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PRIMITIVES IN THE HOPF ALGEBRA OF PROJECTIVE S_n-REPRESENTATIONS*

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We study the $\mathbb{Z}/2\times\mathbb{N}$-graded abelian group which for $4\le n\in\mathbb{N}$ is generated by the irreducible projective representations of A_n (respectively of S_n) in $\mathbb{Z}/2$-grading 0 (respectively 1). This has Hopf algebra structure over $L=\mathbb{Z}[\lambda]/(\lambda^3=2\lambda)$, where the action of λ is inducing and restricting between S_n and A_n. Working over L results in a considerable simplification to the proof of our structure theorem for the above. Our method is similar to Liulevicius' idea for studying the $\mathbb{Z}$-Hopf algebra of ordinary representations of S_n. Among the results is the determination of all primitives in the above L-Hopf algebra.

1. Introduction

In [2], the authors introduced a ring structure in a $\mathbb{Z}/2\times\mathbb{N}$-graded abelian group which is freely generated, except in a few low gradings, by the isomorphism classes of irreducible projective representations of S_n and of A_n. Determining the algebra structure involved some awkward manipulations with Hopf algebras at odd primes and more complicated structure at $p=2$. By working systematically over a certain $\mathbb{Z}/2$-graded ring L rather than over $\mathbb{Z}$, these proofs are much improved. The above ring becomes an $\mathbb{N}$-graded Hopf algebra over L, whose components are $\mathbb{Z}/2$-graded L-modules which admit an L-valued inner product. An interesting feature here is that it becomes unnecessary to construct generators using Clifford modules. Algebra generators arise naturally and uniquely from the formal algebra, once the 'bottom generator' is specified.

We work in the category of L-PSH-algebras, introduced in [1]. Zelevinski introduced PSH-algebras over $\mathbb{Z}$ in [3], where details and references on Hopf algebras may be found.

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Definition 1.1. The ring $L:=\mathbb{Z}[\lambda]/(\lambda^3-2\lambda)$ is graded over $\mathbb{Z}/2$ by $L_{(1)}=\lambda\mathbb{Z}$ and $L_{(0)}=1\mathbb{Z}\oplus\varrho\mathbb{Z}$ where $\varrho:=\lambda^2-1$.

Definition 1.2. An L-PSH-algebra K is an $\mathbb{N}$-graded object which in each degree is a $\mathbb{Z}/2$-graded L-module. K is endowed with L-algebra structure and coproduct $\Delta: K\to K\otimes_L K$, giving a graded connected Hopf algebra over L, where the coproduct and product in $K\otimes_L K$ are defined using the shuffle map

$$a\otimes b\otimes c\otimes d\to\varrho^{\varepsilon\delta+ij}a\otimes c\otimes b\otimes d\quad\text{for } b\in K_{\varepsilon,i},\ c\in K_{\delta,j}.$$

Furthermore in K there is specified a set I of homogeneous elements called *basic elements* such that

(i) $x\in I$ implies $x\neq\varrho x\in I$;

(ii) If $X\subset I$ satisfies $X\cup\varrho X=I$, $X\cap\varrho X$ is empty, then K is the free L-module on X.

The $\mathbb{Z}$-basis $I\cup\lambda I$ defines a positive cone $\mathbb{N}\cdot(I\cup\lambda I)$ and we require that the multiplication, comultiplication, unit, and counit maps be positive, where L and $K\otimes_L K$ are given the obvious positive cones. A symmetric inner product $\langle\ ,\ \rangle: K\otimes_L K\to L$ is determined by requiring X to be orthonormal; this is independent of choice of X. We require the multiplication and comultiplication to be adjoint with respect to $\langle\ ,\ \rangle$. Here the inner product on $K\otimes_L K$ is determined by $\langle\langle a\otimes b, c\otimes d\rangle\rangle=\langle a,c\rangle\langle b,d\rangle$; note that no 'pseudo-sign' $\varrho^{\varepsilon\delta+ij}$ is used.

It follows that the multiplication is necessarily 'pseudo-commutative' in that $xy=\varrho^{\varepsilon\delta+ij}yx$ for $x\in K_{\varepsilon,i}$, $y\in K_{\delta,j}$. By adjointness, Δ is 'pseudo-cocommutative'.

An element $x\in K_{\varepsilon,i}$ will be called odd or even according as $\varepsilon+i$ is odd or even, and $\deg x:=i$.

Let $\{x_\mu:\mu\in M\}$ be a $\mathbb{Z}/2\times\mathbb{N}$-graded set. For later use we record the structure of the free pseudo-commutative L-algebra on this set, denoted $L\lceil x_\mu:\mu\in M\rceil$. With a fixed linear order for M, it is the direct sum of cyclic modules generated by monomials $x_{\mu_1}x_{\mu_2}\cdots x_{\mu_l}$ with $\mu_1\leq\mu_2\leq\cdots\leq\mu_l$. Make this summand isomorphic to L unless there is at least one i with $\mu_i=\mu_{i+1}$ and x_{μ_i} odd, in which case the summand is $L/(\varrho-1)L$. Multiplication is given in the obvious way, using pseudo-commutativity.

A related construction is $B\lceil x\rceil$, where B is some pseudo-commutative L-algebra and x is an object with a given $\mathbb{Z}/2\times\mathbb{N}$ grading. As a module, $B\lceil x\rceil$ is a direct sum of modules $\{Bx^i: i\geq 0\}$. All the summands are isomorphic to B if x is even. When x is odd, $Bx^0\cong B\cong Bx^1$ and, for $i>1$, $Bx^i\cong B/(\varrho-1)B$. One can construct $L\lceil x_\mu:\mu\in M\rceil$ iterating this last construction after giving M a well-ordering.

Definition 1.3. Let $W(\tilde{\Sigma}_n)=\mathrm{GR}^-(\tilde{\Sigma}_n)\oplus R^-(\tilde{\Sigma}_n)$ where $\tilde{\Sigma}_n$ is the double cover of Σ_n, and where R^- and GR^- are Grothendieck groups of negative representations, as defined in [2].

The constructions and the proof of the following theorem are contained in [2], to which the reader should refer for representation-theoretic motivation.

Theorem 1.4. *There is an* L-PSH-*algebra structure on* $H=\bigoplus_{n\geq 0} W(\tilde{\Sigma}_n)$ *which has the following two properties*:

(i) *H has a basic primitive pair* $\{h_1, \varrho h_1\}\subset H_{(0),1}$ *and no other basic primitives in any grading*;

(ii) *H_n is a free $\mathbb{Z}/2$-graded L-module whose rank is at most* $\#\mathscr{D}_n$, *where $\mathscr{D}_n$ is the set of partitions of n into distinct parts.* □

Statement (i) is clear since $\tilde{\Sigma}_1$ is cyclic of order 2 and since for larger n, the restriction of an irreducible representation in $W(\tilde{\Sigma}_n)$ is certainly non-zero in $W(\tilde{\Sigma}_i)\otimes_L W(\tilde{\Sigma}_{n-i})\cong W(\tilde{\Sigma}_i \tilde{\times} \tilde{\Sigma}_{n-i})$. As for (ii), the rank is exactly $\#\mathscr{D}_n$ by the conjugacy class counts in [2], but only the inequality is needed to prove Theorem 1.5 below. For representations over $\mathbb{Q}$, for example, rather than $\mathbb{C}$, the strict inequality will hold, but the coalgebra structure would not exist. Finally, as noted above, the proof of Theorem 1.4 does not require Clifford modules.

Theorem 1.5. *If H is any* L-PSH-*algebra satisfying* (i) *and* (ii) *in Theorem* 1.4, *then there is a sequence* $h_n\in H_{(n+1),n}$ *of basic elements such that, as an algebra,*

$$H\cong L\lceil h_1, h_2, h_3, \ldots\rceil/J$$

where J is the ideal generated by the elements

$$h_n^2-(-1)^{n+1}\lambda\left(h_{2n}+\lambda\sum_{i=1}^{n-1}(-1)^i h_i h_{2n-i}\right)\quad \textit{for } n\geq 1.$$

The coproduct is determined by the formula

$$\Delta h_n = h_n\otimes 1 + 1\otimes h_n + \lambda\sum_{i=1}^{n-1} h_i\otimes h_{n-i}.$$

The rest of this paper is devoted to the proof of Theorem 1.5.

2. Proof of Theorem 1.5

The result classifying atoms in [1] shows that assuming (ii) is unnecessary in Theorem 1.5. However the only known proof of that result depends on Theorems 1.4 and 1.5; more precisely, it depends on the existence of an L-PSH-algebra with structure as given in Theorem 1.5. However [1, Lemma 4.3] depends only on the hypotheses of Theorem 1.5 (with (ii) omitted), so we state this first.

Proposition 2.1. *If H satisfies the hypotheses of Theorem* 1.5, *there is a ϱ-unique sequence* $h_n\in H_{(n+1),n}$ *of basic elements for which*

$$h_1 h_{n-1} = \lambda h_n + u_n,$$

where u_n is a sequence with $u_2 = 0$ and u_n is basic for $n > 2$. (To call h_n ϱ-unique means that ϱh_n is the only other element with the property.)

Furthermore Δh_n and h_n^2 are given by the formulae in the conclusion of Theorem 1.5. □

Let K be the L-algebra generated by $\{h_i: i>0\}$. By Proposition 2.1, K is a sub-Hopf algebra of H, but is not obviously PSH. We shall first prove that K has the algebra structure asserted for H in Theorem 1.5, and then prove that $K = H$.

Definition. In $K_{(0),2k+1}$, define

$$r_{2k+1} = (1 + k\lambda^2) h_{2k+1} + \lambda \sum_{i=1}^{k} (-1)^i (2k - 2i + 1) h_{2k-i+1} h_i.$$

Proposition 2.2. *r_{2k+1} is primitive.*

Proof. Working in $\prod_{n=0}^{\infty} H_n$ with the obvious filtration topology, we have a continuous extension $\hat{\Delta}$ of Δ. If we define $\boldsymbol{h} = 1 + \sum_{n>0} \lambda h_n$; $\boldsymbol{d} = \sum_{n>0} n\lambda h_n$, it is easy to compute that $\hat{\Delta}\boldsymbol{h} = \boldsymbol{h} \hat{\otimes} \boldsymbol{h}$; $\hat{\Delta}\boldsymbol{h}^{-1} = \boldsymbol{h}^{-1} \hat{\otimes} \boldsymbol{h}^{-1}$; $\hat{\Delta}\boldsymbol{d} = \boldsymbol{d} \hat{\otimes} \boldsymbol{h} + \boldsymbol{h} \hat{\otimes} \boldsymbol{d}$. Thus $\hat{\Delta}(\boldsymbol{dh}^{-1}) = \hat{\Delta}(\boldsymbol{d})\hat{\Delta}(\boldsymbol{h}^{-1}) = (\boldsymbol{dh}^{-1}) \hat{\otimes} 1 + 1 \hat{\otimes} (\boldsymbol{dh}^{-1})$. So each homogeneous component of $\boldsymbol{dh}^{-1}$ is primitive, and in grading $2k+1$, this component is λr_{2k+1}. (The components in even degrees are zero by the squaring relations in Theorem 1.5. In fact the arguments below could be altered to make this into another proof of the squaring relations. See [2, 4.10] for more details of this type of algebraic manipulation.) Now let us prove that r_{2k+1} itself is primitive. We have

$$\Delta r_{2k+1} = 1 \otimes r_{2k+1} + r_{2k+1} \otimes 1 + \lambda y$$

for some $y \in \sum_{i,j>0} H_i \otimes H_j$. This follows from the definition of r_{2k+1} and because h_{2k+1} has this property of projecting to a primitive in $H/\lambda H$. But since λr_{2k+1} is primitive we get $0 = \lambda^2 y = (1+\varrho)y$. It follows from freeness of H that $y = (1-\varrho)z$ for some z. Hence $\lambda y = 0$, as required. □

Lemma 2.3. *Let B be a connected pseudo-commutative L-Hopf algebra. Suppose B is a submodule of some free L-module H. Assume B is generated as an algebra by $B' \cap \{x\}$ where B' is a connected sub-Hopf algebra. Let C be the cyclic submodule of B/B' generated by the image of x.*

(i) *If C is free and x is even, then $B = B'\lceil x \rceil$;*

(ii) *If C is free and x is odd, then*

$$B = B'\lceil x \rceil / zx^2 = 0 \quad \forall z \in B' \cap (\varrho - 1)H;$$

(iii) *If $C \cong L/\lambda L$, x is odd and $\lambda x = 0$, then*

$$B = B'\lceil x \rceil / 0 = \lambda x = x^2 = zx \quad \forall z \in B' \cap \lambda H.$$

Proof. Since $B'_0=B_0$ we see $\deg x>0$. Let A be the ideal in B generated by $\bigoplus_{n>0}B'_n$. Define Δ' as the composite

$$B\xrightarrow{\Delta}B\otimes_L B\xrightarrow{1\otimes\eta}B\otimes_L(B/A)$$

where η is the canonical projection. Then Δ' is an algebra morphism, $\Delta'\beta=\beta\otimes 1$ for $\beta\in B'$, and $\Delta'x=x\otimes 1+1\otimes\eta(x)$ since $A_k=B'_k=B_k$ for $0<k<\deg x$. $A_{\deg x}=B'_{\deg x}$ since $A=\{\sum_{i\geq 0}\beta_i x^i:\beta_i\in B',\ \deg\beta_i>0\}$.

(i) We must prove that for $\beta_i\in B'$, if $\sum_{i=0}^r\beta_i x^i=0$, then all $\beta_i=0$. To obtain a contradiction, choose such a relation with $\beta_r\neq 0$ and with r minimal. We have

$$\begin{aligned}0=\Delta'\Big(\sum_i\beta_i x^i\Big)&=\sum_i(\beta_i\otimes 1)[x\otimes 1+1\otimes\eta(x)]^i\\&=\sum_{0\leq j\leq i\leq r}\binom{i}{j}\beta_i x^{i-j}\otimes(\eta(x))^j.\end{aligned}$$

The only term in $B\otimes(B/A)_{\deg x}$ is $(\sum_{i=1}^r i\beta_i x^{i-1})\otimes\eta(x)$. But $\eta(x)$ generates a free cyclic submodule and H is free, so $m\otimes\eta(x)=0$ in $H\otimes(B/B')$ implies $m=0$. Thus $\sum_i i\beta_i x^{i-1}=0$, contradicting minimality of r.

(ii) We must show that $\sum_{i=0}^r\beta_i x^i=0$ with $\beta_i\in B'$ implies $\beta_0=\beta_1=0$ and $\beta_i\in(\varrho-1)H$ for all $i>1$. To obtain a contradiction, suppose we have such a relation with r minimal but with the condition on the β_i not holding. Clearly $r=0$ is impossible, and $r=1$ (so $\beta_1\neq 0$) would contradict the freeness of C. Thus $r>1$ and $\beta_r\notin(\varrho-1)H$. Now

$$\begin{aligned}0=\Delta'\Big(\sum_i\beta_i x^i\Big)&=\sum_i(\beta_i\otimes 1)[x\otimes 1+1\otimes\eta(x)]^i\\&=\sum_{0\leq j\leq i\leq r}(a_{i,j}+b_{i,j}\varrho)\beta_i x^j\otimes\eta(x)^{i-j}\end{aligned}$$

for certain coefficients with $a_{i,j}+b_{i,j}=i+j!/i!j!$. The only term in $B\otimes(B/A)_{\deg x}$ is

$$\sum_{i=1}^r(1+\varrho+\cdots+\varrho^{i-1})\beta_i x^{i-1}\otimes\eta(x).$$

As in the previous case, $\sum_{i=1}^r(1+\varrho+\cdots+\varrho^{i-1})\beta_i x^{i-1}=0$.

By minimality of r, we obtain

$$\begin{aligned}&(1+\varrho)\beta_2=0 &&\text{if } r=2,\\&(1+\varrho+\cdots+\varrho^{r-1})\beta_r\in(\varrho-1)H &&\text{if } r>2.\end{aligned}$$

In both cases, freeness of H easily implies that $\beta_r\in(\varrho-1)H$, giving the contradiction.

(iii) Since $\lambda x=0$, there exists $y\in H$ with $(\varrho-1)y=x$. Now $\varrho y^2=y^2$ by pseudo-commutativity, so $x^2=0$. Thus the map $B'\oplus B'\xrightarrow{\phi}B$ sending (y,z) to $y+zx$ is onto. It remains to prove that $\operatorname{Ker}\phi=0\oplus(\lambda H\cap B')$. If $(y,z)\in\operatorname{Ker}\phi$, then

$$0=\Delta'(y+zx)=y\otimes 1+(z\otimes 1)(x\otimes 1+1\otimes\eta(x))=z\otimes\eta(x).$$

Since H is free, and $\eta(x)$ generates a cyclic module isomorphic to $L/\lambda L$ in B/B', and since $z\otimes\eta(x)=0$ in $H\otimes(B/B')$, we find λ divides z in H. Now $zx=0$, so $y=0$. □

Proposition 2.4. *Let H be a connected pseudo-commutative L-Hopf algebra which is free as an L-module. Let*

$$R=\Big(\bigoplus_{\mu\in M} Lr_\mu\Big)\oplus\Big(\bigoplus_{\nu\in N} Lr_\nu\Big)$$

be a submodule of primitives, where each r is odd and

$$Lr_\mu\cong L \quad \text{and} \quad Lr_\nu\cong L/\lambda L.$$

Then the subalgebra generated by R is

$$L\lceil\{r_\mu:\mu\in M\}\cup\{r_\nu:\nu\in N\}\rceil/\lambda r_\nu=r_\nu^2=r_\mu^2 r_\nu=0.$$

Choose linear orders in M and N so that monomials $\prod_{i=1}^m r_{\mu_i}\prod_{j=1}^n r_{\nu_j}$ are always written with increasing subscripts. Then the module structure of the above subalgebra is the direct sum of cyclic modules generated by the monomials in which all ν_j are distinct, and, if $n>0$, then also all μ_i are distinct. Other monomials are zero. The above cyclic module is isomorphic to

L,	*if* $n=0$ *and all* μ_i *are distinct,*
$L/(\varrho-1)L$,	*if* $n=0$ *and* $\mu_i=\mu_{i+1}$ *for some* i,
$L/\lambda L$,	*if* $n>0$ *and all* μ_i *are distinct.*

Proof. Well-order M. First we prove the case where N is empty by showing inductively on $\mu_0\in M$ that the subalgebra generated by $\{r_\mu:\mu<\mu_0\}$ has structure $L\lceil\{r_\mu:\mu<\mu_0\}\rceil$. The inductive step at a limit ordinal is evident. So we must show that $\{r_\mu:\mu\le\mu_0\}$ generates $L\lceil\{r_\mu:\mu\le\mu_0\}\rceil$ assuming it holds with $<$ in place of $\le$. We use Lemma 2.3(ii), taking $B'=L\lceil r_\mu:\mu<\mu_0\rceil$ and $x=r_{\mu_0}$. We need only verify

(i) $B'\cap(\varrho-1)H=(\varrho-1)B'$, and

(ii) r_{μ_0} projects to a generator of a free cyclic submodule of H/B'.

For (ii), note that B' is the direct sum of the cyclic modules generated by $\{r_{\mu_1}r_{\mu_2}\cdots r_{\mu_s}:\mu_1<\cdots<\mu_s<\mu_0\}$ and

$$\Delta r_\Omega=\sum(a+b\varrho)r_{\Omega'}\otimes r_{\Omega''} \quad \text{for sequences } \Omega',\Omega'',\ \Omega=(\mu_1,\mu_2,\ldots),$$

summation over pairs (Ω',Ω'') with $\Omega'\cup\Omega''=\Omega$, where a and b are non-negative integers, not both zero, depending on (Ω',Ω''). Thus

$$\operatorname{Prim}B'=\operatorname{Span}_L\{r_\mu:\mu<\mu_0\}.$$

But r_{μ_0} is primitive, so the linear independence of $\{r_\mu:\mu\in M\}$ completes the argument.

To verify (i), write $z\in B'\cap(\varrho-1)H$ as a linear combination of the generators $\{\prod r_{\mu_i}\}$ for B'. Since $\lambda z=0$, we find, using the inductive assumption, that the coefficients γ satisfy $\lambda\gamma=0$ when all μ_i are distinct and $\lambda\gamma\in(\varrho-1)L$ otherwise. In both cases, it is easy to deduce that $\gamma\in(\varrho-1)L$, as required.

Now to proceed to the general case where N is non-empty, we know the structure of the algebra generated by $\{r_\mu\colon \mu\in M\}$ and proceed by adding the generators r_ν, $\nu\in N$, one at a time. Note that freeness of H and $(\varrho-1)r_\mu^2=0$ implies $r_\mu^2=\lambda s_\mu$ for some $s_\mu\in H$. Thus $r_\mu^2 r_\nu=0$ since $\lambda r_\nu=0$. Also $\lambda r_\nu=0$ implies $r_\nu=(\varrho-1)t_\nu$ for some $t_\nu\in H$. But $(\varrho-1)t_\nu^2=0$ by pseudo commutativity. Thus $r_\nu^2=(\varrho-1)^2t_\nu^2=0$.

Choose a well-ordering for N. We prove inductively on ν_0 that

$$\mathrm{Alg}_L(\{r_\mu\colon \mu\in M\}\cup\{r_\nu\colon \nu<\nu_0\})$$

has the structure given in the theorem with N replaced by $\{\nu\in N\colon \nu<\nu_0\}$. At limit ordinals the inductive step is trivial. Now in Lemma 2.3, set

$$B'=L\lceil\{r_\mu\colon \mu\in M\}\cup\{r_\nu\colon \nu<\nu_0\}\rceil/r_\mu^2 r_\nu=\lambda r_\nu=r_\nu^2=0$$

and set $x=r_{\nu_0}$. To show that the algebra generated by $B'\cup\{x\}$ is

$$B'\lceil r_{\nu_0}\rceil/r_\mu^2 r_{\nu_0}=\lambda r_{\nu_0}=r_{\nu_0}^2=0,$$

using Lemma 2.3(iii), it remains to show that

(i) r_{ν_0} generates a cyclic submodule of B/B' isomorphic to $L/\lambda L$, and

(ii) for all $z\in B'$, if $z\in\lambda H$, then $z\in\lambda B'+$ the ideal in B' generated by $\{r_\mu^2\colon \mu\in M\}$.

As for (ii), writing z as a linear combination of the monomials $\prod_{i=1}^m r_{\mu_i}\prod_{j=1}^n r_{\nu_j}$ which span B' by the inductive hypothesis, we must show that the coefficients γ are in λL in the cases where all μ_i are distinct. Since $(\varrho-1)z=0$ and using the structure of these cyclic modules we find $(\varrho-1)\gamma=0$ in the case $n=0$ and $(\varrho-1)\gamma\in\lambda L$ in the case $n>0$. It follows that $\gamma\in\lambda L$.

To prove (i), suppose $\alpha\in L$ and $\alpha r_{\nu_0}\in B'$. Then

$$\alpha r_{\nu_0}\in\mathrm{Prim}(B')=\mathrm{Span}_L(\{r_\mu\}_M\cup\{r_\nu\}_{\nu<\nu_0}).$$

By the structure of R we see that $\alpha\in\lambda L$, as required. The above calculation of Prim B' is immediate from the formula

$$\Delta(r_\Omega)=\sum(a+b\varrho)\cdot r_{\Omega'}\otimes r_{\Omega''}$$

as in the earlier part of the proof except that the multi-index Ω may contain both μ's and ν's here. □

Corollary 2.5. *Let* $r_{2k}=(\varrho-1)h_{2k}$. *Then the subalgebra of* H *generated by* $\{r_i\colon i\geq 1\}$ *is* $\bigoplus(r_{i_1}\cdots r_{i_s})L$, *direct sum over all partitions, where, with* $\mathscr{P}^{\mathrm{odd}}$ *denoting the set of partitions into odd parts, we have*

$$r_{i_1}\cdots r_{i_s}L \cong \begin{cases} L & \text{if } (i_1,\dots,i_s)\in\mathscr{P}^{\text{odd}}\cap\mathscr{D},\\ L/(\varrho-1)L & \text{if } (i_1,\dots,i_s)\in\mathscr{P}^{\text{odd}}\setminus\mathscr{D},\\ L/\lambda L & \text{if } (i_1,\dots,i_s)\in\mathscr{D}\setminus\mathscr{P}^{\text{odd}},\\ 0 & \text{otherwise.}\end{cases}$$

Note that r_{2k} is primitive. □

Proposition 2.6. *Let M be an L-module generated by $\{x_1,\dots,x_k\}$. In order to prove M is free over L on $\{x_1,\dots,x_k\}$ it suffices to find submodules N_1 and N_2 such that*

$$N_1 \cong L^a \oplus (L/\lambda L)^b \quad \text{with } a+b\geq k,$$

$$N_2 \cong L^c \oplus (L/(\varrho-1)L)^d \quad \text{with } c+d\geq k.$$

Proof. $L/\lambda L\cong\mathbb{Z}$ and $N_1/\lambda N_1$ is a free submodule of $M/\lambda M$ with $a+b$ generators. Hence $a+b=k$ and $M/\lambda M$ is $L/\lambda L$-free on $\{x_i+\lambda M\colon 1\leq i\leq k\}$. $L/(\varrho-1)L\cong\mathbb{Z}[\sqrt{2}]$ is a principal ideal domain and $N_2/(\varrho-1)N_2$ is a free submodule of $M/(\varrho-1)M$ with $c+d$ generators. Hence $c+d=k$ and $M/(\varrho-1)M$ is $L/(\varrho-1)L$-free on $\{x_i+(\varrho-1)M\colon 1\leq i\leq k\}$. But now an L-linear relation among the x_i's has coefficients all of which project to zero in both $L/\lambda L$ and $L/(\varrho-1)L$. It follows easily that all these coefficients are zero. □

Proposition 2.7. *K has the algebra structure as given in Theorem* 1.5.

Proof. Write each element of $\mathscr{D}$ in decreasing order for definiteness and define $h_\alpha=h_{a_1}h_{a_2}\cdots$ for $\alpha=(a_1,a_2,\dots)\in\mathscr{D}$. We must show K_n is free on $\{h_\alpha\colon \alpha\in\mathscr{D}_n\}$. This is immediate from Proposition 2.6 and Corollary 2.5, taking

$$N_2=\bigoplus\{r_{i_1}\cdot\dots\cdot r_{i_s}L\colon (i_1,\dots,i_s)\in\mathscr{P}_n^{\text{odd}}\},$$

$$N_1=\bigoplus\{r_{i_1}\cdot\dots\cdot r_{i_s}L\colon (i_1,\dots,i_s)\in\mathscr{D}_n\},$$

and recalling that $\#\mathscr{P}_n^{\text{odd}}=\#\mathscr{D}_n$. □

We now proceed with the second half of the proof of Theorem 1.5, that is, the proof that $K=H$.

Proposition 2.8.

(a) $\langle r_{2k+1},h_{2k+1}\rangle=1$;

(b) $\langle(1-\varrho)h_{2k},h_{2k}\rangle=1-\varrho$.

Proof. (b) is trivial, and (a) is an easy calculation using the formula (see [1, 4.3] for similar calculations)

$$\langle h_n,h_{n-i}h_i\rangle=\lambda \quad \text{for } 0<i<n.$$

Alternatively, an argument as in the second half of [2, 4.10] will work here. □

Proposition 2.9. *Let $h_0=1$ and let $h_\sigma=h_{s_1}h_{s_2}\cdots$ for a sequence $\sigma=(s_1,s_2,\ldots)$. Then*

$$\Delta h_\sigma=\sum_{\sigma'+\sigma''=\sigma}\varrho^{\pi(\sigma',\sigma'')}\lambda^{\zeta(\sigma',\sigma'')}h_{\sigma'}\otimes h_{\sigma''},$$

where

(i) *the summation is over all pairs (σ',σ'') of sequences of the same length as σ with entries non-negative integers and entrywise sum σ;*

(ii) *$\pi(\sigma',\sigma'')\in\mathbb{Z}/2$ is defined by*

$$h_{\sigma'}h_{\sigma''}=\varrho^{\pi(\sigma',\sigma'')}h_{a_1}h_{b_1}h_{a_2}h_{b_2}\cdots h_{b_l}$$

where $\sigma'=(a_1,\ldots,a_l)$ and $\sigma''=(b_1,\ldots,b_l)$; and

(iii) *if $nz(\sigma)=\#$non-zero entries in σ, then*

$$\zeta(\sigma',\sigma''):=nz(\sigma')+nz(\sigma'')-nz(\sigma'+\sigma'').$$

Proof. This is straightforward by induction on the length of σ. □

Proposition 2.10. *For each integer prime p, the L/pL-Hopf algebra $K\otimes_L(L/pL)$ has in degree n the following module of primitives:*

$$\mathrm{Prim}_n[K\otimes_L(L/pL)]\cong\begin{cases}L/pL & \text{with generator } r_{2k+1},\ n=2k+1,\\(\varrho-1)L/pL & \text{with generator } (\varrho-1)h_{2k},\ n=2k.\end{cases}$$

Proof. Writing h_α for the basis element $h_\alpha\otimes 1$, $\alpha\in\mathscr{D}$, define

$$\theta:\mathrm{Prim}_n[K\otimes_L(L/pL)]\to L/pL$$

by $\theta(\sum_{\mathscr{D}}\mu_\alpha h_\alpha)=\mu_n$. To prove that θ is injective, proceed by induction on the length $l(\alpha)$ of α to show that ($\sum\mu_\alpha h_\alpha$ primitive and $\mu_n=0$) implies all $\mu_\alpha=0$. If $\alpha=(a_1,a_2,\ldots,a_l)$, then the basis element $h_{a_1}\otimes h_{a_2,\ldots,a_l}$ occurs only in Δh_α and in Δh_β for certain β with $l(\beta)<l(\alpha)$ by Proposition 2.9. Thus

$$0=\text{coefficient of } h_{a_1}\otimes h_{a_2\cdots a_l} \text{ in } \Delta(\textstyle\sum\mu_\beta h_\beta)$$

$$=\mu_\alpha\cdot\text{coefficient of } h_{a_1}\otimes h_{a_2\cdots a_l} \text{ in } \Delta(h_\alpha)$$

since, by induction, $\mu_\beta=0$ for $l(\beta)<l(\alpha)$

$$=\mu_\alpha,\quad \text{by Proposition 2.9, as required.}$$

Now consider the case $n=2k$. The coefficient of $h_k\otimes h_k$ in $\Delta(\sum\mu_\alpha h_\alpha)$ is $\mu_{2k}\lambda$, since $\Delta(h_\alpha)$ has no term $h_k\otimes h_k$ if $\alpha\neq(2k)$. Thus, $\sum\mu_\alpha h_\alpha$ primitive in $K_{2k}\otimes_L(L/pL)$ implies $\mu_{2k}\lambda=0$ in L/pL and thus $\mu_{2k}\in(\varrho-1)(L/pL)$. Since $(\varrho-1)h_{2k}$ is primitive and maps by θ to $\varrho-1$, this completes the proof for $n=2k$.

In the case $n=2k+1$, define

$$\phi : \mathrm{Prim}_n [K \otimes_L (L/pL)] \to L/pL$$

by $\phi(x) = \langle x, h_n \rangle$. Let $x \in \mathrm{Ker}\, \phi$. Then $\theta(x)r_n - \theta(r_n)x$ is in Ker θ, so is zero. By Proposition 2.8, ϕ maps $\theta(x)r_n - \theta(r_n)x$ to $\theta(x)$, so $\theta(x) = 0$. Thus $x = 0$, so ϕ is injective. But ϕ maps r_{2k+1} to 1, so this completes the proof. □

The following is a standard fact.

Proposition 2.11. *Let H' and H'' be graded connected T-Hopf algebras and free as modules over the commutative ring T. If $\psi : H' \to H''$ is a non-injective morphism of Hopf algebras, then* $\mathrm{Ker}\, \psi \cap \mathrm{Prim}\, H' \neq \{0\}$.

Proof. Let x be an element of least degree d in $\mathrm{Ker}\, \psi - \{0\}$. Then

$$0 = \Delta''(\psi(x)) - \psi(x) \otimes 1 - 1 \otimes \psi(x) = (\psi \otimes \psi)[\Delta'(x) - x \otimes 1 - 1 \otimes x].$$

But by choice of d, $\psi \mid H_i'$ is injective for $i < d$. By freeness, $\psi \otimes \psi \mid H_i' \otimes H_j'$ is injective for i and $j < d$. Thus $\Delta'(x) = x \otimes 1 + 1 \otimes x$, so $x \in \mathrm{Ker}\, \psi \cap \mathrm{Prim}\, H'$. □

Completion of the proof of Theorem 1.5. If $K \neq H$, then, since K_n and H_n are free abelian groups of equal rank, there exists an integer prime p such that $K \subset H$ induces a Hopf algebra morphism

$$\psi : K \otimes_L (L/pL) \to H \otimes_L (L/pL)$$

which is not injective. By Propositions 2.10 and 2.11, we have a primitive αr_{2k+1} or $\alpha(1-\varrho)h_{2k}$ which is divisible by p in H but not in K. But then in L, p divides $\langle \alpha r_{2k+1}, h_{2k+1} \rangle = \alpha$ or $\langle \alpha(1-\varrho)h_{2k}, h_{2k} \rangle = \alpha(1-\varrho)$. But then p does divide αr_{2k+1} or $\alpha(1-\varrho)h_{2k}$ in K, a contradiction. □

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