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On the zeta function of divisors for projective varieties with higher rank divisor class group

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ABSTRACT

Given a projective variety X defined over a finite field, the zeta function of divisors attempts to count all irreducible, codimension one subvarieties of X, each measured by their projective degree. When the dimension of X is greater than one, this is a purely p-adic function, convergent on the open unit disk. Four conjectures are expected to hold, the first of which is p-adic meromorphic continuation to all of $\mathbb{C}_p$. When the divisor class group (divisors modulo linear equivalence) of X has rank one, then all four conjectures are known to be true. In this paper, we discuss the higher rank case. In particular, we prove a p-adic meromorphic continuation theorem which applies to a large class of varieties. Examples of such varieties are projective nonsingular surfaces defined over a finite field (whose effective monoid is finitely generated) and all projective toric varieties (smooth or singular).

1. Introduction

This paper continues the study of the zeta function of divisors, a function first proposed by Professor Daqing Wan in [12]. Given a projective variety X defined over a finite field with a fixed embedding into projective space, the zeta function of divisors $Z_{\mathrm{div}}(X,T)$ is a function generated by all irreducible subvarieties P of codimension one, where each subvariety is measured by its projective degree:

$$Z_{\mathrm{div}}(X,T) := \prod_{P \text{ irred}} \frac{1}{1 - T^{\deg(P)}}. \tag{1}$$

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Notice that when X is a curve, this function agrees with the classical Hasse–Weil zeta function since divisors are now points. Unlike the Weil zeta function, which is always a rational function in T by Dwork's theorem, when the dimension of X is greater than one $Z_{\text{div}}(X,T)$ is never a rational function [12]. In fact it has zero radius of convergence over the complex numbers. However, over the p-adic numbers the situation is much better. An immediate consequence of the definition is that $Z_{\text{div}}(X,T)$ is p-adic analytic on the open unit disk. The study of whether this function can be p-adic meromorphically continued makes up the content of this paper.

Denote by $A^1(X)$ the divisor class group of X (divisors modulo linear equivalence) and let $A^+(X)$ be the image in $A^1(X)$ of all effective divisors. We call $A^+(X)$ the effective monoid. *Assume the effective monoid is finitely generated.* Then we have the following four conjectures.

Conjecture I *(p-Adic Meromorphic continuation). $Z_{\text{div}}(X,T)$ is p-adic meromorphic everywhere in T.*

Conjecture II *(Order and rank). $Z_{\text{div}}(X,T)$ has a pole at $T=1$ of order equal to the rank of $A^1(X)$.*

Conjecture III *(Simplicity of zeros and poles). Except for possibly finitely many, all zeros and poles are simple.*

Conjecture IV *(p-Adic Riemann hypothesis). Adjoining the zeros and poles of $Z_{\text{div}}(X,T)$ to the field $\mathbb{Q}_p$ produces a finite extension field of $\mathbb{Q}_p$.*

Conjectures I–III were first stated in [12], while IV was stated in [13].

Previous studies of the zeta function of divisors have focused on the case when $A^1(X)$ is of rank one. In fact, under this condition the conjectures have been completely answered:

Theorem 1. *(See [12,13].) If $A^1(X)$ is of rank one then all four conjectures are true.*

Examples of such varieties are nonsingular complete intersections of dimension 3 or greater and Grassman varieties. When the rank is greater than one, little is known. However, a positive result of Wan's states

Theorem 2. *(See Wan [12].) Suppose the effective monoid $A^+(X)$ is finitely generated over $\mathbb{Z}_{\geqslant 0}$. Let ρ denote the rank of $A^1(X)$ and h the order of its torsion subgroup. Then*

1. *$Z_{\text{div}}(X,T)$ is p-adic meromorphic on the closed unit disk $|T|_p \leqslant 1$.*
2. *Conjecture* II *holds. That is, $Z_{\text{div}}(X,T)$ has a pole of order ρ at $T=1$.*
3. *The special value at $T=1$ is given by*

$$\lim_{T\to 1}(1-T)^{\rho} Z_{\text{div}}(X,T) = \frac{1}{q^{\lambda}-1} hR(X)$$

where there are $q^{\lambda}-1$ roots of unity in the function field of X, and $R(X)$ is a type of regulator (see [12] *for the definition).*

When the rank of $A^1(X)$ is strictly greater than one, only one example currently exists in the literature where all four conjectures have been proven. This is the quadric surface $xw=yz$ in $\mathbb{P}^3$ which has rank 2 [13].

All positive results on the zeta function of divisors so far have been tied with various positive results on the (generalized) Riemann–Roch problem. For an effective divisor D on X, define $l(D) := \dim_{\mathbb{F}_q} H^0(X, \mathcal{O}(D))$. The Riemann–Roch problem asks about the behavior of $l(nD)$ as n varies. For instance, if D is ample then there exists a polynomial which when evaluated at n agrees with $l(nD)$. Often, not one but a finite number of polynomials describe the Riemann–Roch problem. Define a *quasi-polynomial* as a function $f:\mathbb{Z}\to\mathbb{R}$ together with d polynomials $p_1(x),\dots,p_d(x)$ which satisfy

$f(n) = p_i(n)$ if $n \equiv i \bmod d$. For projective nonsingular surfaces defined over $\mathbb{F}_q$, $l(nD)$ equals a quasi-polynomial for all n sufficiently large (see Section 2). This is also true for toric varieties (see Section 3). For the zeta function of divisors, we will need to consider the following generalization.

Definition 1. *We say an effective divisor D has* quasi-polynomial growth *if for each effective divisor E on X, $l(E + nD)$ agrees with a quasi-polynomial for all n sufficiently large.*

The main theorem of this paper is the following.

Theorem 3. *Let X be a projective variety defined over a finite field. Assume the effective monoid $A^+(X)$ is finitely generated and each generator has quasi-polynomial growth. Then*

1. *$Z_{\text{div}}(X, T)$ is p-adic meromorphic.*
2. *The poles of $Z_{\text{div}}(X, T)$ are algebraic over $\mathbb{Q}_p$ of bounded degree.*

As mentioned above, when the dimension of X is greater than one, the zeta function of divisors has zero radius of convergence over the complex numbers. This suggests that a weight should be used to compensate, which is what Atsuchi Moriwaki considers in [7]. First, notice that

$$Z_{\text{div}}(X, T) = \sum_{d=0}^{\infty} M_d T^d$$

where M_d is the number of effective divisors on X of degree d. Moriwaki has shown that the series

$$\sum_{d=0}^{\infty} M_d T^{d^{\dim(X)}}$$

does converge on a small disk around the origin in the complex numbers. It would be interesting to see what further properties this series possesses.

Height zeta function. Let X be an (integral) projective variety defined over $\mathbb{F}_q$, and denote its function field by $\mathbb{F}_q(X)$. With an $\mathbb{F}_q(X)$-rational point $y := [y_0 : y_1 : \cdots : y_n] \in \mathbb{P}^n(\mathbb{F}_q(X))$ we may define the (logarithmic) height

$$h_X(y) := -\deg \inf_i (y_i)$$

where $\inf_i(y_i)$ denotes the greatest divisor D of X such that $D \leqslant (y_i)$ for all i with $y_i \neq 0$. Next, let $Y \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q(X)}$ be a projective variety defined over $\mathbb{F}_q(X)$. Define the height zeta function of $Y/\mathbb{F}_q(X)$ as

$$Z_{\text{ht}}\big(Y\big(\mathbb{F}_q(X)\big), T\big) := \sum_{d=0}^{\infty} N_d(Y) T^d$$

where

$$N_d(Y) := \#\big\{y \in Y\big(\mathbb{F}_q(X)\big) \bigm| h_X(y) = d\big\}.$$

Assuming $A^1(X)$ is of rank one, Wan [11] proved $Z_{\text{ht}}(\mathbb{P}^n(\mathbb{F}_q(X)), T)$ is always p-adic meromorphic; further, it is rational if and only if $\dim(X) = 1$. He conjectured that p-adic meromorphic continuation should hold for any projective variety X defined over $\mathbb{F}_q$ whose effective monoid is finitely generated. From the methods in this paper and [6], we are able to support his conjecture by the following:

Theorem 4. *Let X be a projective variety defined over a finite field. Assume the effective monoid $A^+(X)$ is finitely generated and each generator has quasi-polynomial growth. Then $Z_{\mathrm{ht}}(\mathbb{P}^n(\mathbb{F}_q(X)), T)$ is p-adic meromorphic.*

2. Riemann–Roch problem for surfaces

The zeta function of divisors was initially defined and studied by Wan [12]. In that paper he establishes a method of study based on the Riemann–Roch approach of F.K. Schmidt [10] (see [9, Section 4.3.3] for a historical account) who used this method to prove rationality of the Weil zeta function of curves defined over a finite field. An initial obstruction to this line of investigation is an understanding of the so-called Riemann–Roch problem.

Given a divisor D, the Riemann–Roch problem is concerned with the dimension variation of the space of global sections $H^0(X, \mathcal{O}(nD))$ as n varies. This variation is often very well behaved. Indeed, when D is ample, the dimension $l(nD) := \dim H^0(X, \mathcal{O}(D))$ equals a (Hilbert–Samuel) polynomial of degree $\dim(X)$ evaluated at n for all n sufficiently large. When this type of behavior is expected in general seems to be an open question. However, it certainly does not always occur as an example of Cutkosky and Srinivas demonstrates [2, Section 7]. They describe an effective divisor D on a nonsingular projective 3-fold defined over a finite field for which $l(nD)$ is a polynomial of degree 3 in $\lfloor n(2 - \sqrt{3}/3)\rfloor$.

The quasi-polynomial behavior for surfaces discussed below is a consequence of a deeper algebraic result. For each divisor D define the graded $\mathbb{F}_q$-algebra

$$R[X, D] := \bigoplus_{n\geqslant 0} H^0\big(X, \mathcal{O}(nD)\big).$$

Next, for each divisor E on X define the $R[X, D]$-module

$$R[X, D; E] := \bigoplus_{n\geqslant 0} H^0\big(X, \mathcal{O}(E + nD)\big).$$

If $R[X, D]$ is finitely generated then for all n sufficiently large $l(nD)$ equals a quasi-polynomial in n. Furthermore, when $R[X, D]$ is finitely generated and $R[X, D; E]$ is finite over $R[X, D]$ then $l(E + nD)$ equals a quasi-polynomial in n.

Question 1. *When is $R[X, D]$ finitely generated, and when is $R[X, D; E]$ a finitely generated graded $R[X, D]$-module*?

For D a semi-ample divisor then a positive answer to the latter part of Question 1 is known. Recall, D is a semi-ample divisor if there exists a positive integer m such that the complete linear system $|mD|$ is base point free. In this case, for any projective variety X defined over an arbitrary field, Zariski [14, Theorem 5.1] has shown that for every effective divisor E and semi-ample divisor D, $R[X, D; E]$ is a finitely generated graded $R[X, D]$-module.

Theorem 5. *Let X be a projective nonsingular surface defined over the finite field $\mathbb{F}_q$. Let D be an effective divisor on X. Then,*

1. (*Zariski* [14], *Cutkosky and Srinivas* [2]) *$R[X, D]$ is a finitely generated, graded $\mathbb{F}_q$-algebra.*
2. *$R[X, D; E]$ is a finite $R[X, D]$-module for every effective divisor E on X.*

Proof. That $R[X, D]$ is finitely generated was proven by Zariski [14] and Cutkosky and Srinivas [2]. In order to prove the second part of the theorem, we will need to recall their methods. Let D be an effective divisor of X, and denote by $P_1, \ldots, P_m$ its prime components. We may define a (symmetric) quadratic form ϕ_D by $\sum (P_i \cdot P_j) x_i x_j$. The eigenvalues associated to this form are real, and all but at most one can be positive [14, p. 588]. We say D is of type (s,t) if the associated form has s positive eigenvalues and t negative eigenvalues; that is, D may be of type $(1,t)$ or $(0,t)$.

Case ϕ_D is of type $(0,t)$. In [14, Theorem 11.5], Zariski proves $R[X,D]$ is a finitely generated $\mathbb{F}_q$-algebra. Using his notation, for some $n', e \in \mathbb{Z}_{>0}$, he demonstrates that the complete linear system $|n'eD|$ has no fixed components. By [14, Theorem 6.1], this means $|mn'eD|$ is base point free for all m sufficiently large. It follows from [14, Theorem 5.1] that $R[X,D;E]$ is a finite $R[X,D]$-module.

Case ϕ_D is of type $(1,t)$. By Zariski Decomposition [14, Theorem 7.7] (see [1] for a quick proof) there exists a unique $\mathbb{Q}$-divisor $\mathcal{E}$, called the arithmetically negative part of D, for which by [14, Theorem 10.6] $R[X,D]$ is a finitely generated $\mathbb{F}_q$-algebra if and only if $|m_0(D-\mathcal{E})|$ has no fixed components for some m_0 (hence $D-\mathcal{E}$ is semi-ample). Note, $m_0\mathcal{E}$ is a $\mathbb{Z}$-divisor. In the proof of this, Zariski says $nm_0\mathcal{E}$ is a fixed component of $|nm_0D|$ for all n, which follows from [14, Theorem 8.1]. Thus, we have an equality between linear systems:

$$\bigl\{f \in \mathbb{F}_q(X) \bigm| (f) + nm_0D \geqslant 0\bigr\} = \bigl\{f \in \mathbb{F}_q(X) \bigm| (f) + nm_0D - nm_0\mathcal{E} \geqslant 0\bigr\}.$$

Consequently,

$$R[X, m_0D] = R\bigl[X, m_0(D-\mathcal{E})\bigr].$$

Zariski was unable to prove $D-\mathcal{E}$ is semi-ample. However, Cutkosky and Srinivas in the proof of [2, Theorem 3] indeed show $D-\mathcal{E}$ is semi-ample. Thus, since

$$R[X,D] = \bigoplus_{r=0}^{m_0-1} R[X, m_0D; rD] = \bigoplus_{r=0}^{m_0-1} R\bigl[X, m_0(D-\mathcal{E}); rD\bigr] \tag{2}$$

and $R[X, m_0(D-\mathcal{E}); rD]$ is a finite $R[X, m_0(D-\mathcal{E})]$-module (hence a finite $R[X, m_0D]$-module), we see that $R[X,D]$ is a finite $R[X,m_0D]$-module. Since $R[X,D]$ is an algebra, and $R[X,m_0D]$ is a finitely generated algebra, it follows that $R[X,D]$ is a finitely generated $\mathbb{F}_q$-algebra.

A similar argument follows for $R[X,D;E]$: decompose $R[X,D;E]$ as in (2). Then $R[X, m_0(D-\mathcal{E}); rD+E]$ is a finite $R[X, m_0(D-\mathcal{E})]$-module, hence a finite $R[X,m_0D]$-module, and thus, a finite $R[X,D]$-module. □

Corollary 1. *Let X be a projective nonsingular surface defined over the finite field $\mathbb{F}_q$. All effective divisors D on X have quasi-polynomial growth.*

While divisors on a nonsingular surface defined over a finite field may have quasi-polynomial growth, it may happen that the effective monoid $A^+(X)$ is not finitely generated. For instance, it is known that on a nonsingular projective rational surface whose anticanonical divisor $-K_X$ is effective, $A^+(X)$ is finitely generated if and only if X has only a finite number of (-1)-curves and only a finite number of (-2)-curves; examples where such curves are finite in number may be found in [4]. However, this cannot happen on toric varieties.

In the next section we will see that $A^+(X)$ is always finitely generated for projective toric varieties. Furthermore, every effective divisor D has quasi-polynomial growth.

3. Riemann–Roch problem for toric varieties

For toric varieties, the Riemann–Roch problem has an elegant solution related to counting integer points in polytopes. For any T-invariant Weil divisor D on a complete toric variety, $l(nD)$ equals the number of lattice points in an associated polytope P_{nD}. Since $P_{nD} = nP_D$, we see that the Riemann–Roch problem is equivalent to studying the number of lattice points in a dilation of the polytope P_D. This was extensively studied, and answered, by Ehrhart. In particular, $l(nD)$ is again a quasi-polynomial in n. Let us be more precise.

We will freely use results and terminology from Fulton [5]. While Fulton's results are mostly stated over the complex numbers, only an algebraically closed field is needed for the results we will use.

Let k be an algebraically closed field. Let X be a d-dimensional projective toric variety over k, defined by a complete fan Δ in a lattice $N \cong \mathbb{Z}^d$. Let $M := Hom(N, \mathbb{Z})$ be the dual lattice of N, and denote by $\langle \cdot, \cdot \rangle$ the bilinear pairing on M and N. Let $\Delta(1)$ be the set of one-dimensional rays (cones) in Δ. For each $\rho \in \Delta(1)$, denote by $v_\rho \in N$ the unique generator of $\rho \cap N$.

There is an action of the d-torus $T := N \otimes_{\mathbb{Z}} k^* = Hom_{\mathbb{Z}}(N, k^*)$ on X. For every $\rho \in \Delta(1)$ there corresponds an irreducible T-invariant Weil divisor D_ρ which is the orbit closure of T acting on the distinguished point associated to ρ. Further, every T-invariant Weil divisor is a linear combination of these.

It follows from the following exact sequence that the group $A^1(X)$ is completely determined by the torus action:

$$0 \to M \to \bigoplus_{\rho \in \Delta(1)} \mathbb{Z} D_\rho \to A^1(X) \to 0 \tag{3}$$

where the second map is $m \mapsto \sum_{\rho \in \Delta(1)} \langle m, v_\rho \rangle D_\rho$. This exact sequence also helps determine the effective monoid $A^+(X)$ as Theorem 6 below demonstrates.

Associated to each T-invariant Weil divisor $D = \sum_{\rho \in \Delta(1)} b_\rho D_\rho$ is a polyhedron P_D defined by

$$P_D := \{ m \in M \otimes_{\mathbb{Z}} \mathbb{R} \colon \ \langle m, v_\rho \rangle \geqslant -b_\rho \text{ for each } \rho \}. \tag{4}$$

This polyhedron is compact since the fan Δ is complete. There is a bijection between a basis of the Riemann–Roch space $H^0(X, \mathcal{O}(D))$ and the number of lattice points in the polyhedron P_D:

$$l(D) = \#(P_D \cap M). \tag{5}$$

Theorem 6. *For X a projective toric variety, the effective monoid $A^+(X)$ is finitely generated. In particular, it is generated by the image of $\bigoplus_{\rho \in \Delta(1)} \mathbb{Z}_{\geqslant 0} D_\rho$ in $A^1(X)$.*

Proof. Let E be an effective divisor on X. By (3), E is linearly equivalent to a T-invariant divisor D, thus $l(D) = \#(P_D \cap M) \geqslant 1$. Next, write $D = \sum_{\rho \in \Delta(1)} b_\rho D_\rho$. Let A be the matrix with rows the vectors v_ρ and let b be the column vector with entries b_ρ. We say $v \geqslant 0$ if its entries are each greater than or equal to zero. With this new notation, and viewing the pairing between N and M as an inner product, we may rewrite (4) as

$$P_D = \{ m \in M \otimes_{\mathbb{Z}} \mathbb{R} \colon \ Am \geqslant -b \}$$

where we represent D as the vector b. Since $P_D \cap M$ is nonempty, there exist vectors $m \in M$ and $q \geqslant 0$ such that $Am + b = q$. In terms of divisors, this means D is linearly equivalent to the divisor D_q defined by q. The theorem follows since D_q is an element of the effective divisors $\bigoplus_{\rho \in \Delta(1)} \mathbb{Z}_{\geqslant 0} D_\rho$. $\square$

Theorem 7. *Let E and D be effective T-invariant divisors on a projective toric variety of dimension d. Then $l(E + nD)$ equals a quasi-polynomial, whose degree is bounded below by* 1 *and above by d, for all nonnegative integers n.*

Proof. From (5) we have $l(E + nD) = \#(P_{nD+E} \cap M)$ for every n. Write $D = \sum_{\rho \in \Delta(1)} b_\rho D_\rho$ and $E = \sum_{\rho \in \Delta(1)} a_\rho D_\rho$ with $b_\rho, a_\rho \in \mathbb{Z}_{\geqslant 0}$. As in the proof of Theorem 6, let A be the matrix with rows the vectors v_ρ, let b be the column vector with entries b_ρ, and let a be the column vector with entries a_ρ.

Then

$$P_{nD+E} = \{m \in M \otimes_{\mathbb{Z}} \mathbb{R}\colon\ Am \geqslant -nb - a\}.$$

Using terminology from [8], since $P_{nD} = nP_D$, the polytope P_{E+nD} constitutes a *bordered system*. By [8, Theorem 2], the number of lattice points in P_{nD+E} equals a quasi-polynomial of degree bounded above by d. Furthermore, this holds for all n. Since the dimension of P_{nD} is at least one, the degree of the quasi-polynomial is at least one. □

While Theorem 7 allows us to prove p-adic meromorphic continuation for the zeta function of divisors, the proof is rather unsatisfying since the author believes the stronger result is true, that $R[X, D; E]$ is finite over $R[X, D]$. Note, Elizondo [3] has already shown that $R[X, D]$ is finitely generated for any effective divisor D.

4. *p*-Adic meromorphic continuation

In this section, we will prove a p-adic meromorphic continuation theorem for the zeta function of divisors which applies whenever the generators of $A^+(X)$ have quasi-polynomial growth. The proof will also show that adjoining all the poles of the zeta function to $\mathbb{Q}_p$ creates a finite extension field.

When the effective monoid $A^+(X)$ is finitely generated, Wan [12] reduces the zeta function of divisors to a finite sum of both rational functions, and series of the form

$$Z_E(X, T) := \sum_{n_1,\dots,n_r \geqslant 0} q^{l(E+n_1D_1+\cdots+n_rD_r)} T^{e+n_1d_1+\cdots+n_rd_r}$$

where $E, D_1, D_2, \dots, D_r$ are effective divisors, $e := \deg(E)$, and $d_i := \deg(D_i)$. Thus, we need only study the p-adic analytic behavior of these series.

Lemma 1. *Let X be a projective variety. Let $D_1, \dots, D_r$ be effective divisors on X, each having quasi-polynomial growth. Then for every effective divisor E on X there exists a positive integer N such that $l(E + n_1D_1 + \cdots + n_rD_r)$ equals a quasi-polynomial in the variables $n_1, \dots, n_r$ for all $n_i \geqslant N$.*

Proof. This was first proven in [8, Theorem 2] in the context of counting lattice points in polytopes. For completeness, we will present it here. The proof follows by induction on r. Assume the result for $r - 1$ divisors. Fixing positive integers $n_1, \dots, n_{r-1}$ and letting n_r vary, since D_r has quasi-polynomial growth there exists N_r such that

$$l(E + n_1D_1 + \cdots + n_rD_r) = \alpha_d n_r^d + \alpha_{d-1} n_r^{d-1} + \cdots + \alpha_0 \quad \text{for all } n_r \geqslant N_r,$$

where the right-hand side is a quasi-polynomial. Note, the coefficients α_i depend on the fixed $n_1, \dots, n_{r-1}$. Plugging in values for n_r, we obtain a linear system

$$\begin{cases} \alpha_d(N_r+0)^d + \alpha_{d-1}(N_r+0)^{d-1} + \cdots + \alpha_0 = l\big(E + n_1D_1 + \cdots + (N_r+0)D_r\big), \\ \alpha_d(N_r+1)^d + \alpha_{d-1}(N_r+1)^{d-1} + \cdots + \alpha_0 = l\big(E + n_1D_1 + \cdots + (N_r+1)D_r\big), \\ \quad\vdots \end{cases}$$

By Cramer's rule, each coefficient α_i is a linear combination of elements from the right-hand side. Since, by the induction hypothesis, each member of the right-hand side is a quasi-polynomial for all large $n_1, \dots, n_{r-1}$, we see that α_i will also be of this form. □

From this lemma and the discussion at the beginning of this section, the zeta function of divisors may be reduced to a finite sum of both rational functions, and series of the form

$$\sum_{x_1,\dots,x_n \geqslant 0} q^{l(x_1,\dots,x_n)} T^{d_1x_1+\cdots+d_nx_n}$$

where $l(x_1,\dots,x_n)$ is an increasing polynomial with rational coefficients in the variables $x_1,\dots,x_n$.

Theorem 8. *Let $f \in \mathbb{Q}[x_1,\dots,x_n]$ be an increasing polynomial; that is,*

$$f(x_1,\dots,x_i+1,\dots,x_n) \geqslant f(x_1,\dots,x_i,\dots,x_n) \quad \text{for all } x_1,\dots,x_n \in \mathbb{Z}_{\geqslant 0}.$$

Let $d_1,\dots,d_n$ be positive integers. Then

$$F(T) := \sum_{x_1,\dots,x_n \geqslant 0} q^{f(x_1,\dots,x_n)} T^{d_1x_1+\cdots+d_nx_n}$$

is p-adic meromorphic on $\mathbb{C}_p$. Furthermore, adjoining the poles to $\mathbb{Q}_p$ creates a finite extension field of $\mathbb{Q}_p$.

The proof will consist of the rest of this section and will follow by induction on the number of variables n. In the case when there is only one variable, $n = 1$, the result follows by either an application of the geometric series (when f is a linear polynomial) in which case $F(T)$ is meromorphic, else it is entire (when f has degree at least two).

Suppose the result is true for $1, 2, \dots, n-1$ number of variables. Let S_n denote the symmetric group on $\{1,\dots,n\}$. A permutation $\sigma \in S_n$ acts on points by permuting the coordinates:

$$\sigma(x_1,\dots,x_n) := (x_{\sigma(1)},\dots,x_{\sigma(n)}).$$

Define

$$E^{(n)} := \bigl\{(k_1+k_2+\cdots+k_n, k_2+k_3+\cdots+k_n,\dots,k_n) \bigm| k_1,\dots,k_n \in \mathbb{Z}_{\geqslant 0}\bigr\}.$$

For each $\sigma \in S_n$, define the set $E_\sigma^{(n)} := \sigma(E^{(n)})$. Notice that $\mathbb{Z}^n_{\geqslant 0} = \bigcup_{\sigma \in S_n} E_\sigma^{(n)}$, and that points in $\mathbb{Z}^n_{\geqslant 0}$ may be in several $E_\sigma^{(n)}$. We would like to use the inclusion–exclusion principle on this collection. To accomplish this, let $\mathcal{E}_n$ denote the collection of all possible intersections between elements of $\{E_\sigma^{(n)}\}_{\sigma \in S_n}$ (that is, $\mathcal{E}_n$ is the closure of the set $\{E_\sigma^{(n)}\}_{\sigma \in S_n}$ by the operation of intersection). Observe that elements in $\mathcal{E}_n$ are of the form:

$$\bigl\{\sigma(k_1+k_2+\cdots+k_r,\dots,k_r) \bigm| k_1,\dots,k_r \in \mathbb{Z}_{\geqslant 0}\bigr\},$$

for some $1 \leqslant r \leqslant n$ and $\sigma \in S_n$ with possible repetition of the coordinates in the middle. For example, with $n = 3$ and $r = 2$, $\mathcal{E}_3$ contains a set of the form

$$\bigl\{(k_1+k_2, k_1+k_2, k_2) \bigm| k_1, k_2 \in \mathbb{Z}_{\geqslant 0}\bigr\}.$$

Consequently, using the inclusion–exclusion principle, we may write

$$F(T) = \sum_{H \in \mathcal{E}_n} c_H \sum_{(x_1,\dots,x_n) \in H} q^{f(x_1,\dots,x_n)} T^{d_1x_1+\cdots+d_nx_n} \tag{6}$$

where c_H is some constant coming from the inclusion–exclusion. Since $\mathcal{E}_n$ is a finite set, we are reduced to proving $\sum_{(x_1,\dots,x_n)\in H} q^{f(x_1,\dots,x_n)} T^{d_1x_1+\dots+d_nx_n}$ is p-adic meromorphic for each $H \in \mathcal{E}_n$.

For a typical sum in (6), we have

$$\sum_{k_1,\dots,k_r\geqslant 0} q^{f(k_1+\dots+k_r,\dots,k_r)} T^{d_1(k_1+\dots+k_r)+\dots+d_n(k_r)}. \tag{7}$$

Notice that, unless H equals $E_\sigma^{(n)}$ for some σ, we may write

$$f(k_1+\dots+k_r,\dots,k_r) = g(k_1,\dots,k_r)$$

where $1 \leqslant r \leqslant n-1$ and g is an increasing polynomial in the variables $k_1,\dots,k_r$. It follows by the induction hypothesis that (7) is p-adic meromorphic. Thus, we need only consider the case when $H = E_\sigma^{(n)}$.

Without loss of generality, suppose $H = E^{(n)}$. Thus, with

$$\begin{aligned} x_1 &= k_1 + k_2 + k_3 + \dots + k_n, \\ x_2 &= \phantom{k_1 + {}} k_2 + k_3 + \dots + k_n, \\ &\vdots \qquad\qquad \ddots \\ x_n &= \phantom{k_1 + k_2 + k_3 + \dots + {}} k_n, \end{aligned} \tag{8}$$

let us prove

$$F_1(T) := \sum_{k_1,\dots,k_n\geqslant 0} q^{f(k_1+\dots+k_n,\dots,k_n)} T^{d_1(k_1+\dots+k_n)+\dots+d_n(k_n)} \tag{9}$$

is p-adic meromorphic. Our argument breaks up into two parts: when $\deg_{x_1}(f) \geqslant 2$ and when $\deg_{x_1}(f) = 1$.

Let us first consider the case when $\deg_{x_1}(f) \geqslant 2$. We need the following technical lemma.

Lemma 2. *For each $i = 1,\dots,n$, write*

$$f(k_1+\dots+k_n,\dots,k_n) = A_{r_i}^{(i)} k_i^{r_i} + A_{r_i-1}^{(i)} k_i^{r_i-1} + \dots + A_0^{(i)}.$$

Then $A_{r_i}^{(i)}$ is an increasing polynomial which depends only on $k_{i+1},\dots,k_n$. Furthermore, $A_{r_i}^{(i)}$ is always nonnegative, and $r_i \geqslant \max_{1\leqslant j\leqslant i} \deg_{x_j}(f)$ for each i.

Proof. Notice that f is an increasing polynomial in $k_1,\dots,k_n$. It is not hard to show that the coefficient $A_{r_i}^{(i)}$ of the largest power of k_i is a polynomial in $k_{i+1},\dots,k_n$. Let us show it is an increasing polynomial. Write $A_{r_i}^{(i)} = g_{r_i}(k_{i+1},\dots,k_n)$. Writing $f(k_1,\dots,k_n)$ for $f(k_1+\dots+k_n,\dots,k_n)$, then for any $j > i$ we have

$$\begin{aligned} f(k_1,\dots,k_j+1,\dots,k_n) - f(k_1,\dots,k_n) &= \bigl(g_{r_i}(k_{i+1},\dots,k_j+1,\dots,k_n) - g_{r_i}(k_{i+1},\dots,k_n)\bigr)k_i^{r_i} \\ &\quad + (\text{lower terms}). \end{aligned}$$

Since f is an increasing function, this must be nonnegative for all $k_1,\dots,k_n \geqslant 0$, and so the coefficient of $k_i^{r_i}$ must be nonnegative. This shows g_{r_i} is an increasing polynomial.

Next, let us show $r_i \geqslant \max_{1\leqslant j\leqslant i} \deg_{x_j}(f)$ and g_{r_i} is not identically zero. Write

$$f(x_1,\dots,x_n)=\sum_{j_1,\dots,j_n} C_{j_1,\dots,j_n} x_1^{j_1}\cdots x_n^{j_n}$$

where the $C_{j_1,\dots,j_n}$ are nonzero rational numbers. Let R_i be the maximum of all $j_1+\cdots+j_i$. We will show $R_i=r_i$. Write

$$f(x_1,\dots,x_n)=\sum_{j_1+\cdots+j_i=R_i} H_{j_1,\dots,j_i}(x_{i+1},\dots,x_n)x_1^{j_1}\cdots x_i^{j_i}+(\text{lower terms}) \tag{10}$$

where $H_{j_1,\dots,j_i}$ is a polynomial in $x_{i+1},\dots,x_n$. Note, "lower terms" means $x_1^{l_1}\cdots x_n^{l_n}$ with $l_1+\cdots+l_n<R_i$. Using the substitution (8), we have

$$\begin{aligned} f(k_1,\dots,k_n)=\sum_{j_1+\cdots+j_i=R_i} & H_{j_1,\dots,j_i}(k_{i+1}+\cdots+k_n,\dots,k_n)(k_1+\cdots+k_n)^{j_1}\cdots(k_i+\cdots+k_n)^{j_i} \\ & +(\text{lower terms}). \end{aligned} \tag{11}$$

Notice that the largest power of k_i to appear is $k_i^{R_i}$, and furthermore, by the definition of R_i, must satisfy $R_i\geqslant \max_{1\leqslant j\leqslant i}\deg_{x_j}(f)$. Let us show the coefficient of $k_i^{R_i}$ is nonzero (and hence $r_i=R_i$).

The coefficient of $k_i^{R_i}$ in (11) is

$$\sum_{j_1+\cdots+j_i=R_i} H_{j_1,\dots,j_i}(k_{i+1}+\cdots+k_n,\dots,k_n),$$

and this contains the unique polynomial

$$\sum_{j_1+\cdots+j_i=R_i} H_{j_1,\dots,j_i}(k_{i+1},\dots,k_n)$$

which is a nonzero polynomial by (10). □

Let us show F_1 is p-adic meromorphic when $\deg_{x_1}(f)\geqslant 2$. By Lemma 2, the coefficient of the largest power of k_n is a positive rational number, and since the degree of x_1 is at least 2, the degree of k_n is also at least two. Consider the coefficient of the largest power of k_{n-1} in f. By Lemma 2, this is a nonzero increasing polynomial in k_n. Find k_n' such that this polynomial is positive for all $k_n\geqslant k_n'$. Then

$$F_1(T)=\Biggl(\sum_{\substack{k_1,\dots,k_{n-1}\geqslant 0\\ 0\leqslant k_n<k_n'}}+\sum_{\substack{k_1,\dots,k_{n-1}\geqslant 0\\ k_n\geqslant k_n'}}\Biggr)q^{f(k_1+\cdots+k_n,\dots,k_n)}T^{d_1k_1+\cdots+d_nk_n}.$$

Notice that the first sum is p-adic meromorphic by the induction hypothesis (it has fewer than n variables). Thus, let us proceed with the second summation.

By Lemma 2, the coefficient of the largest power of k_{n-2} is a nonzero increasing polynomial in k_{n-1} and k_n. Find constants k_{n-1}' and k_n'' such that this polynomial is always positive whenever $k_{n-1}\geqslant k_{n-1}'$ and $k_n\geqslant k_n''$. Then we may write

$$(\text{second sum}) = \left(\sum_{\substack{k_1,\dots,k_n \geqslant 0 \\ 0 \leqslant k_{n-1} < k'_{n-1} \text{ or } k'_n \leqslant k_n < k''_n}} + \sum_{\substack{k_1,\dots,k_{n-2} \geqslant 0 \\ k_{n-1} \geqslant k'_{n-1} \text{ and } k_n \geqslant k''_n}} \right) q^{f(k_1+\cdots+k_n,\dots,k_n)} T^{d_1k_1+\cdots+d_nk_n}.$$

As before, the first summation is p-adic meromorphic by the induction hypothesis.

Continuing this argument, we are finally left with a series of the form

$$\sum_{k_1,\dots,k_n \gg 0} q^{f(k_1+\cdots+k_n,\dots,k_n)} T^{d_1(k_1+\cdots+k_n)+\cdots+d_n(k_n)}$$

where $\deg_{k_i}(f_i) \geqslant 2$ and the coefficient of the largest power of k_i is always positive. It follows from the proof of Lemma 4 below that this is a p-adic entire function. This finishes the proof of the result when $\deg_{x_1}(f) \geqslant 2$.

Consider now the case when $\deg_{x_1}(f) = 1$. In this case, if we write

$$f(x_1,\dots,x_n) = g_1(x_2,\dots,x_n)x_1 + g_0(x_2,\dots,x_n),$$

then by an application of geometric series,

$$F_1(T) = \sum_{k_2,\dots,k_n \geqslant 0} \frac{q^{g_1(k_2+\cdots+k_n,\dots,k_n)(k_2+\cdots+k_n)+g_0(k_2+\cdots+k_n,\dots,k_n)} T^{d'_2k_2+\cdots+d'_nk_n}}{1 - q^{g_1(k_2+\cdots+k_n,\dots,k_n)} T^{d_1}}$$

where $d'_2 := d_1 + d_2$, $d'_3 := d_1 + d_2 + d_3$, etc. This will be a meromorphic function by the following lemma. (Note: g_0 is an increasing polynomial by taking $x_1 = 0$.)

Lemma 3. *Let g_1 and g_0 be increasing polynomials. Then*

$$F_1(T) := \sum_{k_2,\dots,k_n \geqslant 0} \frac{q^{g_1(k_2+\cdots+k_n,\dots,k_n)(k_2+\cdots+k_n)+g_0(k_2+\cdots+k_n,\dots,k_n)} T^{d_2k_2+\cdots+d_nk_n}}{1 - q^{g_1(k_2+\cdots+k_n,\dots,k_n)} T^{d_1}}$$

is p-adic meromorphic.

Proof. We will proceed by induction on the number of variables. Suppose $n = 2$, then F_1 takes the form

$$\sum_{k_2 \geqslant 0} \frac{q^{g_1(k_2)k_2+g_0(k_2)} T^{d_2k_2}}{1 - q^{g_1(k_2)} T^{d_1}}.$$

If either $\deg_{k_2}(g_1) \geqslant 1$ or $\deg_{k_2}(g_0) \geqslant 2$ then this summation is p-adic meromorphic by Lemma 4 below (note: we are using the fact that g_1 and g_0 are increasing polynomials, and so the coefficients of the largest powers of k_2 are necessarily positive). If g_1 is a constant (perhaps zero) and $\deg_{k_2}(g_0) = 1$ then the series is meromorphic by an application of geometric series.

Let us assume the result is true whenever there are fewer than n variables. For convenience, let us define

$$\begin{aligned} u(k_2,\dots,k_n) &:= \text{the exponent of } q \text{ in the numerator of } F_1 \\ &= g_1(k_2+\cdots+k_n,\dots,k_n)(k_2+\cdots+k_n) + g_0(k_2+\cdots+k_n,\dots,k_n). \end{aligned}$$

In the following, we will move progressively backwards through the variables: $k_n, k_{n-1}, \dots, k_2$.

By Lemma 2 we know the degree of k_n in u is at least one and the coefficient of this largest degree term is a positive rational number. If the degree is equal to one, then g_1 must be identically zero and g_0 is linear in $k_2, \ldots, k_n$. It follows that F_1 is meromorphic.

Suppose the degree of k_n in u is at least two. By Lemma 2, the only way for k_{n-1} to not appear in u is if g_1 is identically zero and g_0 depends only on k_n. (Again, we are using the fact that g_1 and g_0 are increasing polynomials, and so the coefficient of the largest power of k_{n-1} in both g_1 and g_0 must be nonnegative. In particular, these terms cannot eliminate one another.) It follows from Lemma 4 below that this is a meromorphic function. Thus, let us suppose the degree of k_{n-1} in u is at least one. By Lemma 2 the coefficient of the largest degree term of k_{n-1} is a nonzero increasing polynomial in k_n. Find k_n' large enough such that for all $k_n \geqslant k_n'$ this polynomial takes positive values. From this, we may write

$$F_1(T) = \Bigg(\sum_{\substack{k_2,\ldots,k_{n-1}\geqslant 0\\ 0\leqslant k_n<k_n'}} + \sum_{\substack{k_2,\ldots,k_{n-1}\geqslant 0\\ k_n\geqslant k_n'}} \Bigg) \frac{q^{u(k_2,\ldots,k_n)}T^{d_2k_2+\cdots+d_nk_n}}{1-q^{g_1(k_2+\cdots+k_n,\ldots,k_n)}T^{d_1}}.$$

Notice that the first summation takes on a similar form to $F_1(T)$ but with variables $k_2, \ldots, k_{n-1}$. Hence, it is meromorphic by the induction hypothesis. Let us consider the second summation

$$F_2(T) := \sum_{\substack{k_2,\ldots,k_{n-1}\geqslant 0\\ k_n\geqslant k_n'}} \frac{q^{u(k_2,\ldots,k_n)}T^{d_2k_2+\cdots+d_nk_n}}{1-q^{g_1(k_2+\cdots+k_n,\ldots,k_n)}T^{d_1}}$$

where the degree of k_{n-1} in u is at least one and the degree of k_n in u is at least two, and the coefficients of their largest power is always positive. Suppose the degree of k_{n-1} is one. Then g_1 must be a nonnegative constant c_1 and g_0 cannot contain any $k_2, \ldots, k_{n-2}$. Thus $F_2(T)$ takes the form

$$\sum_{\substack{k_2,\ldots,k_{n-1}\geqslant 0\\ k_n\geqslant k_n'}} \frac{q^{c_1(k_2+\cdots+k_n)+c_2k_{n-1}+h(k_n)}T^{d_2k_2+\cdots+d_nk_n}}{1-q^{c_1}T^{d_1}}$$

where h is a quadratic polynomial in k_n, and c_2 is a constant. From geometric series and Lemma 4 below, this is a meromorphic function. Thus, we may suppose that the degrees of both k_{n-1} and k_n in u in $F_2(T)$ are at least two, and the coefficients of their largest powers are always positive.

A similar argument works for k_{n-2} as follows. By Lemma 2, the only way for k_{n-2} to not appear in u is if g_1 is identically zero and g_0 is a polynomial in k_{n-1} and k_n. By our assumptions on u, it follows from Lemma 4 below that this is a meromorphic function. Thus, let us suppose the degree of k_{n-2} in u is at least one.

By Lemma 2 the coefficient of the largest degree term of k_{n-2} is a nonzero increasing polynomial in k_{n-1} and k_n. Find k_{n-1}' and k_n'' large enough such that whenever both $k_{n-1} \geqslant k_{n-1}'$ and $k_n \geqslant k_n''$ this polynomial takes only positive values. From this, assuming $k_n' \geqslant k_n''$, we may write

$$F_2(T) = \Bigg(\sum_{\substack{k_2,\ldots,k_n\geqslant 0\\ 0\leqslant k_{n-1}<k_{n-1}' \text{ or } k_n'\leqslant k_n<k_n''}} + \sum_{\substack{k_2,\ldots,k_{n-2}\geqslant 0\\ k_{n-1}\geqslant k_{n-1}' \text{ and } k_n\geqslant k_n''}} \Bigg) \frac{q^{u(k_2,\ldots,k_n)}T^{d_2k_2+\cdots+d_nk_n}}{1-q^{g_1(k_2+\cdots+k_n,\ldots,k_n)}T^{d_1}}.$$

Notice that the first summation takes on a similar form to $F_1(T)$ but with $n-3$ variables. Hence, it is meromorphic by the induction hypothesis. Let us consider the second summation

$$F_3(T) := \sum_{\substack{k_2,\ldots,k_{n-2}\geqslant 0\\ k_{n-1}\geqslant k_{n-1}' \text{ and } k_n\geqslant k_n''}} \frac{q^{u(k_2,\ldots,k_n)}T^{d_2k_2+\cdots+d_nk_n}}{1-q^{g_1(k_2+\cdots+k_n,\ldots,k_n)}T^{d_1}}$$

where the degree of k_{n-2} in u is at least one and the degree of k_{n-1} and k_n in u is at least two, and the coefficients of their largest powers always take positive values. Suppose the degree of k_{n-2} is one. Then g_1 must be a nonnegative constant and g_0 cannot contain any $k_2, \dots, k_{n-3}$. Thus $F_3(T)$ takes the form

$$\sum_{\substack{k_2,\dots,k_{n-2}\geqslant 0\\ k_{n-1}\geqslant k'_{n-1} \text{ and } k_n\geqslant k''_n}} \frac{q^{c_1(k_2+\cdots+k_n)+c_2k_{n-2}+h(k_{n-1},k_n)}T^{d_2k_2+\cdots+d_nk_n}}{1-q^{c_1}T^{d_1}}$$

where c_1 and c_2 are constants. From geometric series and Lemma 4 below, this is a meromorphic function. Thus, we may suppose the u in $F_3(T)$ has the degrees of k_{n-2}, k_{n-1}, and k_n being at least two, and the coefficients of their largest powers are always positive.

Continuing this argument, we are left with a series of the form

$$\sum_{k_2\geqslant k'_2,\dots,k_n\geqslant k'_n} \frac{q^{u(k_2,\dots,k_n)}T^{d_2k_2+\cdots+d_nk_n}}{1-q^{g_1(k_2+\cdots+k_n,\dots,k_n)}T^{d_1}}$$

where $\deg_{k_i}(u)\geqslant 2$ for each $i=2,\dots,n$ and the leading coefficient of k_i in u is a polynomial in $k_{i+1},\dots,k_n$ which takes positive values for all $k_{i+1}\geqslant k'_{i+1},\dots,k_n\geqslant k'_n$. It follows from Lemma 4 below that this is p-adic meromorphic, finishing the proof of the result. $\square$

Lemma 4. *Let $u(k_1,\dots,k_n)$ be an increasing polynomial with $\deg_{k_i}(u)\geqslant 2$ for every $i=1,\dots,n$, and the leading coefficient of k_i in u takes positive values for all $k_1,\dots,k_n\geqslant 0$. Let $v(k_1,\dots,k_n)$ be an increasing polynomial. Then*

$$\sum_{k_1,\dots,k_n\geqslant 0} \frac{q^{u(k_1,\dots,k_n)}T^{d_1k_1+\cdots+d_nk_n}}{1-q^{v(k_1+\cdots+k_n,\dots,k_n)}T} \tag{12}$$

is p-adic meromorphic.

Proof. We will proceed by writing (12) as a quotient of two entire functions. First, the denominator of (12) takes the form

$$G(T) := \prod\big(1-q^{v(k_1+\cdots+k_n,\dots,k_n)}T\big)$$

where the product runs over all $k_1,\dots,k_n\geqslant 0$ with distinct values under v; that is,

$$v\big(k'_1+\cdots+k'_n,\dots,k'_n\big)\neq v(k_1+\cdots+k_n,\dots,k_n)$$

whenever $k'_i\neq k_i$ for any i. This will be an entire function by Lemma 5 below, and its Newton polygon will be bounded below by a quadratic polynomial. That is, if we write

$$G(T)=\sum_{r\geqslant 0} M_rT^r$$

then there exist $c>0$ and d such that

$$\operatorname{ord}_p(M_r)\geqslant cr^2+d \quad \text{for all } r\geqslant 0.$$

Let us now consider the numerator of (12), which takes the form

$$\sum_{k_1,\ldots,k_n \geqslant 0} q^{u(k_1,\ldots,k_n)} T^{d_1k_1+\cdots+d_nk_n} G_{k_1,\ldots,k_n}(T) \tag{13}$$

where

$$G_{k_1,\ldots,k_n}(T) := G(T)/\bigl(1 - q^{v(k_1+\cdots+k_n,\ldots,k_n)}\bigr).$$

Since $G_{k_1,\ldots,k_n}$ is a factor of $G(T)$, its Newton polygon is bounded below by the same quadratic polynomial as that of $G(T)$.

Write

$$u(k_1,\ldots,k_n) = g_r(k_2,\ldots,k_n)k_1^r + \cdots + g_0(k_2,\ldots,k_n).$$

By hypothesis, $r \geqslant 2$ and g_r takes positive values for all $k_2,\ldots,k_n \geqslant 0$. Let a_1 satisfy $0 < a_1 < g_r(0,\ldots,0)$. Then there exists a constant b_1 such that

$$u(k_1,\ldots,k_n) \geqslant a_1k_1^2 + b_1 \quad \text{for all } k_1,\ldots,k_n \geqslant 0.$$

Find similar constants a_i and b_i for each variable x_i. Let $a > 0$ be the minimum of the a_i and b be the minimum of the b_i. Then, for each i,

$$u(k_1,\ldots,k_n) \geqslant ak_i^2 + b \quad \text{for all } k_1,\ldots,k_n \geqslant 0.$$

Now, for each $G_{k_1,\ldots,k_n}$ write

$$G_{k_1,\ldots,k_n}(T) = \sum_{r\geqslant 0} M_r^{(k_1,\ldots,k_n)} T^r.$$

As mentioned above, the Newton polygon is uniformly bounded below:

$$\operatorname{ord}_p\bigl(M_r^{(k_1,\ldots,k_n)}\bigr) \geqslant cr^2 + d \quad \text{for all } r \geqslant 0 \text{ and } k_1,\ldots,k_n \geqslant 0.$$

Write the numerator (13) as a series $\sum_{s\geqslant 0} A_sT^s$ where

$$A_s = \sum_{r=0}^{s} \sum q^{u(k_1,\ldots,k_n)} M_{s-r}^{(k_1,\ldots,k_n)}$$

the second sum running over all $d_1k_1+\cdots+d_nk_n = s$. Now, if $d_1k_1+\cdots+d_nk_n = s$ then there exists k_i such that $k_i \geqslant \frac{s}{d_in} \geqslant \frac{s}{d'n}$ where $d' := \max\{d_1,\ldots,d_n\}$. It follows that

$$\operatorname{ord}_p(A_r) \geqslant \min_{0\leqslant r\leqslant s} \left\{a\left(\frac{s}{d'n}\right)^2 + b + c(s-r)^2 + d\right\} \geqslant \epsilon_1 r^2 + \epsilon_2$$

for some $\epsilon_1 > 0$ and $\epsilon_2 \in \mathbb{R}$. This shows the numerator of (12) is entire, finishing the proof. $\square$

Lemma 5. *Let $v(x_1,\dots,x_n)$ be an increasing polynomial with $\deg_{x_i}(v) \geqslant 1$ for each i. Consider*

$$G(T) := \prod \bigl(1 - q^{v(k_1+\dots+k_n,\dots,k_n)}T\bigr)$$

where the product runs over all $k_1,\dots,k_n \geqslant 0$ such that

$$v\bigl(k_1' + \dots + k_n', \dots, k_n'\bigr) \neq v(k_1 + \dots + k_n, \dots, k_n)$$

whenever $k_i' \neq k_i$ for any i. Writing $G(T) = \sum_{r \geqslant 0} M_r T^r$ there exist constants $c > 0$ and d such that

$$\operatorname{ord}_p(M_r) \geqslant cr^2 + d \quad \text{for all } r \geqslant 0. \tag{14}$$

Proof. We will proceed by induction on the number of variables n. Suppose $n = 1$. Write $v(x) = a_m x^m + \dots + a_0$ where $a_m > 0$ and $m \geqslant 1$. Let c satisfy $0 < c < a_m$. Find $x' \geqslant 0$ such that for all $x \geqslant x'$, we have $v(x) - cx \geqslant 0$. Let $d := \min\{v(x) - cx\colon\ 0 \leqslant x < x'\}$. Then $v(x) \geqslant cx + d$ for all $x \geqslant 0$. Replacing x with k, we see that

$$M_r = \sum (-1)^r q^{v(k^{(1)}) + \dots + v(k^{(r)})}$$

where the sum runs over all nonnegative integers $k^{(1)},\dots,k^{(r)}$ such that their values under v are distinct. Using that v is an increasing polynomial and (14)

$$\operatorname{ord}_p(M_r) \geqslant v(0) + v(1) + \dots + v(r-1) \geqslant \frac{cr(r-1)}{2} + dr$$

which proves the result for $n = 1$.

Let us assume the result whenever there are fewer than n variables. By Lemma 2, $\deg_{k_i}(v) \geqslant 1$ for every i. As we did for u in the proof of Lemma 4, find constants $c > 0$, $d \in \mathbb{R}$, and $k_1',\dots,k_n' \geqslant 0$ such that

$$v(k_1 + \dots + k_n, \dots, k_n) \geqslant ck_i + d \quad \text{whenever } k_i \geqslant k_i' \text{ for all } i. \tag{15}$$

Write

$$G(T) = G_1(T)G_2(T)$$

where

$$G_1(T) = \prod_{\substack{k_1,\dots,k_n \geqslant 0, \text{ and} \\ \exists i \text{ such that } 0 \leqslant k_i < k_i' \\ (\text{distinct } v \text{ values})}} \bigl(1 - q^{v(k_1+\dots+k_n,\dots,k_n)}T\bigr)$$

and

$$G_2(T) = \prod_{\substack{k_i \geqslant k_i' \text{ for every } i=1,\dots,n \\ (\text{distinct } v \text{ values})}} \bigl(1 - q^{v(k_1+\dots+k_n,\dots,k_n)}T\bigr).$$

By the induction hypothesis, $G_1(T)$ is an entire function whose Newton polygon is bounded below by a quadratic polynomial. For convenience, let us make the change of variables $k_i \mapsto k_i - k_i'$ in $G_2(T)$. Thus, consider

$$G_2(T) = \prod_{\substack{k_1,\ldots,k_n \geqslant 0 \\ \text{(distinct } v \text{ values)}}} \left(1 - q^{v(k_1+\cdots+k_n,\ldots,k_n)}T\right).$$

Notice that the set

$$\mathcal{A} := \left\{(k_1+\cdots+k_n,\ldots,k_n)\colon\ k_i \geqslant 0\right\} = \left\{(a_1,\ldots,a_n)\colon\ a_1 \geqslant \cdots \geqslant a_n \geqslant 0\right\}$$

is linearly ordered by lexicographic ordering, and has a minimal element $(0,\ldots,0)$. For example, when $n=3$ we have

$$(0,0,0) < (1,0,0) < (1,1,0) < (1,1,1) < (2,0,0) < \cdots.$$

It follows that if $a^{(0)} < a^{(1)} < \cdots < a^{(r)}$, where

$$a^{(i)} = \left(a_1^{(i)},\ldots,a_n^{(i)}\right) \in \mathcal{A},$$

then $a_1^{(i)} \geqslant \frac{i}{n}$ for each i. Now, write $G_2(T) = \sum_{r\geqslant 0} \overline{M}_r T^r$ with

$$\overline{M}_r := \sum (-1)^r q^{v(a^{(1)})+\cdots+v(a^{(r)})}$$

where the sum runs over all $a^{(i)} \in \mathcal{A}$ satisfying $v(a^{(i)}) \neq v(a^{(j)})$ for $i \neq j$. Ordering the $a^{(i)}$ if necessary we may assume $a_1^{(i)} \geqslant \frac{i-1}{n}$ for every i. Since

$$a_1^{(i)} = k_1^{(i)} + \cdots + k_n^{(i)},$$

we see that there must be a j such that $k_j^{(i)} \geqslant \frac{i-1}{n^2}$. Consequently, from (15) we see that

$$\operatorname{ord}_p(\overline{M}_r) \geqslant \sum_{i=1}^{r}\left(\frac{c(i-1)}{n^2} + d\right) = \frac{cr(r-1)}{2n^2} + dr \quad \text{for all } r \geqslant 0.$$

This proves the result for $G_2(T)$. Since the result is true for both G_1 and G_2, it is true for their product. □

This finish the proof of Theorem 8.

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