n, \alpha} a_{m, \beta} N_{\theta\alpha, \theta\beta}
\]

\[
a_{m+n, \alpha+\beta} = \pm N_{\alpha, \beta}^{-1} N_{\theta\alpha, \theta\beta}
\]

because by assumption \( a_{n, \alpha} = \pm 1 \) and \( a_{m, \beta} = \pm 1 \). Using [4, Theorem 4.12], it is easy to see that

\[
N_{\alpha, \beta} = N_{\theta\alpha, \theta\beta}^2
\]

So \( a_{m+n, \alpha+\beta} = \pm 1 \) and (5.6) is proved. Let us see that

\[
\theta(\Re(Y_{n, \alpha} - Y_{-n, -\alpha}) + \Im(Y_{n, \alpha} + Y_{-n, -\alpha}))
\]

\[
\subseteq (\Re(Y_{\theta(n, \alpha)} - Y_{\theta(-n, -\alpha)}) + \Im(Y_{\theta(n, \alpha)} + Y_{\theta(-n, -\alpha)})) \quad (5.7)
\]

If \( z = x + yi \), where \( x, y \) are real, then \( x(Y_{n, \alpha} - Y_{-n, -\alpha}) + yi(Y_{n, \alpha} + Y_{-n, -\alpha}) = zY_{n, \alpha} - \bar{z}Y_{-n, -\alpha} \). Thus (5.7) amounts to the assertion that the expression \( \theta(zY_{n, \alpha} - \bar{z}Y_{-n, -\alpha}) = za_{n, \alpha} Y_{\theta(n, \alpha)} - \bar{z}a_{-n, -\alpha} Y_{\theta(-n, -\alpha)} \) must be
of the form \( wY_{\theta(n, \alpha)} - \bar{w}Y_{\theta(-n, -\alpha)} \), and this follows from (5.5) and (5.6). Since \( \theta(\Delta) = \Delta \), hence

\[
\theta \left( \sum_{\alpha} \mathbb{R}i h_{\alpha} + \mathbb{R}ic + \mathbb{R}i d \right) \subset \sum_{\alpha} \mathbb{R}i h_{\alpha} + \mathbb{R}ic + \mathbb{R}i d. \tag{5.8}
\]

Now we want to see that

\[
\theta \left( \sum_{n, \alpha} \mathbb{R}(t^n h_{\alpha} - h_{\alpha} t^{-n}) \right) \subset \sum_{n, \alpha} \mathbb{R}(t^n h_{\alpha} - h_{\alpha} t^{-n}), \tag{5.9}
\]

and

\[
\theta \left( \sum_{n, \alpha} \mathbb{R}i(t^n h_{\alpha} + h_{\alpha} t^{-n}) \right) \subset \sum_{n, \alpha} \mathbb{R}i(t^n h_{\alpha} + h_{\alpha} t^{-n}). \tag{5.10}
\]

For that using \([4, \text{Theorem 4.12(iii)}]\), we have

\[
[Y_{r, a}, Y_{m, -a}] = t^n h_{a}, \text{ where } r + m = n \neq 0.
\]

\[
\theta(t^n h_{a}) = \theta[Y_{r, a}, Y_{m, -a}]
\]

\[
= [a_{r, a} Y_{\theta(r, a)}, a_{m, -a} Y_{\theta(m, -a)}]
\]

\[
= a_{r, a} a_{m, -a} t^{(r+m)} H_{\theta a}
\]

\[
\Rightarrow a_{r, a} a_{m, -a} t^{\theta n} H_{\theta a}.
\]

Similarly \( \theta(t^{-n} h_{a}) = a_{-r, a} a_{m, -a} t^{-\theta n} H_{\theta a} \). Again using \([4, \text{Theorem 4.12(iii)}]\), we have that \( t^{\theta n} H_{\theta a} \) is an integral linear combination of \( t^{\theta n} H_{1}, t^{\theta n} H_{2}, \ldots, t^{\theta n} H_{n} \) and \( t^{-\theta n} H_{\theta a} \) is an integral linear combination of \( t^{-\theta n} H_{1}, t^{-\theta n} H_{2}, \ldots, t^{-\theta n} H_{n} \). Using (5.6) and the above argument we see that

\[
\theta \left( \sum_{n, \alpha} \mathbb{R}(t^n h_{\alpha} - h_{\alpha} t^{-n}) \right) \subset \sum_{n, \alpha} \mathbb{R}(t^n h_{\alpha} - h_{\alpha} t^{-n}).
\]

Similarly we can show (5.10). Using (5.1) and (5.7)–(5.10), we see that \( \theta(u_0) = u_0 \).

Let \( f \) and \( p \) be the +1 and −1 eigenspaces for \( \theta \) in \( g \), so that \( g = f \oplus p \). Since \( \theta(u_0) = u_0 \), we have \( u_0 = (u_0 \cap f) \oplus (u_0 \cap p) \). We define \( f_0 = u_0 \cap f \) and \( p_0 = i(u_0 \cap f) \), so that

\[
u_0 = f_0 \oplus i p_0.
\]

Since \( u_0 \) is a real form of \( g \) as a vector space, so is \( g_0 = f_0 \oplus p_0 \). Since \( \theta(u_0) = u_0 \) and \( \theta \) is an involution, we have the following bracket relations.

\[
[f_0, f_0] \subset f_0, \quad [f_0, p_0] \subset p_0, \quad [p_0, p_0] \subset f_0.
\]
Therefore $\mathfrak{g}_0$ is closed under brackets and is a real form of $\mathfrak{g}$ as a Lie algebra. The involution $\theta$ is 1 on $\mathfrak{f}_0$ and $-1$ on $\mathfrak{p}_0$; it is a Cartan involution of $\mathfrak{g}_0$, since $\mathfrak{u}_0 = \mathfrak{f}_0 \oplus i\mathfrak{p}_0$ is the compact real form of $\mathfrak{g}$.

By (5.8), $\theta$ maps $\mathfrak{u}_0 \cap \mathfrak{h}$ to itself; therefore
\[
\mathfrak{u}_0 \cap \mathfrak{h} = (\mathfrak{u}_0 \cap \mathfrak{f} \cap \mathfrak{h}) \oplus (i\mathfrak{p}_0 \cap \mathfrak{h}) = (\mathfrak{f}_0 \cap \mathfrak{h}) \oplus (i\mathfrak{p}_0 \cap \mathfrak{h}) = (\mathfrak{f}_0 \cap \mathfrak{h}) \oplus i(\mathfrak{p}_0 \cap \mathfrak{h}).
\]
The abelian subspace $\mathfrak{u}_0 \cap \mathfrak{h}$ is a real form of $\mathfrak{h}$, and hence $\mathfrak{h}_0 = (\mathfrak{f}_0 \cap \mathfrak{h}) \oplus (\mathfrak{p}_0 \cap \mathfrak{h})$.

The subspace $\mathfrak{h}_0$ is contained in $\mathfrak{g}_0$, and it is therefore a $\theta$-stable Cartan subalgebra of $\mathfrak{g}_0$. A real root $\alpha$ relative to $\mathfrak{h}_0$ has the property that $\theta(\alpha) = -\alpha$. Since $\theta$ preserves the positivity relative to $\Delta^+$, there are no real roots.

So by [1, Proposition 3.8], $\mathfrak{h}_0$ is maximally compact. Our definitions of $\theta$ on $\mathfrak{h}^*$ and on the $Y_{n,\alpha}$ for $\alpha$ simple make it clear that the Vogan diagram of $(\mathfrak{g}_0, \mathfrak{h}_0, \Delta^+)$ coincides with the given data. This completes the proof. 

**REFERENCES**