On the Riesz fusion bases in Hilbert spaces

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Abstract In this paper we investigate the connection between fusion frames and obtain a relation between indexes of the synthesis operators of a Besselian fusion frame and associated frame to it. Next we introduce a new notion of a Riesz fusion bases in a Hilbert space. We show that any Riesz fusion basis is equivalent with a orthonormal fusion basis. We also obtain generalizations of Theorem 4.6 of [1]. Our results generalize results obtained for Riesz bases in Hilbert spaces. Finally we obtain some results about stability of fusion frame sequences under small perturbations.

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1. Introduction

A frame is a redundant set of vectors in a Hilbert space with the property that provide usually non-unique representations of vectors in terms of the frame elements. Frames for Hilbert spaces were first defined by Duffin and Schaeffer [2] in 1952 and reintroduced in 1986 by Daubechies, Grossmann and Meyer [3]. Fusion frames are a generalization of frames in Hilbert spaces, were introduced by Casazza and Kutyniok in [1,4] and with a different focus were undertaken in [5–7]. Frames and fusion frames play important roles in many applications in mathematics, science, and engineering, including coding theory, filter bank theory, and applications to sensor networks, and many other areas. The paper is organized as follows: Section 2, contains a few elementary definitions and results from standard fusion frame theory. In this section we introduce the concept of Besselian fusion frame and obtain the connection between index of the synthesis operators of a Besselian fusion frame and associated frame to it. In Section 3 we study Riesz fusion bases in Hilbert spaces. We introduce a new definition of Riesz fusion basis and then give some characterizations of Riesz fusion bases. In Section 4 we study the stability of fusion frame sequences under small perturbations.

Throughout this paper, \(H\), \(K\) are separable Hilbert spaces and \(I, J, J_i\) denote the countable (or finite) index sets and \(\pi_W\) denotes the orthogonal projection of a closed subspace \(W\) of \(H\). We will always use \(\{\varphi_j\}_{j \in I}\) and \(\{\delta_i\}_{i \in J}\) to denote orthonormal bases for \(H\) and \(\ell^2(I)\) respectively. The range and the null spaces of an operator \(T \in \mathcal{B}(H, K)\) denoted by \(R_T\) and \(N_T\), respectively.

There are several ways of defining the tensor product of Hilbert spaces. Folland in [8], Kadison and Ringrose in [9] have represented the tensor product of Hilbert spaces \(H\) and \(K\) as a certain linear space of operators. Since we used their results firstly we state some of the definitions.
We consider the set of all bounded antilinear maps from $\mathcal{K}$ to $\mathcal{H}$. The operator norm of an antilinear map $T : \mathcal{K} \to \mathcal{H}$ is defined as in the linear case:

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$ 

The adjoint of a bounded antilinear map $T$ is defined by

$$\langle T^*f, g \rangle = \langle f, g \rangle \quad \forall f \in \mathcal{H}, \; g \in \mathcal{K}. \quad (1)$$

Note that the map $T \to T^*$ is linear rather than antilinear. Suppose that $(u_i)_{i \in I}$ is an orthonormal basis for $\mathcal{K}$, then by the Parseval identity we have

$$\sum_{j \in J} \|Tu_j\|^2 = \sum_{i \in I} \|T^*e_i\|^2.$$ 

This shows that $\sum_{i \in I} \|Tu_i\|^2$ is independent of the choice of basis $(u_i)_{i \in I}$.

**Definition 1.1.** The tensor product of $\mathcal{H}$ and $\mathcal{K}$ is the set $\mathcal{H} \otimes_\pi \mathcal{K}$ of all antilinear maps $T : \mathcal{K} \to \mathcal{H}$ such that $\sum_{i \in I} \|Tu_i\|^2 < \infty$ for some, and hence every orthonormal basis of $\mathcal{K}$. Moreover for every $T \in \mathcal{H} \otimes_\pi \mathcal{K}$ we set

$$\|T\|^2 = \sum_{i \in I} \|Tu_i\|^2.$$ 

By Theorem 7.12 of [8], $\mathcal{H} \otimes \mathcal{K}$ is a Hilbert space with the norm $\|\cdot\|$ and associated inner product

$$\langle Q, T \rangle = \sum_{i \in I} \langle Q u_i, T u_i \rangle \quad \forall Q, T \in \mathcal{H} \otimes \mathcal{K}. \quad (3)$$

Moreover, if for every $f \in \mathcal{H}, g \in \mathcal{K}$ we define $f \otimes g$ by

$$f \otimes g (g') = \langle f, g' \rangle, \quad \forall g' \in \mathcal{K},$$

Then $f \otimes g \in \mathcal{H} \otimes \mathcal{K}$ and for all $T \in \mathcal{H} \otimes \mathcal{K}$ we have

$$\|T\|^2 = \|T\|.$$ 

**2. Characterization of fusion frames by frames**

A family of vectors $F = \{f_i\}_{i \in I}$ is called a frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that,

$$A\|f\|^2 \leq \inf_{i \in I} \langle f_i f, f \rangle \leq B\|f\|^2 \quad \forall f \in \mathcal{H}. \quad (5)$$

The constants $A$ and $B$ are called frame bounds. If we only have the right-hand inequality of $(5)$, we call $F$ a Bessel sequence. The representation space associated with a frame is $\ell^2(I)$. If $F = \{f_i\}_{i \in I}$ is a Bessel sequence, the synthesis operator for $F$ is the bounded linear operator $T_F : \ell^2(I) \to \mathcal{H}$, given by $T_F (\{e_i\}_{i \in I}) = \sum_{i \in I} e_i f_i$. The analysis operator for $F$ is $T_F^* : \mathcal{H} \to \ell^2(I)$ and satisfies $T_F T_F^* = \|f_i f\|_I$. By composing $T_F$ and $T_F^*$ we obtain the frame operator

$$S_F : \mathcal{H} \to \mathcal{H}, \quad S_F = T_F T_F^* = \sum_{i \in I} \langle f_i f, f_i \rangle f_i,$$

which is a positive, self-adjoint and invertible operator and the following reconstruction formula holds for all $f \in \mathcal{H}$:

$$f = \sum_{i \in I} \langle f_i f, f_i \rangle f_i = \sum_{i \in I} \langle f_i f, f_i \rangle f_i,$$

where $f = S_F^{-1} f$ (if $I$ is finite). Also $\tilde{F} = \{\tilde{f}_i\}_{i \in I}$ is a frame for $\mathcal{H}$ and called the canonical dual frame of $F = \{f_i\}_{i \in I}$. In general, the Bessel sequence $G = \{g_i\}_{i \in I}$ is called a dual of the frame $F = \{f_i\}_{i \in I}$ if the following formula holds

$$f = T_F T_F^*(f) = \sum_{i \in I} \langle f, g_i \rangle f_i \quad \forall f \in \mathcal{H}.$$ 

Moreover a Riesz basis for $\mathcal{H}$ is a family of the form $\{U(e_i)\}_{i \in I}$, where $\{e_i\}_{i \in I}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator. For more details about the theory and applications of frames and Riesz bases we refer the reader to Casazza and Kutyniok [10], Christensen [11], Feichtinger [12] and Holub [13].

By Proposition 7.14 of [8], the tensor product of two orthonormal bases in $\mathcal{H}$ and $\mathcal{K}$ is an orthonormal basis of $\mathcal{H} \otimes \mathcal{K}$. We generalized this Proposition to frame situation in [14].

**Theorem 2.1.** Let $F_i = \{f_{ik}\}_{i \in I}$, $(i = 1, \ldots, n)$ be Bessel sequences for $\mathcal{H}_i$, $(1 \leq i \leq n$, respectively). Then $F = \{f_{ij}\} \otimes \cdots \otimes f_{ij}, f_{ij} \in F_i, 1 \leq i \leq n$ is a frame for $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ if and only if each $F_i = \{f_{ij}\}_{i \in I}$ is a frame for $\mathcal{H}_i$. Moreover $T_F = T_{F_1} \otimes \cdots \otimes T_{F_n}$.

**Definition 2.2.** Let $W = \{W_i\}_{i \in I}$ be a sequence of closed subspaces in $\mathcal{H}$, and let $A = \{x_i\}_{i \in I}$ be a family of weights, i.e., $x_i > 0$ for all $i \in I$. We say that $W_a = \{(W_i, x_i)\}_{i \in I}$ is a fusion frame for $\mathcal{H}$ if there exist constants $0 < C \leq D < \infty$ such that,

$$C\|f\|^2 \leq \sum_{i \in I} x_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2 \quad \forall f \in \mathcal{H}. \quad (6)$$

The numbers $C, D$ are called the fusion frame bounds. The family $W_a$ is called a $C$-tight fusion frame if $C = D$, it is a Parseval fusion frame if $C = D = 1$, and a $\pi$-uniform if $x = x_i = x_j$ for all $i, j \in I$. If the right-hand inequality of $(6)$ holds, then we say that $W_a$ is a Bessel fusion sequence with Bessel bound $D$. The family $W_a$ is called a fusion frame sequence if it is a fusion frame for $\mathcal{H}_a = \sigma_{\operatorname{ran}}(W_i)_{i \in I}$. Moreover $W = \{W_i\}_{i \in I}$ is called an orthonormal fusion basis for $\mathcal{H}_a$ if $\mathcal{H}_a = \bigoplus_{i \in I} W_i$.

**Theorem 2.3.** $W_a = \{(W_i, x_i)\}_{i \in I}$ is a fusion frame for $\mathcal{H}$ if and only if there exists a bounded linear operator $T_{W_a}$ from $\ell^2(I) \otimes \mathcal{H}$ onto $\mathcal{H}$ for which $T_{W_a}(\delta_i \otimes e_j) = x_i \pi_{W_i}(e_j)$ for all $i \in I, j \in J$.

**Proof.** Let $W_a$ be a fusion frame with fusion frame bounds $C, D$ for $\mathcal{H}$. Then we can define a bounded linear operator

$$U : \mathcal{H} \to \ell^2(I) \otimes \mathcal{H} \quad U(f) = \sum_{i \in I} x_i L_i \pi_{W_i}(f),$$

where for all $i \in I$ the partial isometry $L_i : \mathcal{H} \to \ell^2(I) \otimes \mathcal{H}$ is defined by $L_i f = \delta_i \otimes f$. Clearly $\|U(f)\|^2 \geq C\|f\|^2$, so $U$ is bounded below on $\mathcal{H}$, and hence has closed range. It follows that $T_{W_a} = U^* \ell^2(I) \otimes \mathcal{H}$ onto $\mathcal{H}$ and operator

$$\langle f, T_{W_a}(\delta_i \otimes e_j) \rangle = \sum_{i,j} x_i \langle x_i \delta_i \otimes \pi_{W_i}(f), \delta_i \otimes e_j \rangle = x_i \langle \pi_{W_i}(f), e_j \rangle = \langle f, \pi_{W_i}(e_j) \rangle,$$

for all $i, j \in I$ and $f \in \mathcal{H}$ hence $T_{W_a}(\delta_i \otimes e_j) = x_i \pi_{W_i}(e_j)$. For the converse, suppose that $T_{W_a} : \ell^2(I) \otimes \mathcal{H} \to \mathcal{H}$ is a bounded surjective operator such that $T_{W_a}(\delta_i \otimes e_j) = x_i \pi_{W_i}(e_j)$ for all $i \in I, j \in J$. Then for all $f \in \mathcal{H}$ we have
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\[ T_{W_i}(f) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle T_{W_i}(f), \delta_i \otimes e_j \rangle \delta_i \otimes e_j \]

\[ = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle f, x_i \pi_{W_i}(e_j) \rangle \delta_i \otimes e_j. \]

If we set \( D = \| T_{W_i} \| \), it follows that

\[ \sum_{i \in I} \| \pi_{W_i}(f) \|^2 = \sum_{i \in I} \| \langle f, x_i \pi_{W_i}(e_j) \rangle \| \leq D \| f \|^2. \]

Moreover since \( T_{W_i} \) is surjective, thus \( T_{W_i} \) is one-to-one with closed range and hence is bounded below. This, is there is some positive number \( C \) such that

\[ \sum_{i \in I} \| \pi_{W_i}(f) \|^2 \geq C \| f \|^2 \].

In order to analyze a signal \( f \in H \), we denote the representation space associated with a fusion frame by \( \mathcal{F}(I) \otimes H \). The synthesis operator of a Bessel fusion sequence is defined by

\[ T_{W_i} : \mathcal{F}(I) \otimes H \to H \quad T_{W_i} = \sum_{i \in I} \pi_{W_i} L_i^*, \]

where the partial isometry \( L_i \) is defined by

\[ L_i : H \to \mathcal{F}(I) \otimes H \quad L_i f = \delta_i \otimes f. \]

It is easy to see that

\[ L_i^* L_i = \begin{cases} I_{H}, & i = j \\ 0, & i \neq j \end{cases} \]

and

\[ \sum_{i \in I} L_i^* L_i = I_{\mathcal{F}(I)} \otimes I_{H}. \]

The associated adjoint operator given by

\[ T_{W_i} : H \to \mathcal{F}(I) \otimes H \quad T_{W_i} = \sum_{i \in I} \pi_{W_i} L_i, \]

is called the analysis operator. The fusion frame operator \( S_{W_i} \) for \( W_i \) is defined by

\[ S_{W_i} : H \to H \quad S_{W_i}(f) = T_{W_i}^* T_{W_i}(f) = \sum_{i \in I} \pi_{W_i}^* \pi_{W_i}(f), \]

which is a bounded, invertible, and positive operator. This provides the reconstruction formula

\[ f = \sum_{i \in I} \pi_{W_i}^* \pi_{W_i}(f) = \sum_{i \in I} \pi_{W_i}^* S_{W_i}(f) \quad \forall f \in H. \]

The unitary operator

\[ \Theta : \mathcal{F}(I \times J) \to \mathcal{F}(I) \otimes H \quad \Theta\{c_{ij}\}_{i \in I, j \in J} = \sum_{i \in I} \sum_{j \in J} c_{ij} \delta_i \otimes e_j \]

gives a connection between the synthesis operators of a fusion frame and the associated frame to it. Also the connection between the reconstruction formulas is exposed.

**Proposition 2.4.** \( W_i = \{(W_i, z_i)\}_{i \in I} \) is a fusion frame for \( H \) if and only if \( \mathcal{F}_{W_i} = \{x_i W_i(e_j)\}_{i \in I, j \in J} \) is a frame for \( H \). In this case \( T_{W_i} \Theta = \Theta T_{\mathcal{F}_{W_i}} \) and for all \( f \in H \) we have

\[ \sum_{i \in I} \| \pi_{W_i}(f) \|^2 = \sum_{i \in I} \sum_{j \in J} \langle f, x_i \pi_{W_i}(e_j) \rangle \pi_{W_i}^* \pi_{W_i}(e_j). \]

**Proof.** See Proposition 3.17 of [1]. □

Let \( W_i = \{(W_i, z_i)\}_{i \in I} \) be a fusion frame for \( H \), then the operator

\[ P_{W_i} = T_{W_i} S_{W_i}^* T_{W_i} \]

is called the fusion frame projection of \( W_i \). If \( P_{\mathcal{F}_{W_i}} \) is the frame projection of \( \mathcal{F}_{W_i} = \{x_i \pi_{W_i}(e_j)\}_{i \in I, j \in J} \), then by the Proposition 2.4 we have \( P_{\mathcal{F}_{W_i}} = \Theta P_{W_i} \Theta \). Furthermore two sequences \( \{f_{ij}\}_{i \in I, j \in J} \) in \( H \) and \( K \) respectively, are called equivalent if there is an isomorphism \( U : H \to K \) such that \( U f_i = g_i \) for all \( i \in I \).

**Theorem 2.5.** Let \( W_i = \{(W_i, z_i)\}_{i \in I} \) be a fusion frame for \( H \), then the frame \( \mathcal{F}_{W_i} = \{x_i \pi_{W_i}(e_j)\}_{i \in I, j \in J} \) is equivalent to a frame in \( N_{\mathcal{F}_{W_i}}. \)

**Proof.** Since the set \( \{P_{W_i}(\delta_i \otimes e_j)\}_{i \in I, j \in J} \) is a frame for \( N_{\mathcal{F}_{W_i}} \) and the operator \( T_{W_i} \) is an isomorphism of \( N_{\mathcal{F}_{W_i}} \) onto \( H \) thus by Theorem 2.3 we have

\[ T_{W_i}(P_{W_i}(\delta_i \otimes e_j)) = T_{W_i}(\delta_i \otimes e_j) = x_i \pi_{W_i}(e_j) \]

for all \( i \in I, j \in J \). From this the claim follows as required. □

**Definition 2.6.** Let \( W_i = \{(W_i, z_i)\}_{i \in I} \) be a fusion frame and let \( \mathcal{F} = \{f_{ij}\}_{i \in I, j \in J} \) be a frame for \( H \). Then

(i) \( W_i \) is called a Besselian fusion frame if whenever \( \sum_{i \in I} x_i \pi_{W_i}(T \delta_i) \) is convergent for some antilinear operator \( T \) of \( H \) in \( \mathcal{F}(I) \), then \( T \in \mathcal{F}(I) \otimes H \).

(ii) \( \mathcal{F} \) is called a Besselian frame if whenever \( \sum_{i \in I} c_i f_i \) converges, then \( \{c_{ij}\}_{i \in I} \in \mathcal{F}(J) \).

**Proposition 2.7.** \( W_i = \{(W_i, z_i)\}_{i \in I} \) is a Besselian fusion frame if and only if \( \mathcal{F}_{W_i} = \{x_i \pi_{W_i}(e_j)\}_{i \in I, j \in J} \) is a Bessel frame for \( H \).

**Proof.** This follows immediately from Proposition 2.4. □

**Theorem 2.8.** Let \( W_i = \{(W_i, z_i)\}_{i \in I} \) be a Besselian fusion frame for \( H \), then the synthesis operator \( T_{W_i} \) has finite-dimensional kernel.

**Proof.** By Proposition 2.7 \( \mathcal{F}_{W_i} = \{x_i \pi_{W_i}(e_j)\}_{i \in I, j \in J} \) is a Besselian frame for \( H \) and hence the synthesis operator \( T_{\mathcal{F}_{W_i}} \) has finite-dimensional kernel by Theorem 2.3 of [13]. Now by Proposition 2.4 we also have \( \Theta(N_{\mathcal{F}_{W_i}}) = N_{T_{W_i}} \) which shows that \( T_{W_i} \) has finite-dimensional kernel. □

Recall that a bounded linear operator \( U : H \to K \) is called a Fredholm operator if \( N_U \) is finite-dimensional and \( U(\mathcal{H}) \) is finite-codimensional in \( K \). For a Fredholm operator, we define \( \text{ind}(U) = \dim N_U - \text{codim} U(\mathcal{H}). \)

**Corollary 2.9.** Let \( W_i = \{(W_i, z_i)\}_{i \in I} \) be a Besselian fusion frame for \( H \), then the synthesis and analysis operators \( T_{W_i}, T_{W_i}^* \), and \( T_{\mathcal{F}_{W_i}} = T_{\mathcal{F}_{W_i}}^* \) are Fredholm operators and

\[ \text{ind}(T_{W_i}) = \text{ind}(T_{\mathcal{F}_{W_i}}) \quad \text{ind}(T_{W_i}^*) = \text{ind}(T_{\mathcal{F}_{W_i}}^*). \]
3. Fusion bases and Riesz fusion bases

In this section, we introduce a new definition of Riesz fusion basis in Hilbert spaces, and as a consequence we give some characterizations of Riesz fusion bases. We show that any Riesz fusion basis is equivalent with an orthonormal fusion basis in a Hilbert space.

**Definition 3.1.** Let \( \mathcal{W} = \{W_i\}_{i \in I} \) be a sequence of closed subspaces of \( \mathcal{H} \). Then \( \mathcal{W} = \{W_i\}_{i \in I} \) is called a **Riesz fusion basis** for \( \mathcal{H} \), if \( \text{span}\{W_i\}_{i \in I} = \mathcal{H} \) and there exist constants \( C, D \) such that for every \( T \in \ell^2(I) \otimes \mathcal{H} \),

\[
C \sum_{i \in I} \|\pi_{W_i}(T)\|_2^p \leq \left\| \sum_{i \in I} \pi_{W_i}(T) \right\|^2 \leq D \sum_{i \in I} \|\pi_{W_i}(T)\|_2^p.
\]

The numbers \( C, D \) are called the Riesz fusion basis bounds.

**Example 3.2.** Let \( \{f_i\}_{i \in I} = \{U(e_i)\}_{i \in I} \) be a Riesz basis for \( \mathcal{H} \) and let \( W_i = \text{span}\{f_i\} \). Then \( \mathcal{W} = \{W_i\}_{i \in I} \) is a Riesz fusion basis for \( \mathcal{H} \).

Thus for any \( T \in \ell^2(I) \otimes \mathcal{H} \) we have

\[
\|U\|^2 \|U^{-1}\|^{-2} \sum_{i \in I} \|\pi_{W_i}(T)\|_2^2 \leq \left\| \sum_{i \in I} \pi_{W_i}(T) \right\|^2 \leq D \sum_{i \in I} \|\pi_{W_i}(T)\|_2^2.
\]

**Remark 3.3.** In general, (14) does not imply that \( \mathcal{F}_{\mathcal{W}} = \{\pi_{W_i}(e_j)\}_{i,j \in I} \) is a Riesz basis for \( \mathcal{H} \). The following is a counterexample.

Let \( \mathcal{H} = \mathbb{R}^3 \) and \( \{e_i: 1 \leq i \leq 3\} \) be standard orthonormal basis for \( \mathcal{H} \). Define

\[
N_1 = \{(x, x, x) : x \in \mathbb{R}\}, \quad N_2 = \{(y, -2y, y) : y \in \mathbb{R}\}, \quad N_3 = \{(z, 0, -z) : z \in \mathbb{R}\}.
\]

Then for all \( T \in B(\mathcal{H}) \) we have

\[
\left\| \sum_{i=1}^3 \pi_{N_i}(T e_i) \right\|_2^2 = \sum_{i=1}^3 \left\| \pi_{N_i}(T e_i) \right\|^2.
\]

Hence (14) is satisfied with \( C = D = 1 \). But \( \{\pi_{N_i}(e_j)\}_{j=1}^3 \) is not a Riesz basis for \( \mathcal{H} \).

**Notation 3.4.** For each family of subspaces \( \mathcal{W} = \{W_i\}_{i \in I} \) of \( \mathcal{H} \), we define the space

\[
\left\{ \sum_{i \in I} L_i W_i \right\}_\varphi = \left\{ \sum_{i \in I} L_i f_i \in W_i \text{ and } \left\| f_i \right\|^2 < \infty \right\}.
\]

where \( L_i : \mathcal{H} \to \ell^2(I) \otimes \mathcal{H} \) is defined by \( L_i f = \delta_i \otimes f \) for all \( i \in I \). It is clear that \( \left\{ \sum_{i \in I} L_i W_i \right\}_\varphi \) is a closed subspace of \( \ell^2(I) \otimes \mathcal{H} \) and also

\[
\left( \sum_{i \in I} L_i W_i \right)_\varphi = \left( \sum_{i \in I} L_i W_i^\perp \right)_\varphi.
\]

**Theorem 3.5.** If \( \mathcal{W} = \{W_i\}_{i \in I} \) is a Riesz fusion basis for \( \mathcal{H} \) then \( \mathcal{W} = \{(W_i, 1)\}_{i \in I} \) is a 1-uniform fusion frame for \( \mathcal{H} \).

**Proof.** Let \( C, D > 0 \) be the Riesz fusion basis bounds for \( \mathcal{W} = \{W_i\}_{i \in I} \), then we have

\[
\|T_{\mathcal{W}}(T)\|^2 \leq D \sum_{i \in I} \|\pi_{W_i}(T)\|^2 \leq D \|T\|^2.
\]

It follows that \( \mathcal{W} = \{(W_i, 1)\}_{i \in I} \) is a Bessel sequence with Bessel fusion bound \( D \). For the lower fusion frame bound we prove that \( T_{\mathcal{W}} \) is onto. First we show that \( \mathcal{R}_{T_{\mathcal{W}}} \) is closed. Suppose that \( g \in \mathcal{R}_{T_{\mathcal{W}}} \), then we can find a sequence \( \{T_{\mathcal{W}}(T_k)\}_{k=1}^\infty \) in \( \ell^2(I) \otimes \mathcal{H} \) such that \( g = \lim_{k \to \infty} T_{\mathcal{W}}(T_k) \). Now by the definition of Riesz fusion basis the sequence \( \{\sum_{i \in I} L_i \pi_{W_i}(T_k)\}_{k \in \mathbb{N}} \) is a Cauchy sequence in \( \{\sum_{i \in I} L_i W_i\}_\varphi \). Therefore there exists \( \sum_{i \in I} L_i f_i \in \{\sum_{i \in I} L_i W_i\}_\varphi \) such that

\[
\lim_{k \to \infty} \sum_{i \in I} L_i \pi_{W_i}(T_k) = \sum_{i \in I} L_i f_i.
\]

Now by continuity of \( T_{\mathcal{W}} \) we have \( T_{\mathcal{W}}(\sum_{i \in I} L_i f_i) = g \), thus \( \mathcal{R}_{T_{\mathcal{W}}} \) is closed. Moreover if \( g \in \mathcal{R}_{T_{\mathcal{W}}} \) then for all \( i \in I \) we have \( 0 = \langle T_{\mathcal{W}}(g), e_i \rangle = \|\pi_{W_i}(g)\|^2 \) and hence \( g \in \text{span}\{W_i\}_{i \in I} \) = \{0\}, this shows that \( \mathcal{R}_{T_{\mathcal{W}}} = \mathcal{H} \). Using Theorem 2.2 of [15], \( \mathcal{W} = \{(W_i, 1)\}_{i \in I} \) is a 1-uniform fusion frame for \( \mathcal{H} \) and for all \( f \in \mathcal{H} \) we have

\[
\|T_{\mathcal{W}}^\perp f\|^2 \leq \sum_{i \in I} \|\pi_{W_i}(f)\|^2.
\]

where \( T_{\mathcal{W}}^\perp \) is the pseudo-inverse of \( T_{\mathcal{W}} \). □

To check Riesz fusion baseness for any sequence \( \mathcal{W} = \{W_i\}_{i \in I} \), we derive the following useful characterization.

**Theorem 3.6.** Let \( \mathcal{W} = \{W_i\}_{i \in I} \) be a family of closed subspaces of \( \mathcal{H} \) and for each \( i \in I \) let \( \{e_j\}_{j \in J_i} \) be an orthonormal basis for each subspace \( W_i \). Then the following conditions are equivalent.

(i) \( \mathcal{W} = \{W_i\}_{i \in I} \) is a Riesz fusion basis for \( \mathcal{H} \).

(ii) \( \mathcal{F} = \{e_j\}_{i,j \in J_i} \) is a Riesz basis for \( \mathcal{H} \).

**Proof.** If (i) is satisfied, then by Theorem 3.5 and Theorem 3.2 of [1], \( \{e_j\}_{i,j \in J_i} \) is a frame for \( \mathcal{H} \). Now assume that \( \{e_j\}_{i,j \in J_i} \in \ell^2(\bigcup_{i \in I} J_i) \) and \( \sum_{i \in I} \sum_{j \in J_i} c_{ij} e_j = 0 \). If we define

\[
T = \sum_{i \in I} \sum_{j \in J_i} c_{ij} e_j \otimes e_j,
\]

then

\[
\sum_{i \in I} \sum_{j \in J_i} \pi_{W_i}(T) = \sum_{i \in I} \sum_{j \in J_i} c_{ij} e_j = 0.
\]

This yields \( \pi_{W_i}(T) = 0 \) for all \( i \in I \). It follows that \( c_{ij} = 0 \) for all \( i \in I \land j \in J_i \). Applying Theorem 6.1.1 of [11] the conclusion (ii) follows. On the other hand suppose that (ii) holds, then...
by definition we can write \( \{ e_i \}_{i \in I} = \{ U(u_i) \}_{i \in I} \) where \( U : \mathcal{H} \to \mathcal{H} \) is a bounded bijective operator and \( \{ u_i \}_{i \in I} \) is an orthonormal basis for \( \mathcal{H} \). For all \( T \in \ell^2(I) \otimes \mathcal{H} \) we have

\[
\| \sum_{i \in I} \pi_w(T \delta_i) \| = \sup_{\{ e_i \}_{i \in I}} \left( \sum_{i \in I} \pi_w(T \delta_i, e_i) \right)^2 = U \left( \sum_{i \in I} \pi_w(T \delta_i) \right) U^*.
\]

This yields

\[
\| \sum_{i \in I} \pi_w(T \delta_i) \|^2 = \| U \|^2 \sum_{i \in I} \| (T \delta_i, e_i) \|^2 = \| U \|^2 \sum_{i \in I} \| \pi_w(T \delta_i) \|^2.
\]

Similarly, we obtain a lower Riesz fusion bound for \( W \). Moreover by Lemma 3.5 of [1] we have

\[
\overline{\text{span}} \{ W_i \}_{i \in I} = \mathcal{H}.
\]

From this the result follows. □

**Definition 3.7.** Let \( \{ W_i \}_{i \in I} \) and \( \{ Z_i \}_{i \in I} \) be sequences of closed subspaces for \( \mathcal{H} \) and \( \mathcal{K} \) respectively. Then we will say \( \{ W_i \}_{i \in I} \) and \( \{ Z_i \}_{i \in I} \) are equivalent if there exists a bounded invertible operator \( U : \mathcal{H} \to \mathcal{K} \) such that \( UW_i = Z_i \) for every \( i \in I \).

In the following we show that every Riesz fusion basis is equivalent with an orthonormal fusion basis in \( \mathcal{H} \). For this, we first need a technical lemma, which is taken from [16].

**Lemma 3.8.** Let \( V \) be a closed subspace of \( \mathcal{H} \) and let \( T \) be a bounded operator on \( \mathcal{H} \), then \( \pi_T V = \pi_T \pi_T^* \).

**Theorem 3.9.** A family of closed subspaces \( \mathcal{W} = \{ W_i \}_{i \in I} \) is a Riesz fusion basis for \( \mathcal{H} \) if and only if there exists an orthonormal fusion basis \( \{ N_i \}_{i \in I} \) for \( \mathcal{H} \) and a bounded bijective operator \( U : \mathcal{H} \to \mathcal{H} \) for which \( U N_i = W_i \) for all \( i \in I \).

**Proof.** First, assume that \( \mathcal{W} = \{ W_i \}_{i \in I} \) is a Riesz fusion basis for \( \mathcal{H} \), and let \( \{ e_i \}_{i \in I} \) be an orthonormal basis for \( W_i \) for all \( i \in I \). Then by Theorem 3.6, \( \{ e_i \}_{i \in I} \) is a Riesz basis for \( \mathcal{H} \) and hence it is the form \( \{ U(u_i) \}_{i \in I} \), where \( \{ u_i \}_{i \in I} \) is an orthonormal basis for \( \mathcal{H} \) and \( U : \mathcal{H} \to \mathcal{H} \) is a bounded bijective operator. Define \( N_i = \overline{\text{span}} \{ u_i \}_{i \in I} \) for all \( i \in I \), then \( \{ N_i \}_{i \in I} \) is an orthonormal fusion basis for \( \mathcal{H} \) and \( U N_i = W_i \) for all \( i \in I \). For the converse, suppose that \( \{ W_i \}_{i \in I} = \{ U N_i \}_{i \in I} \) for some orthonormal fusion basis \( \{ N_i \}_{i \in I} \) for \( \mathcal{H} \) and a bounded bijective operator \( U : \mathcal{H} \to \mathcal{H} \). First note that \( \{ W_i \}_{i \in I} \) is complete. In order to show the Riesz fusion basis property, notice that applying Lemma 3.8 to \( W_i \) and \( U^{-1} \) yields \( \pi_w = \pi_w(U^{-1}) \pi_{U^{-1}} U^{-1} \) by taking adjoint we have \( \pi_w = U_{\mathcal{H}} \pi_{U^{-1}} \pi_w \) for all \( i \in I \). From this we have

\[
\| \sum_{i \in I} \pi_w(T \delta_i) \|^2 = \sup_{\{ e_i \}_{i \in I}} \left( \sum_{i \in I} \pi_w(T \delta_i, e_i) \right)^2 \leq \sup_{\{ e_i \}_{i \in I}} \left( \sum_{i \in I} \pi_{U^{-1}} \pi_w(T \delta_i, g) \right)^2 \leq \| U \|^2 \sum_{i \in I} \| \pi_{U^{-1}} \pi_w(T \delta_i) \|^2 \leq \| U \|^2 \sum_{i \in I} \| \pi_w(T \delta_i) \|^2.
\]

This completes the proof. □

Let \( V \) be a closed subspace of \( \mathcal{H} \) then a linear mapping \( P_V : \mathcal{H} \to V \) is called an oblique projection on \( V \), if \( P_V^2 = P_V \).

An important property is that the adjoint \( P_V^* \) is also an oblique projection from \( \mathcal{H} \) onto \( N^\perp \).

**Theorem 3.10.** Let \( \mathcal{W} = \{ W_i \}_{i \in I} \) be a Riesz fusion basis for \( \mathcal{H} \), then there exists an Riesz fusion basis \( \mathcal{Z} = \{ Z_i \}_{i \in I} \) for \( \mathcal{H} \) and a family of oblique projections \( \{ P_{Z_i} \}_{i \in I} \) such that

\[
f = \sum_{i \in I} P_{Z_i} \pi_{W_i}(f) \forall f \in \mathcal{H}.
\]

Moreover \( W_i \bot Z_i \) for all \( i, k \in I, i \neq k \).

**Proof.** For all \( i \in I \) let \( \{ e_i \}_{j \in J} \) be an orthonormal basis for \( W_i \) then by Theorem 3.6 \( \{ e_i \}_{j \in J} \) is a Riesz basis for \( \mathcal{H} \) and hence we can write \( \{ e_i \}_{j \in J} = \{ U(u_i) \}_{j \in J} \). Put \( N_i = \overline{\text{span}} \{ u_i \}_{j \in J} \) for all \( i \in I \), then \( U N_i = W_i \). Since \( (U^{-1})^* \) is bounded and bijective thus by Theorem 3.9 \( \{ Z_i \}_{i \in J} = \{ (U^{-1})^* N_i \}_{i \in J} \) is a Riesz fusion basis for \( \mathcal{H} \) and for every \( i \in I \)

\[
P_{Z_i}(f) = \sum_{j \in J} \langle f, e_i \rangle (U^{-1})^* u_j \forall f \in \mathcal{H}
\]

is an oblique projection onto \( Z_i \). In addition to we compute

\[
\sum_{i \in I} P_{Z_i} \pi_{W_i}(f) = \sum_{i \in I} \sum_{j \in J} \langle \pi_{W_i}(f), e_i \rangle (U^{-1})^* u_j = \sum_{i \in I} \sum_{j \in J} \langle f, e_i \rangle (U^{-1})^* u_j = f.
\]

For the moreover part, let \( i, k, l \in I, i \neq k \) then for any \( f \in \mathcal{H} \) we have

\[
\langle \pi_{Z_l} \pi_{W_i}(f), j \rangle = \sum_{j \in J} \langle f, e_{i} \rangle \langle e_{i}, \pi_{Z_l}(f) \rangle \langle e_{i}, (U^{-1})^* u_j \rangle = \sum_{j \in J} \langle f, e_{i} \rangle \langle e_{i}, \pi_{Z_l}(f) \rangle \langle u_{i}, u_j \rangle = 0.
\]

From this the result follows. □

A fusion frame \( W_k = \{ (W_i, z_i) \}_{i \in I} \) is called exact, if it ceases to be a fusion frame whenever anyone of its element is deleted.
Theorem 3.11. Let \( \mathcal{W} = \{W_i\}_{i=1}^\infty \) be a Riesz fusion basis for \( \mathcal{H} \), then \( \mathcal{W} = \{\{W_i, 1\}\}_{i=1}^\infty \) is a 1-uniform exact fusion frame for \( \mathcal{H} \). But the opposite implication is not valid.

Proof. Let \( \{e_i\}_{i=1}^\infty \) be an orthonormal basis for \( W_i \) for all \( i \in I \). Then by Theorem 3.6, \( \{e_i\}_{i=1}^\infty \) is a Riesz basis for \( \mathcal{H} \) and hence it is an exact frame. Now by Lemma 4.5 of [1] \( \mathcal{W} = \{(W_i, 1)\}_{i=1}^\infty \) is a 1-uniform exact fusion frame for \( \mathcal{H} \). For the opposite implication it is not valid suppose that \( \{e_i\}_{i=1}^\infty \) is an orthonormal basis for \( \mathcal{H} \) and for each \( i \in I \) define the subspaces \( W_1 \) and \( W_2 \) by

\[
W_1 = \text{span}(e_{i^0}) \quad \text{and} \quad W_2 = \text{span}(e_{i^0}).
\]

Then \( \{(W_1, 1), (W_2, 1)\} \) is a 1-uniform exact fusion frame but is not a Riesz fusion basis for \( \mathcal{H} \). □

Remark 3.12. If \( \mathcal{W} = \{(W_i, x_i)\}_{i=1}^\infty \) is a fusion frame for \( \mathcal{H} \) then \( \mathcal{W} = \{(\sum_{i=1}^\infty L_iW_i^j)_j\}_{j=1}^\infty \subseteq \mathcal{N}_{\text{ran}} \). In general not \( \mathcal{W} = \{(\sum_{i=1}^\infty L_iW_i^j)_j\}_{j=1}^\infty \subseteq \mathcal{N}_{\text{ran}} \). For example, let \( \{e_i\}_{i=1}^\infty \) be a Riesz normal basis for \( \mathcal{H} \) and for each \( i \in \mathbb{N} \) define the subspaces \( \{W_i\}_{i=1}^\infty \) by \( W_i = \text{span}(e_{i^0}) \). Then it is easily checked that \( \mathcal{W} = \{(W_i, 1)\}_{i=1}^\infty \) is a 1-uniform fusion frame with fusion frame bounds \( A = 1 \) and \( B = 2 \) for \( \mathcal{H} \) but it is not a Riesz fusion basis. Moreover if we take

\[
f_1 = \frac{1}{2}(e_1 + e_2), \quad f_2 = -\frac{1}{2}(e_1 + e_2), \quad f_3 = \frac{1}{2}(e_3 + e_4)
\]

Then \( \sum_{i=1}^\infty \delta_i \otimes f_i \neq \left( \sum_{i=1}^\infty L_iW_i^j \right)_j \) but \( \mathcal{W}_V, (\sum_{i=1}^\infty \delta_i \otimes f_i) = 0 \) consequently \( \mathcal{N}_{\text{ran}} \neq \left( \sum_{i=1}^\infty L_iW_i^j \right)_j \).

A family of subspaces \( \{W_i\}_{i=1}^\infty \) of \( \mathcal{H} \) is called minimal, if for each \( j \in I \)

\[
W_j \cap \text{span}(W_i)_{i \neq j} = \{0\}.
\]

Also a fusion frame \( \mathcal{W} = \{(W_i, x_i)\}_{i=1}^\infty \) is called a Riesz decomposition of \( \mathcal{H} \), if for every \( f \in \mathcal{H} \) there is a unique choice of \( f_i \in W_i \) so that \( f = \sum_{i=1}^\infty f_i \).

Theorem 3.13. Let \( \mathcal{W} = \{(W_i, 1)\}_{i=1}^\infty \) be a 1-uniform fusion frame for \( \mathcal{H} \). Then the following conditions are equivalent:

(i) \( \mathcal{W} = \{(W_i, 1)\}_{i=1}^\infty \) is a Riesz fusion basis for \( \mathcal{H} \).

(ii) The synthesis operator \( T_W \) is bounded, surjective and \( \mathcal{N}_{\text{ran}} = \left( \sum_{i=1}^\infty L_iW_i^j \right)_j \).

(iii) The analysis operator \( T_W \) is injective and \( \mathcal{R}_{\text{ran}} = \left( \sum_{i=1}^\infty L_iW_i^j \right)_j \).

(iv) \( \mathcal{W} = \{(W_i, 1)\}_{i=1}^\infty \) is a Riesz decomposition of \( \mathcal{H} \).

(v) \( \mathcal{W} = \{(W_i, 1)\}_{i=1}^\infty \) is minimal.

Proof.

(i) \( \Rightarrow \) (ii) Suppose that \( T \in \mathcal{N}_{\text{ran}} \), by the definition of Riesz fusion basis there exists \( C > 0 \) such that

\[
C \sum_{i=1}^\infty \|\pi_{W_i}(T \delta_i)\|^2 \leq \sum_{i=1}^\infty \|\pi_{W_i}(T \delta_i)\|^2 = 1.
\]

Then \( \{\pi_{W_i}(T \delta_i)\}_{i=1}^\infty \) and \( \{\pi_{W_i}(T \delta_i)\}_{i=1}^\infty \) exist for all \( i \) and

\[
\sum_{i=1}^\infty \|\pi_{W_i}(T \delta_i)\|^2 = \sum_{i=1}^\infty \|\pi_{W_i}(T \delta_i)\|^2 = 0.
\]

4. Stability of fusion frame sequences under perturbations

Suppose that the operator \( T \in B(\mathcal{H}, K) \) has closed range. Then there exist a unique bounded operator \( \gamma : K \rightarrow \mathcal{H} \) satisfying:

\[
T^* T = T, \quad T^* T T^* = T, \quad (T^* T)^* = T^* T, \quad (T^* T)^* = T^* T.
\]

The operator \( T^* \) is called the pseudo-inverse of \( T \). If \( T \) is a bounded invertible operator, then \( T^* = T^{-1} \). It is well known [17] that the operator \( T \in B(\mathcal{H}, \mathcal{K}) \) has closed range if and only if

\[
\gamma(T) = \inf_{f \in N_T^\perp} \|Tf\| > 0.
\]

\[
\|f\| = 1
\]
It can be shown that if $\mathcal{R}_T$ is closed, then

$$\gamma(T) = \gamma(T) = \gamma(T)^2$$

and $\gamma(T) = \|T\|^{-1}$.

A sequence $\mathcal{W}_s = \{(W_i, \pi_i)\}_{i \in I}$ is called a fusion frame sequence if it is a fusion frame for $\mathcal{H}_W = \overline{\text{span}}\{W_i\}_{i \in I}$. Theorem 2.2 from [15] leads to a statement about fusion frame sequence.

Corollary 4.1. A sequence $\mathcal{W}_s = \{(W_i, \pi_i)\}_{i \in I}$ is a fusion frame sequence if and only if

$$T_{W_i} : \ell^2(I) \otimes \mathcal{H} \to \mathcal{H}, \quad T_{W_i} = \sum_{i \in I} \pi_i W_i^*,$$

is a well-defined bounded operator with closed range.

Corollary 4.2. The optimal fusion frame bounds for $\mathcal{W}_s$ are

$$C = \gamma(T_{W_i}) = \|T_{W_i}\|^{-2}$$

and $D = \|S_{W_i}\| = \|T_{W_i}\|^2$.

Proof. See Lemma 2.4 of [15]. □

Let $W, Z$ be closed subspaces of $\mathcal{H}$. If $W \neq 0$, the gap between $W$ and $Z$ is defined by:

$$\Delta(W, Z) = \sup_{f \in W} \text{dist} (f, Z) = \sup_{f \in W} \inf_{g \in Z} \|f - g\|$$

$$= \sup_{f \in W} \|f - \pi_Z f\|, \quad \|f\| = 1$$

As a convention we use $\Delta(0, Z) = 0$ and for all $T, U \in B(\mathcal{H})$ we set

$$\Delta_T = \Delta(N_T, N_U).$$

Using this notation Christensen [18] proved the following stability result for the closure of the range of an operator.

Proposition 4.3. Let $T, U \in B(\mathcal{H})$. Suppose that $\Delta_T' < 1$ and that there exist numbers $\lambda_1 \in [0; 1], \lambda_2 \in [1; \infty)$ and $\mu \geq 0$ such that

$$\|T' - Uf\| \leq \lambda_1 \|T\| + \lambda_2 \|U\| + \mu \|f\| \quad \forall f \in \mathcal{H}.$$

Then

$$\gamma(U) \geq \frac{(1 - \lambda_1)(1 - \lambda_2)}{\gamma(T)(1 - \lambda_2)}.$$

(ii) If $\mathcal{R}_T$ is closed and $\lambda_1 + \frac{\mu}{\gamma(T)(1 - \lambda_2)} < 1$, then $\mathcal{R}_U$ is closed and

$$\|U\| \leq \frac{(1 + \lambda_2)\|T\|}{(1 - \lambda_1)(1 - \lambda_2) - \mu \|T\|}.$$ 

The stability of frame sequences is important in practice and is therefore studied by Christensen in [18]. In this section we study the stability of fusion frame sequences.

Theorem 4.4. Let $\mathcal{W}_s = \{(W_i, \pi_i)\}_{i \in I}$ be a fusion frame sequence with bounds $C, D$ for $\mathcal{H}$. Let $\{Z_i\}_{i \in I}$ be a family of closed subspaces in $\mathcal{H}$, and suppose that there exists numbers $\lambda_2 \in [0; 1]$ and $\lambda_1, \mu \geq 0$ such that

$$\sum_{i \in I} \pi_i (\pi_i W_i - \pi_i Z_i) L_i^*(T) \leq \lambda_1 \sum_{i \in I} \pi_i W_i^* L_i^*(T)$$

$$+ \lambda_2 \sum_{i \in I} \pi_i Z_i L_i^*(T) + \mu \left( \sum_{i \in I} \|T' (\delta_i)\|^2 \right)^{1/2}$$

for any finite subset $I_0 \subseteq I$ and $T \in \ell^2(I) \otimes \mathcal{H}$. Then $Z_s = \{(Z_i, \pi_i)\}_{i \in I}$ is a Bessel frame sequence with Bessel bound $D \left(1 + \frac{\lambda_1 + \lambda_2 + \mu}{\gamma(T)(1 - \lambda_2)}\right)^2$. If furthermore $\lambda_2 < 1$ and $\lambda_1 + \frac{\mu}{\gamma(T)(1 - \lambda_2)} < 1$, then $Z_s = \{(Z_i, \pi_i)\}_{i \in I}$ is a fusion frame sequence with lower fusion frame bound

$$C(1 - \Delta_T') \left(1 - \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\gamma(T)(1 - \lambda_2)}}{1 + \lambda_2}\right)^2.$$ 

Proof. Since $\mathcal{W}_s$ is a frame fusion sequence, hence the operator

$$T_{W_i} : \ell^2(I) \otimes \mathcal{H} \to \mathcal{H}, \quad T_{W_i} = \sum_{i \in I} \pi_i W_i^*$$

is a well-defined bounded operator with $\|T_{W_i}\| \leq \sqrt{D}$ and $\mathcal{R}_T$ is closed. Let $T \in \ell^2(I) \otimes \mathcal{H}$ and fix $J \subset I$ with $\|J\| < \infty$. Then the condition (17) implies that

$$\left\| \sum_{i \in J} \pi_i Z_i L_i^*(T) \right\| \leq \left\| \sum_{i \in J} \pi_i (\pi_i W_i - \pi_i Z_i) L_i^*(T) \right\|$$

$$+ \left\| \sum_{i \in J} \pi_i Z_i L_i^*(T) \right\| + \mu \left( \sum_{i \in J} \|T' (\delta_i)\|^2 \right)^{1/2}.$$ 

Thus

$$\left\| \sum_{i \in J} \pi_i Z_i L_i^*(T) \right\| \leq 1 + \frac{\lambda_1}{1 - \lambda_2} \sum_{i \in J} \pi_i W_i^* L_i^*(T)$$

$$+ \lambda_2 \sum_{i \in J} \pi_i Z_i L_i^*(T) + \mu \left( \sum_{i \in J} \|T' (\delta_i)\|^2 \right)^{1/2}.$$ 

It follows that $\sum_{i \in J} \pi_i Z_i L_i^*(T)$ is weakly unconditionally Cauchy and hence unconditionally convergent in $\mathcal{H}$. If we set

$$T_{Z_i} : \ell^2(I) \otimes \mathcal{H} \to \mathcal{H}, \quad T_{Z_i} = \sum_{i \in I} \pi_i Z_i^* L_i^*,$$

we have

$$\|T_{W_i}(T) - T_{Z_i}(T)\| \leq \lambda_1 \|T_{W_i}(T)\| + \lambda_2 \|T_{Z_i}(T)\| + \mu \|T\| \quad \forall T \in \ell^2(I) \otimes \mathcal{H}.$$
This yields
\[
\|T_{Z_a}(T)\| \leq \frac{1 + \lambda_1}{1 - \lambda_2} \|T_{W_a}(T)\| + \frac{\mu}{1 - \lambda_2} \|T\| \leq (1 + \lambda_1)\sqrt{D} + \frac{\mu}{1 - \lambda_2} \|T\| \quad \forall T \in \ell^2(I) \otimes H.
\]

It follows that \(Z_a = \{(Z_i, x_i)\}_{i \in I}\) is a Bessel fusion sequence with Bessel bound \(D = \left(1 + \frac{\lambda_1 + \lambda_2 + \sqrt{\lambda_1^2 - \lambda_2^2}}{1 - \lambda_2}\right)^2\). Now suppose that \(D < 1\) and \(\lambda_1 + \frac{\mu}{\sqrt{\lambda_1(1 - \lambda_2^2)}} < 1\). Since \(R_{W_{a}}\) is closed so by Corollary 4.2 we have \(\gamma(T_{W_a}) = \|T_{W_a}\|^{-1} \geq \sqrt{C}\) which implies that \(\lambda_1 + \frac{\mu}{\gamma(T_{W_a})(1 - \lambda_2^2)} < 1\). By Proposition 4.3 \(R_{Z_a}\) is closed and by Corollary 4.1 \(Z_a = \{(Z_i, x_i)\}_{i \in I}\) is a fusion frame sequence. The optimal lower fusion frame bounds of \(Z_a\) is
\[
\|T_{W_a}\|^{-2} = \gamma(T_{Z_a})^2 \quad \text{this shows that}
\]
\[
\gamma(T_{Z_a})^2 \geq \left(\frac{1 - \lambda_1}{1 + \lambda_2}\right) \sqrt{1 - \lambda_2^2}^2 + \frac{\mu}{\sqrt{1 - \lambda_2^2}}^2 \quad \text{which finishes the proof.}
\]

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