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The Commutant Modulo C_p of Co-prime Powers of Operators on a Hilbert Space

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Let H be a separable infinite-dimensional complex Hilbert space and let $A, B \in B(H)$, where $B(H)$ is the algebra of operators on H into itself. Let $\delta_{A,B}$: $B(H) \to B(H)$ denote the generalized derivation $\delta_{AB}(X) = AX - XB$. This note considers the relationship between the commutant of an operator and the commutant of co-prime powers of the operator. Let m, n be some co-prime natural numbers and let $\mathscr{C}_p$ denote the Schatten p-class, $1 \le p < \infty$. We prove (i) If $\delta_{A^m B^m}(X) = 0$ for some $X \in B(H)$ and if either of A and B^* is injective, then a necessary and sufficient condition for $\delta_{AB}(X) = 0$ is that $A^r X B^{n-r} - A^{n-r} X B^r = 0$ for (any) two consecutive values of r, $1 \le r < n$. (ii) If $\delta_{A^m B^m}(X)$ and $\delta_{A^n B^n}(X) \in \mathscr{C}_p$ for some $X \in B(H)$, and if $m = 2$ or 3, then either $\delta^n_{AB}(X)$ or $\delta^{n+3}_{AB}(X) \in \mathscr{C}_p$; for general m and n, if A and B^* are normal or subnormal, then there exists a natural number t such that $\delta_{AB}(X) \in \mathscr{C}_{2^{tm}p}$. (iii) If $\delta_{A^m B^m}(X)$ and $\delta_{A^n B^n}(X) \in \mathscr{C}_p$ for some $X \in B(H)$, and if either A is semi-Fredholm with $\operatorname{ind} A \le 0$ or $1 - A^*A \in \mathscr{C}_p$, then $\delta_{AB}(X) \in \mathscr{C}_p$.

1. INTRODUCTION

Let H be a separable infinite-dimensional complex Hilbert space and let $B(H)$ denote the algebra of operators (= bounded linear transformations) on H. Let $A, B \in B(H)$ and let δ_{AB}: $B(H) \to B(H)$ denote the generalized derivation $\delta_{AB}(X) = AX - XB$. Then X is in the commutant of A and B if and only if $\delta_{AB}(X) = 0$. Let m, n be (relatively) co-prime natural numbers, denoted $(m, n) = 1$, with $1 < m < n$, and suppose that $\delta_{A^m B^m}(X) = 0 = \delta_{A^n B^n}(X)$ for some $X \in B(H)$. Then $\delta_{A^{tm+n} B^{tm+n}}(X) = 0$ for all $t = 1, 2, \dots$ and either $m + n$ or $2m + n$ is an odd natural number.

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Suppose, for definiteness, that $m+n$ is odd. Then

$$\delta_{AB}^{m+n}(X) = \delta_{AB}(\delta_{AB}^{m+n-1}(X)) = \sum_{r=0}^{m+n}(-1)^r \binom{m+n}{r} A^{m+n-r}XB^r$$
$$= (A^{m+n}X - XB^{m+n}) - \sum_{r=1}^{(m+n-1)/2} \binom{m+n}{r}$$
$$\times\{A^{m+n-r}XB^r - A^rXB^{m+n-r}\},$$

and the simplest way for $\delta_{AB}^{m+n}(X)$ to be equal to 0 is that

$$A^{m+n-r}XB^r - A^rXB^{m+n-r} = 0$$

for all $1 \le r \le \frac{m+n-1}{2}$. Assuming now that $\delta_{AB}^{m+n}(X) = 0$, and that A and B are normal, it follows from [8, Lemma 1] that $\delta_{AB}(X) = 0$, i.e., X is in the commutant of A and B.

Relationships between the commutant of an operator A and the commutant of the powers of the operator have been investigated by a number of authors, among them Al-Moajil [1], Embry [3], and Kittaneh [6]. Al-Moajil [1] has shown that if A is a normal operator such that $\delta_{A^2}(X)(= A^2X - XA^2) = 0 = \delta_{A^3}(X)$ for some $X \in B(H)$, then $\delta_A(X) = 0$. This result was extended to subnormal operators A and B^* for which $\delta_{A^2B^2}(X) = 0 = \delta_{A^3B^3}(X)$ for some $X \in B(H)$ by Kittaneh [6], who also considered commutants modulo $\mathscr{C}_p(= \mathscr{C}_p(H))$, the Schatten p-class, of A^2 and A^3. This note considers the relationship between the commutant (including commutant modulo $\mathscr{C}_p$) of an operator and the commutant of co-prime powers of the operator. Thus, let m, n be co-prime natural numbers. It is proved that (i) If $\delta_{A^mB^m}(X) = 0$ for some $X \in B(H)$ and if either of A and B^* is injective, then a necessary and sufficient condition for $\delta_{AB}(X) = 0$ is that $A^rXB^{n-r} - A^{n-r}XB^r = 0$ for (any) two consecutive values of $r, 1 \le r < n$. (ii) If $\delta_{A^mB^m}(X)$ and $\delta_{A^nB^n}(X) \in \mathscr{C}_p$ for some $X \in B(H)$ and if $m = 2$ or 3, then either $\delta_{AB}^n(X)$ or $\delta_{AB}^{n+3}(X) \in \mathscr{C}_p$; for general $(m, n) = 1$, if A and B^* are normal or subnormal, then there exists a natural number t such that $\delta_{AB}(X) \in \mathscr{C}_{2^{tm}p}$. We prove also that if $\delta_{A^mB^m}(X)$ and $\delta_{A^nB^n}(X) \in \mathscr{C}_p$ for some $X \in B(H)$, and if either A is semi-Fredholm with ind $A \le 0$ or $1 - A^*A \in \mathscr{C}_p$, then $\delta_{AB}(X) \in \mathscr{C}_p$.

In the following we shall denote the set of natural numbers by $\mathscr{N}$. The spectrum and the point spectrum of an operator A will be denoted by $\sigma(A)$ and $\sigma_{\mathrm{p}}(A)$, respectively. Most of the other notation that we employ in the following is standard and is usually explained at the first instance of occurrence.

2. RESULTS

We assume in the following that m, n are co-prime natural numbers with $1 < m < n$. Although it will not always be required (as, for example, in Theorem 1), we assume in the following that our Hilbert space H is separable. We shall denote the ideal of compact operators by $\mathcal{K}$ $(= \mathcal{K}(H))$; thus when we discuss the ideals $\mathcal{C}_p$ it will be assumed that $1 \leq p < \infty$. Recall that a Banach space operator T has dense range if and only if T^* is injective, and that if T^* has dense range then T is injective [9, pp. 94–96].

Let $A, B \in B(H)$ and suppose that $\delta_{A^m B^m}(X) = 0 = \delta_{A^n B^n}(X)$ for some $X \in B(H)$. Then $\delta_{A^{tm} B^{tm}}(X) = 0$ for all $t \in \mathcal{N}$. Also, since $(m, n) = 1$, there exist integers p and q, with $pq < 0$, such that $pm + qn = 1$. Suppose for definiteness that $p < 0$; then

$$\begin{aligned}\delta_{A^n B^n}(X) = 0 \Longrightarrow A^{1-mp} X - X B^{1-mp} \\ = \delta_{AB}(X) B^{-mp} = 0 = A^{-mp} \delta_{AB}(X),\end{aligned}$$

and hence if either of A or B^* is injective, then $\delta_{AB}(X) = 0$. It is clear that if $\delta_{AB}(X) = 0$, then

$$A^r X B^{n-r} - A^{n-r} X B^r = 0 \tag{1}$$

for all $0 \leq r \leq n$. The following theorem shows that if $\delta_{A^m B^m}(X) = 0$ for some $X \in B(H)$, then a sufficient condition for $\delta_{AB}(X) = 0$ is that B has dense range and (1) holds for (any) two consecutive values of r $(1 \leq r < n)$.

THEOREM 1. *Let $A, B \in B(H)$ and suppose that either A or B^* is injective. If $\delta_{A^m B^m}(X) = 0$ for some $X \in B(H)$, then a necessary and sufficient condition for $\delta_{AB}(X) = 0$ is that* (1) *holds for some $r = r_0$ and $r = r_0 + 1$, $1 \leq r_0 < n$.*

Proof. Let us, for brevity's sake, set $\delta_{A^m B^m}(X) = T_m$ and $A^r X B^{n-r} - A^{n-r} X B^r = S_r$. If $T_1 = \delta_{AB}(X) = 0$, then (upon letting $n = pm + s$) $S_r = T_n = A^s T_{pm} + A^s X B^{pm} - X B^{pm+s} = 0$ for all $0 \leq r \leq n$. Hence to prove the theorem it will suffice to show that if hypothesis (1) is satisfied for (any) two consecutive values of r, then $T_n = 0$. The hypothesis $(m, n) = 1$ implies that m cannot divide both r and $n - r$ for all $1 \leq r < n$. Also, since $T_n = 0$ if either $m | r$ or $m | n - r$, we may assume that $m \nmid r$ and $m \nmid n - r$. Then $r \equiv b_1 \pmod{m}$ and $n - r \equiv c_1 \pmod{m}$ for some $b_1, c_1 \in \mathcal{N}$ such that $b_1, c_1 < m$. Suppose now that B has dense range. (The proof for the case in which A is injective is similar.) Then $T_m = 0$ implies that $S_r = S_{b_1} = 0$, and so $A^{b_1} X B^{c_1} = A^{c_1} X B^{b_1}$. We have two possibilities: either $b_1 = c_1$ (i.e., $r \equiv n - r \pmod{m}$ or $b_1 \neq c_1$. We consider these cases separately.

Case $b_1 = c_1$. For a fixed $r(= r_0)$, $1 \le r < n$, the hypotheses imply that $S_{r+1} = 0$ also. If either $m | r+1$ or $m | n-r-1$, then $T_n = 0$ and we are done. If, on the other hand, $m \nmid r+1$ and $m \nmid n-r-1$, then either $r+1 \equiv n-r-1 \pmod m$ or $r+1 \not\equiv n-r-1 \pmod m$. Since $r+1 \equiv n-r-1 \pmod m$ implies that $2(r+1)-2r = 2 \equiv 0 \pmod m$, $m = 2$ and $2|(n-2r)$. But then $2|n$ and $(m,n) = 2$. This contradiction implies that $r+1 \not\equiv n-r-1 \pmod m$. Arguing as above, it then follows that there exist $b_2, c_2 \in \mathcal{N}$, $b_2 \neq c_2$, such that $A^{b_2}XB^{c_2} = A^{c_2}XB^{b_2}$. This reduces the proof to the case $b_1 \neq c_1$, which we consider next.

Case $b_1 \neq c_1$. We may assume that $b_1 > c_1$. Let $b_1 - c_1 = t$ and let $m = c_1 + d$ for some $d \in \mathcal{N}$. Then

$$\begin{aligned} A^{b_1}XB^{c_1} = A^{c_1}XB^{b_1} \Longrightarrow A^{b_1+d}XB^{c_1} &= A^t A^m XB^{c_1} = A^t XB^{m+c_1} \\ &= A^{c_1+d}XB^{b_1} = XB^{b_1+m} = XB^t B^{m+c_1} \\ &\Longrightarrow A^t X = XB^t \end{aligned}$$

(since B has dense range). Now if $(m,t) = 1$, then $T_t = 0 = T_m$ implies that $T_1 = \delta_{AB}(X)$, and hence also $T_n = 0$. If, on the other hand, $(m,t) \neq 1$, then let $(m,t) = k$. There exist integers p and q, with $pq < 0$, such that $mp + qt = k$. Assume, for definiteness, that $p < 0$; then

$$A^{tq}X = XB^{tq} \Longrightarrow XB^{k-mp} = A^{k-mp}X = A^k XB^{-mp} \Longrightarrow A^k X = XB^k.$$

Thus, upon letting $t = kt_1$ and $m = km_1$, $k \in \mathcal{N}$, and $(m_1, t_1) = 1$, we have that

$$XB^{kt_1} = XB^t = A^t X = A^{kt_1}X = A^{t_1}XB^{(k-1)t_1} \Longrightarrow A^{t_1}X = XB^{t_1}$$

and

$$XB^{km_1} = XB^m = A^m X = A^{km_1}X = A^{m_1}XB^{(k-1)m_1} \Longrightarrow A^{m_1}X = XB^{m_1}.$$

Consequently, T_1 and, hence, T_n equal 0.

Remarks. (1). Theorem 1 fails in the absence of the hypothesis that $(m,n) = 1$. To see this, let $\{e_n\}_{-\infty}^{\infty}$ be an orthonormal basis for H. Define $A \in B(H)$ by $Ae_{2n} = \frac{1}{2}e_{2n+1}$ and $Ae_{2n+1} = 2e_{2n+2}$. Then A^2 is unitary. Now choose $X \in B(H)$ to be the bilateral shift $Xe_n = e_{n+1}$; then $A^r XA^{4-r} = A^{4-r}XA^r$ for all $0 \le r \le 4$ but $\delta_A(X) = AX - XA \neq 0$.

(2). Let $\sigma_{\mathrm{jp}}(A)$ denote the joint point spectrum of A (i.e., $\sigma_{\mathrm{jp}}(A) = \{\lambda \in \sigma_{\mathrm{p}}(A) : (A-\lambda)x = 0 \Longleftrightarrow (A-\lambda)^*x = 0\}$. A number of classes of operators, for example, those consisting of normal, subnormal, hyponormal, and dominant operators (see [8] for the definition of a dominant operator), have the property that $\sigma_{\mathrm{p}}(A) = \sigma_{\mathrm{jp}}(A)$ for operators A in these classes. If $0 \notin \sigma_{\mathrm{p}}(B^*)$ (or $0 \notin \sigma_p(A)$) for operators B^* (resp., A) belonging to one of these classes, then B has dense range (resp., A is injective). If, on the other hand, $0 \in \sigma_{\mathrm{p}}(A)$ and $0 \in \sigma_{\mathrm{p}}(B^*)$, then A and B^* have decompositions $A = 0 \oplus A_1$ and $B^* = 0 \oplus B_1^*$ (where the operators A_1 and B_1^* are injective). Letting X have the corresponding matrix representation $X = [X_{ij}]_{i,j=1}^2$, it follows that if $T_m = 0$, then $X = 0 \oplus X_{22}$ and $\delta_{A_1^m B_1^m}(X_{22}) = 0$. Similarly, if $S_r = 0$ for two consecutive values of r $(1 \leq r < n)$, then $A_1^r X_{22} B_1^{n-r} - A_1^{n-r} X_{22} B_1^r = 0$ for two consecutive values of r. Consequently, $\delta_{A_1 B_1}(X_{22})$, and so also T_1, equal 0. We note here that the hypothesis that A and B^* belong to a class for which $\sigma_{\mathrm{p}}(.) = \sigma_{\mathrm{jp}}(.)$ cannot be replaced by the hypothesis that A and B belong to the class (see [1, 3.2 Remark(a)] for a counterexample).

(3). The hypothesis that $S_r = 0$ for two consecutive values of r (or even all r, $1 \leq r < n$) is not sufficient to guarantee $\delta_{AB}(X) = 0$. To see this, let A be a pure quasi-normal operator, and let $B = A$ and $X = A^*$. Then, since $AA^*A = A^*A^2$, $A^r A^* A^{n-r} = A^{n-r} A^* A^r$ for all integers $n \geq 3$ and $1 \leq r < n$, but $\delta_{A^m}(A^*) = A^m A^* - A^* A^m \neq 0$ for any $m \in \mathcal{N}$.

Commutators Modulo $\mathcal{C}_p$.

Let π: $B(H) \to B(H)\backslash\mathcal{K}$ denote the Calkin map. If $\delta_{A^m B^m}(X)$ and $\delta_{A^n B^n}(X) \in \mathcal{K}$ for all $X \in B(H)$, then $\delta_{A^m B^m}(X) = 0 = \delta_{A^n B^n}(X)$ (see [5, Example 1]), and it follows from Theorem 1 that $\delta_{AB}(X) = 0$ for all $X \in B(H)$ whenever either of A or B^* is injective. More generally, if $\delta_{A^m B^m}(X)$ and $\delta_{A^n B^n}(X) \in \mathcal{K}$ for some $X \in B(H)$, and if either $\pi(A)$ or $\pi(B^*)$ is injective, then $\pi(\delta_{AB}(X)) = 0$, and so $\delta_{AB}(X) \in \mathcal{K}$. The situation is not as straightforward in the case in which $\delta_{A^m B^m}(X)$ and $\delta_{A^n B^n}(X) \in \mathcal{C}_p$, even in the case in which $m = 2$ and $n = 3$ (see [6]). Here $\delta_{A^m B^m}(X)$ and $\delta_{A^n B^n}(X)$ may belong to $\mathcal{C}_p$ without $\delta_{AB}(X)$ belonging to $\mathcal{C}_p$, as the following example shows. Let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis for H. Define the operators A, B, and X by

$$Ae_j = 2j^{-2/(3p)} e_j, \quad Be_j = (j+1)^{-2/(3p)} e_j, \quad \text{and} \quad Xe_j = e_{j+1}. \tag{2}$$

Then $\delta_{A^2 B^2}(X)$ and $\delta_{A^3 B^3} \in \mathcal{C}_p$, but $\delta_{AB}(X) \notin \mathcal{C}_p$.

Let $\delta_{A^m B^m}(X)$ and $\delta_{A^n B^n}(X) \in \mathcal{C}_p$ for some $X \in B(H)$. We prove in the following that (i) if A and B^* are normal or subnormal operators, then there exists a $t \in \mathcal{N}$ such that $\delta_{AB}(X) \in \mathcal{C}_{2^{tm} p}$, and (ii) if either $1 - A^*A$ or $1 - B^*B \in \mathcal{C}_p$, then $\delta_{AB}(X) \in \mathcal{C}_p$. (This generalizes Theorems 3 and 5 of [6] to the case $(m, n) = 1$.) But before that we consider the case in which $m = 2$ or 3.

THEOREM 2. *Let $m = 2$ or 3. If $\delta_{A^m B^m}(X)$ and $\delta_{A^n B^n}(X) \in \mathcal{C}_p$ for some $X \in B(H)$, then either $\delta_{AB}^n(X)$ or $\delta_{AB}^{n+3}(X) \in \mathcal{C}_p$.*

Here $\delta^n_{AB}(X)$ denotes $\delta_{AB}(\delta^{n-1}_{AB}(X))$. The following lemmas will be required in the proof of the theorem. (Retaining the notation of the proof of Theorem 1, we henceforth let $T_m = \delta_{A^m B^m}(X)$ and $S_r = A^r X B^{n-r} - A^{n-r} X B^r$.)

LEMMA 3. *If $T_m \in \mathscr{C}_p$ for some $X \in B(H)$, then $T_{sm} \in \mathscr{C}_p$ for all $s \in \mathscr{N}$.*

Proof. We have

$$\begin{aligned} T_{sm} = A^{sm}X - XB^{sm} &= A^{(s-1)m}T_m + A^{(s-1)m}XB^m - XB^{sm} \\ &= A^{(s-1)m}T_m + A^{(s-2)m}T_mB^m \\ &\quad + A^{(s-2)m}XB^{2m} - XB^{sm} \\ &\quad \cdots \\ &= A^{(s-1)m}T_m + A^{(s-2)m}T_mB^m \\ &\quad + \cdots + A^mT_mB^{(s-2)m} + T_mB^{(s-1)m} \\ &\in \mathscr{C}_p. \end{aligned}$$

LEMMA 4. *Let $m = 2$ or 3 and suppose that $T_m \in \mathscr{C}_p$ for some $X \in B(H)$.*

(i) *If $T_n \in \mathscr{C}_p$, then $S_r \in \mathscr{C}_p$ for all $0 \le r \le n$.*

(ii) *If $S_r \in \mathscr{C}_p$ for some $r = r_0$ and $r = r_0 + 1$, $1 \le r_0 < n$, then $T_n \in \mathscr{C}_p$.*

Proof. (i) The hypothesis $m = 2$ or 3 implies that either $n \equiv 1 \pmod{m}$ or $n \equiv 2 \pmod{m}$ (in which case m is necessarily equal to 3). We consider these cases separately.

Case $n \equiv 1 \pmod{m}$. In this case either $r \equiv 0 \pmod{m}$ and $n - r \equiv 1 \pmod{m}$ or $r \equiv 1 \pmod{m}$ and $n - r \equiv 0 \pmod{m}$ or $r \equiv 2 \pmod{m}$ and $n - r \equiv 2 \pmod{m}$. Notice that

$$\begin{aligned} S_r &= T_rB^{n-r} - T_n + A^{n-r}T_r \\ &= -A^rT_{n-r} + T_n - T_{n-r}B^r; \end{aligned}$$

hence if either $r \equiv 0 \pmod{m}$ or $n - r \equiv 0 \pmod{m}$, then $S_r \in \mathscr{C}_p$ (by Lemma 3). If $r \equiv 2 \pmod{m}$, then ($m = 3$ and) $S_r = -A^rT_{n-r-2}B^2 + A^{n-r}T_{r-2}B^2 \in \mathscr{C}_p$ (since $n - r \equiv 2 \pmod{m}$).

Case $n \equiv 2 \pmod{m}$. Once again either $r \equiv 0 \pmod{m}$ and $n - r \equiv 2 \pmod{m}$ or $r \equiv 1 \pmod{m}$ and $n - r \equiv 1 \pmod{m}$ or $r \equiv 2 \pmod{m}$ and $n - r \equiv 0 \pmod{m}$. The argument above shows that $S_r \in \mathscr{C}_p$ if either $r \equiv 0 \pmod{m}$ or $n - r \equiv 0 \pmod{m}$. Let $r \equiv 1 \pmod{m}$; then $n - r - 1 \equiv 0 \pmod{m}$ and

$$S_r = A^rT_{n-r-1}B + AT_{n-2}B + AT_{n-r-1}B^r \in \mathscr{C}_p.$$

(ii) The proof here is similar to that of part (i), except for the case in which $n \equiv 1 \pmod{m}$, $r \equiv 2 \pmod{m}$ and $n - r \equiv 2 \pmod{m}$, where we require the hypothesis that $S_{r_0+1} \in \mathscr{C}_p$ (along with the hypothesis that $S_{r_0} \in \mathscr{C}_p$). Recall that $m = 3$ in this case, and so $r_0 \equiv 2 \pmod{m}$ implies that $r_0 + 1 \equiv 0 \pmod{m}$ and $n - r_0 - 1 \equiv 1 \pmod{m}$. Applying Lemma 3 to

$$T_n = A^{n-r_0-1}T_{r_0+1} - S_{r_0+1} + T_{r_0+1}B^{n-r_0-1},$$

$T_n \in \mathscr{C}_p$ follows.

Proof of Theorem 2. Assume initially that $n = 2t + 1$ for some $t \in \mathscr{N}$. Then

$$\delta^n_{AB}(X) = \sum_{r=0}^{n}(-1)^r \binom{n}{r} A^{n-r}XB^r$$
$$= T_n - \sum_{r=1}^{t} \binom{n}{r} S_{n-r},$$

where each $S_{n-r} \in \mathscr{C}_p$ (by Lemma 4). Hence $\delta^n_{AB}(X) \in \mathscr{C}_p$ in this case. Now if n is even, then ($m = 3$ and) $n + m$ is odd with $(m, n + m) = 1$. Since

$$\delta^{n+m}_{AB}(X) = T_{n+m} - \sum_{r=1}^{(n+m-1)/2} \binom{n+m}{r} \{A^{n+m-r}XB^r - A^rXB^{n+m-r}\},$$

and since

$$A^{n+m-r}XB^r - A^rXB^{n+m-r} = A^mS_{n-r} + A^rT_mB^{n-r} \in \mathscr{C}_p,$$

$\delta^{n+m}_{AB}(X) \in \mathscr{C}_p$. This completes the proof.

Remark 4. For general co-prime $m, n \in \mathscr{N}$, T_m and $T_n \in \mathscr{C}_p$ implies that $T_{m+n} \in \mathscr{C}_p$. Let $n_0 = n$ if n is odd and $n_0 = m + n$ if n is even. Then $(m, n_0) = 1$ and

$$\pi(\delta^{n_0}_{AB}(X) - T_{n_0}) = \pi\left(\sum_{r=1}^{t}(-1) \binom{n_0}{r} \{A^{n_0-r}XB^r - A^rXB^{n_0-r}\}\right) = 0,$$

where $t = [n_0/2]$ denotes the largest integer less than $n_0/2$. This implies the existence of a compact operator K such that

$$\delta^{n_0}_{AB}(X) + K \in \mathscr{C}_p.$$

Assume now that A and B^* are normal (or subnormal) operators. Applying [8, Lemma 1] (resp., [8, Corollary 1]), it then follows that $\pi(\delta_{AB}(X))(= \pi(\delta_{A^*B^*}(X)) = 0$. In particular, $\delta_{AB}(X)$ is a compact operator. Does $\delta_{AB}(X) \in \mathscr{C}_p$? The answer to this question is negative, as follows from a consideration of the operators A, B, and X of (2).

We prove next the first of our promised results.

THEOREM 5. *If A, B^* are normal or subnormal operators, and if T_m and $T_n \in \mathscr{C}_p$ for some $X \in B(H)$, then there exists a $t \in \mathscr{N}$ such that $\delta_{AB}(X) \in \mathscr{C}_{2^{tn}p}$.*

We note here that the choice of t depends upon the number s such that $n \equiv s \pmod{m}$. If $s = 1$, then $t = 1$ and $\delta_{AB}(X) \in \mathscr{C}_{2^n p}$. The proof of the theorem (is an extension of the argument used by Kittaneh to prove [6, Theorem 3] and) depends on the following result of Weiss [10, p. 114].

LEMMA 6. *If N and $X \in B(H)$, N is normal, and $\mathscr{I}$ is any two-sided ideal of $B(H)$, then $NX \in \mathscr{I}$ ($XN \in \mathscr{I}$) implies $N^*X \in \mathscr{I}$ (resp., $XN^* \in \mathscr{I}$).*

Proof of Theorem 5. If A and B^* are subnormal, then let $\widetilde{A}$ and $\widetilde{B}^*$ denote their minimal normal extensions on $\mathscr{H} \supset H$, say. Let $\widetilde{X} = X \oplus 0$ on $H \oplus (\mathscr{H} \ominus H)$. Then $\delta_{\widetilde{A^m}\widetilde{B^m}}(\widetilde{X})$ and $\delta_{\widetilde{A^n}\widetilde{B^n}}(\widetilde{X}) \in \mathscr{C}_p(\mathscr{H})$. Consequently it will suffice to consider the case of normal A and B.

Let $n \equiv s \pmod{m}$. Then there exists a $t \in \mathscr{N}$ such that $t < m$, $st \equiv 1 \pmod{m}$ and $(m, tn) = 1$. By Lemma 4, $T_{tn} \in \mathscr{C}_p$ and (since $tn \equiv 1 \pmod{m}$) $T_{tn-1} \in \mathscr{C}_p$. Hence

$$A^{tn-1}T_1 = T_{tn} - T_{tn-1}B \in \mathscr{C}_p$$

and

$$T_1 B^{tn-1} = T_{tn} - AT_{tn-1} \in \mathscr{C}_p.$$

The operator A being normal, Lemma 6 applied to $AA^{tn-2}T_1 \in \mathscr{C}_p$ implies that $A^*A^{tn-2}T_1 \in \mathscr{C}_p$. This implies that $T_1^* A^{*tn-2} A^{tn-2} T_1 \in \mathscr{C}_p$, and hence that $A^{tn-2}T_1 \in \mathscr{C}_{2p}$. Repeating this argument another $tn - 3$ times, it now follows that $AT_1 \in \mathscr{C}_{2^{tn-2}p}$. A similar argument applied to $T_1 B^{tn-1} \in \mathscr{C}_p$ implies that $T_B \in \mathscr{C}_{2^{tn-2}p}$. Consider now

$$T_1 T_1^* T_1 = T_1(X^*A^* - B^*X^*)T_1 = T_1 X^*(A^*T_1) - (T_1 B^*)X^*T_1.$$

Since AT_1 and $T_1 B \in \mathscr{C}_{2^{tn-2}p}$, A^*T_1 and $T_1 B^* \in \mathscr{C}_{2^{tn-2}p}$. Hence $(T_1^*T_1)^2 \in \mathscr{C}_{2^{tn-2}p}$, and so $T_1^*T_1 \in \mathscr{C}_{2^{tn-1}p}$. This implies that $T_1 \in \mathscr{C}_{2^{tn}p}$, and the proof is complete.

Remark 5. If $m = 2$ in Theorem 5, then $n \equiv 1 \pmod{m}$, and it follows that $\delta_{AB}(X) \in \mathscr{C}_{2^n p}$; if $m = 3$, then $\delta_{AB}(X) \in \mathscr{C}_{2^n p}$ in the case in which $n \equiv 1 \pmod{m}$ and $\delta_{AB}(X) \in \mathscr{C}_{2^{2n}p}$ in the case in which $n \equiv 2 \pmod{m}$. The values $2^n p$ (or $2^{2n}p$) are not the best possible; this follows from the example of operators A, B, and X defined in (2).

As seen in the proof of Theorem 5, if T_m and $T_n \in \mathscr{C}_p$ for some $X \in B(H)$, then there exists a $t \in \mathscr{N}$ such that $A^{tn-1}T_1$ and $T_1B^{tn-1} \in \mathscr{C}_p$. This implies that if either A is left-invertible or B is right-invertible, then $T_1 \in \mathscr{C}_p$. Recall that the operator A is said to be left-Fredholm (right-Fredholm) if A is left-invertible (resp., right-invertible) in the Calkin algebra $B(H)\backslash\mathscr{K}$. Let A be a semi-Fredholm operator (i.e., A is either left- or right-Fredhom) with Fredholm index, $\operatorname{ind} A, \leq 0$. Then there exists a finite rank operator F such that $A+F$ is injective with $\operatorname{ind}(A+F) = \operatorname{ind} A$ (see [2; p. 366, Proposition 3.21]). Set $A+F=C$; then C is bounded below and hence is left-invertible. Suppose further that T_m and $T_n \in \mathscr{C}_p$ for some operator X. Then $\delta_{C^mB^m}(X)$ and $\delta_{C^nB^n}(X) \in \mathscr{C}_p$, and hence there exists a $t \in \mathscr{N}$ such that $C^{tn-1}\delta_{CB}(X) \in \mathscr{C}_p$. The operator C being left-invertible, $\delta_{CB}(X) \in \mathscr{C}_p$. Since $T_1 = \delta_{CB}(X) - FX$, we conclude that $T_1 \in \mathscr{C}_p$.

COROLLARY 7. *If T_m and $T_n \in \mathscr{C}_p$ for some $X \in B(H)$, and A (or B) is semi-Fredholm with* $\operatorname{ind} A \leq 0$ (*resp.*, $\operatorname{ind} B \geq 0$), *then* $T_1 \in \mathscr{C}_p$.

Proof. B^* is semi-Fredholm with $\operatorname{ind} B^* \leq 0$. Since $\delta_{AB}(X)B^{tn-1} \in \mathscr{C}_p$, the proof follows as above.

Remark 6. If $\operatorname{ind} A = 0$ in Corollary 7, then the operator C is invertible. Corollary 7 is proved in [6, Theorem 6] for the case in which $A = B$, $p = 2$, $m = 2$, and $n = 3$.

Let $A \in B(H)$ be such that $1 - A^*A \in \mathscr{C}_p$. Then $\pi(A)$ is an isometry V, and there exists an operator $K \in \mathscr{C}_p$ such that $A = V + K$ (see [7, p. 70]; see also [4, Theorem (6.2)]). Here we may take the isometry V to be a unitary in the case where $\sigma(A)$ does not contain the open unit disc.

THEOREM 8. *Let $A, B \in B(H)$ be such that either $1 - A^*A$ or $1 - B^*B \in \mathscr{C}_p$. If T_m and $T_n \in \mathscr{C}_p$ for some $X \in B(H)$, then $T_1 \in \mathscr{C}_p$.*

Proof. We consider the case in which $1 - A^*A \in \mathscr{C}_p$; the other case is similarly dealt with. Letting $A = V + K$, where V is an isometry and $K \in \mathscr{C}_p$, it follows from the hypotheses that both $\delta_{V^mB^m}$ and $\delta_{V^nB^n} \in \mathscr{C}_p$. Arguing as in the proof of Theorem 5, it follows that there exists a $t \in \mathscr{N}$ such that $V^{tn-1}\delta_{VB}(X) \in \mathscr{C}_p$. Since $\delta_{VB}(X) = V^{*tn-1}V^{tn-1}\delta_{VB}(X)$, $\delta_{VB}(X) \in \mathscr{C}_p$. Hence, since $\delta_{AB}(X) = (V+K)X - XB = (VX - XB) + KX$ and $KX \in \mathscr{C}_p$, $T_1 = \delta_{AB}(X) \in \mathscr{C}_p$.

The Elementary Operator $X \to AXB - X$

Given A and $B \in B(H)$, the elementary operator Δ_{AB}: $B(H) \to B(H)$ is defined by $\Delta_{AB}(X) = AXB - X$. We close this note with a remark about the analogues of Theorems 1 and 5 for the operator $\Delta_{AB}(X)$. It turns out that these analogues are trivial, and the results correspondingly uninteresting. It is easily seen that if $\Delta_{A^m B^m}(X) = 0$ for some $X \in B(H)$, then $\Delta_{A^{sm} B^{sm}}(X) = 0$ for all $s \in \mathscr{N}$. Also, if $\Delta_{A^m B^m}(X) = 0 = \Delta_{A^n B^n}(X)$ for some $X \in B(H)$, then $\Delta_{AB}(X) = 0$ and $A^{n-r}XB^{n-r} - A^r XB^r = 0$ for all $0 \le r \le n$ (without any additional hypotheses on A and B). Conversely, if $\Delta_{A^m B^m}(X) = 0$ for some $X \in B(H)$ and if $A^{n-r}XB^{n-r} - A^r XB^r = 0$ holds for (any) two consecutive values of r, $1 \le r < n$, then $\Delta_{AB}(X) = 0$ (once again, no additional hypotheses on A and B). This is seen as follows.

Choose an r, $1 \le r < n$, and suppose that $A^{n-r}XB^{n-r} - A^r XB^r = 0 = A^{n-r-1}XB^{n-r-1} - A^{r+1}XB^{r+1} = 0$. Then $A^{r+2}XB^{r+2} = A^r XB^r$. Let $r \equiv s \pmod{m}$; then there exists a $t \in \mathscr{N}$ such that $tr (\equiv ts) \equiv 1 \pmod{m}$, $A^{tr+2}XB^{tr+2} = A^{tr}XB^{tr}$, and so $A^3XB^3 = AXB$. Now if $m \equiv 0 \pmod 3$ or $m \equiv 2 \pmod 3$, then $X = A^m XB^m = AXB$. If, on the other hand, $m \equiv 1 \pmod 3$, then $X = A(A^{m-1}XB^{m-1})B = A^2XB^2$. This implies that if m is odd, then $X = A(A^{m-1}XB^{m-1})B = AXB$, and we are left with the case m is even to consider. Now if m is even, then $(m, n) = 1$ implies $(2, n) = 1$, $n \equiv 1 \pmod 2$, and either $r \equiv 0 \pmod 2$ and $n - r \equiv 1 \pmod 2$ or $r \equiv 1 \pmod 2$ and $n - r \equiv 0 \pmod 2$. In either case, $A^2XB^2 = X$ and $A^{n-r}XB^{n-r} = A^r XB^r$ together imply that $AXB - X = 0$.

Suppose now that $\Delta_{A^m B^m}(X)$ and $\Delta_{A^n B^n}(X) \in \mathscr{C}_p$ for some $X \in B(H)$. Then there exists a $t \in \mathscr{N}$ such that $tn \equiv 1 \pmod{m}$ and both $\Delta_{A^{tn-1}B^{tn-1}}(X)$ and $\Delta_{A^{tn}B^{tn}} \in \mathscr{C}_p$ (see the proof of Theorem 5). Hence

$$\Delta_{AB}(X) = \Delta_{A^{tn}B^{tn}}(X) - A(\Delta_{A^{tn-1}B^{tn-1}}(X))B \in \mathscr{C}_p$$

(with no additional hypotheses on A and B).

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