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Degenerate Bernoulli polynomials, generalized factorial sums, and their applications

Paul Thomas Young

Department of Mathematics, College of Charleston, Charleston, SC 29424, USA

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Abstract

We prove a general symmetric identity involving the degenerate Bernoulli polynomials and sums of generalized falling factorials, which unifies several known identities for Bernoulli and degenerate Bernoulli numbers and polynomials. We use this identity to describe some combinatorial relations between these polynomials and generalized factorial sums. As further applications we derive several identities, recurrences, and congruences involving the Bernoulli numbers, degenerate Bernoulli numbers, generalized factorial sums, Stirling numbers of the first kind, Bernoulli numbers of higher order, and Bernoulli numbers of the second kind.

Keywords: Bernoulli numbers; Degenerate Bernoulli numbers; Power sum polynomials; Generalized factorials; Stirling numbers of first kind; Bernoulli numbers of second kind

1. Introduction

Carlitz [3,4] defined the *degenerate Bernoulli polynomials* $\beta_m(\lambda, x)$ for $\lambda \neq 0$ by means of the generating function

$$\left(\frac{t}{(1+\lambda t)^{\mu}-1}\right)(1+\lambda t)^{\mu x}=\sum_{m=0}^{\infty}\beta_m(\lambda, x)\frac{t^m}{m!} \tag{1.1}$$

E-mail address: paul@math.cofc.edu.

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where $\lambda\mu = 1$. These are polynomials in λ and x with rational coefficients; we often write $\beta_m(\lambda)$ for $\beta_m(\lambda, 0)$, and refer to the polynomial $\beta_m(\lambda)$ as a *degenerate Bernoulli number*. The first few are $\beta_0(\lambda, x) = 1$, $\beta_1(\lambda, x) = x - \frac{1}{2} + \frac{1}{2}\lambda$, $\beta_2(\lambda, x) = x^2 - x + \frac{1}{6} - \frac{1}{6}\lambda^2$, $\beta_3(\lambda, x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x - \frac{3}{2}\lambda x^2 + \frac{3}{2}\lambda x + \frac{1}{4}\lambda^3 - \frac{1}{4}\lambda$. One combinatorial significance these polynomials have found is in expressing sums of generalized falling factorials $(i|\lambda)_m$; specifically, we have

$$\sum_{i=0}^{a-1}(i|\lambda)_m = \frac{1}{m+1}\bigl[\beta_{m+1}(\lambda, a) - \beta_{m+1}(\lambda)\bigr] \tag{1.2}$$

for all integers $a > 0$ and $m \geqslant 0$ (cf. [4, Eq. (5.4)]), where $(i|\lambda)_m = i(i-\lambda)(i-2\lambda)\cdots(i-(m-1)\lambda)$.

The (usual) Bernoulli polynomials $B_m(x)$ may be defined by the generating function

$$\left(\frac{t}{e^t - 1}\right)e^{xt} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!} \tag{1.3}$$

and their values at $x = 0$ are called the Bernoulli numbers and denoted B_m. Since $(1+\lambda t)^\mu \to e^t$ as $\lambda \to 0$ it is evident that $\beta_m(0, x) = B_m(x)$; letting $\lambda \to 0$ in (1.2) yields the familiar identity

$$\sum_{i=0}^{a-1} i^m = \frac{1}{m+1}\bigl[B_{m+1}(a) - B_{m+1}\bigr] \tag{1.4}$$

expressing power sums in terms of Bernoulli polynomials.

A major theme of the present paper is that the degenerate Bernoulli polynomials provide useful ways to study the Bernoulli numbers, their various other generalizations, and other important sequences. In Section 3 we prove the polynomial identity

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k-1}\beta_k(a\lambda, ax)\sigma_{n-k}(b\lambda, b-1)\\ &\quad= \sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^{k-1}\beta_k(b\lambda, bx)\sigma_{n-k}(a\lambda, a-1)\end{aligned} \tag{1.5}$$

for all positive integers a, b, and n, where

$$\sigma_m(\lambda, c) = \sum_{i=0}^{c}(i|\lambda)_m \tag{1.6}$$

is a generalized falling factorial sum. We will use (1.5) to demonstrate several other combinatorial connections between the polynomials $\beta_n(\lambda, x)$ and $\sigma_m(\lambda, c)$, and show how several known identities [3–6,10,11,13] are special cases of (1.5). In Sections 4 and 5 we derive several congruences and identities from (1.5), some of which extend and generalize results in [1–3,9–12]. These give indication of the vast arithmetic interplay between the various generalizations of Bernoulli numbers and other combinatorially important sequences; for example, in Section 4 we prove that if c and d are any two divisors of the odd integer n, then

$$\sum_{k=1}^{n+1} B_k s(n+1,k-1)\left(\frac{c^k-d^k}{k}\right)=0, \tag{1.7}$$

where $s(n,k)$ denotes the Stirling number of the first kind, defined in (2.2). A similar identity, with $c=2$ and $d=1$, was recently established in [12, Corollary 2]. We also show that if d is any divisor of the odd integer n, then

$$\sum_{k=1}^{n+2} s(n+1,k-1)\left(\frac{n^k-2B_k d^k}{k}\right)=0. \tag{1.8}$$

2. Notation and preliminaries

Throughout this paper p will denote a prime number, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Z}_p^\times$ the multiplicative group of units in $\mathbb{Z}_p$, and $\mathbb{Q}_p$ the field of p-adic numbers. For a rational number $x=r/s$ we have $x\in\mathbb{Z}_p$ if and only if p does not divide s, and $x\in\mathbb{Z}_p^\times$ if and only if p divides neither r nor s. The p-adic valuation ord_p is defined by setting $\operatorname{ord}_p(x)=k$ if $x=p^k y$ with $y\in\mathbb{Z}_p^\times$, and $\operatorname{ord}_p(0)=+\infty$. A congruence $x\equiv y \pmod{m\mathbb{Z}_p}$ is equivalent to $\operatorname{ord}_p(x-y)\geqslant \operatorname{ord}_p m$, and if x and y are rational numbers this congruence for all primes p is equivalent to the definition of congruence $x\equiv y \pmod m$ given in [11, §2]. The symbols λ and μ will generally represent elements of $\mathbb{Q}_p$ satisfying $\lambda\mu=1$, although $\lambda=0$ will also be allowed and λ will sometimes be regarded as an indeterminate.

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n=\prod_{j=0}^{n-1}(x-j\lambda) \tag{2.1}$$

for positive integers n, with the convention $(x|\lambda)_0=1$; we may also write

$$(x|\lambda)_n=\sum_{k=0}^{n} s(n,k)x^k\lambda^{n-k} \tag{2.2}$$

where the integers $s(n,k)$ are the *Stirling numbers of the first kind*. Note that $(x|\lambda)_n$ is a homogeneous polynomial in λ and x of degree n, so if $\lambda\neq 0$ then $(x|\lambda)_n=\lambda^n(\lambda^{-1}x|1)_n$. Clearly $(x|0)_n=x^n$. The generalized factorial sum $\sigma_m(\lambda,c)$ is defined for integers $c\geqslant 0$ by (1.6); note that $\sigma_0(\lambda,c)=c+1$. The identity (1.2) shows that $\sigma_m(\lambda,c)$ is a polynomial in λ and c; when $\lambda=0$ the sum $\sigma_m(0,c)$ is called a power sum polynomial. Our main identity in Section 3 will be derived from (1.1) and the generating function

$$\frac{(1+\lambda t)^{(c+1)\mu}-1}{(1+\lambda t)^{\mu}-1}=\sum_{m=0}^{\infty}\sigma_m(\lambda,c)\frac{t^m}{m!}; \tag{2.3}$$

this generating function follows from the definition (1.6) and the binomial expansion

$$(1+\lambda t)^{\mu y} = \sum_{n=0}^{\infty} (y|\lambda)_n \frac{t^n}{n!}. \tag{2.4}$$

For our identities we will rely on the following well-known property of exponential generating functions:

$$\left(\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!}\right) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} \quad \text{where } c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}. \tag{2.5}$$

For example, taking the product of the generating function (1.1) with the binomial expansion (2.4) yields the identity

$$\beta_n(\lambda, x+y) = \sum_{k=0}^{n} \binom{n}{k} \beta_k(\lambda, x)(y|\lambda)_{n-k} \tag{2.6}$$

for the degenerate Bernoulli polynomials (cf. [4, Eq. (5.12)]).

3. A symmetric identity for the degenerate Bernoulli polynomials

Most of the results of this paper are derived from the following symmetric identity.

Theorem 3.1. *For all positive integers a and b and all nonnegative integers n we have*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k-1} \beta_k(a\lambda, ax)\sigma_{n-k}(b\lambda, b-1) \\ &\quad = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k-1} \beta_k(b\lambda, bx)\sigma_{n-k}(a\lambda, a-1) \end{aligned}$$

as an identity in the polynomial ring $\mathbb{Q}[\lambda, x]$.

Proof. We consider the generating function

$$F(t) = \frac{t(1+\lambda t)^{ab\mu x}((1+\lambda t)^{ab\mu} - 1)}{((1+\lambda t)^{a\mu} - 1)((1+\lambda t)^{b\mu} - 1)}. \tag{3.1}$$

We first use (1.1) and (2.3) to expand $F(t)$ as

$$\begin{aligned} F(t) &= \frac{t(1+\lambda t)^{ab\mu x}}{((1+\lambda t)^{a\mu} - 1)} \cdot \frac{(1+\lambda t)^{ab\mu} - 1}{((1+\lambda t)^{b\mu} - 1)} \\ &= \left(a^{-1} \sum_{n=0}^{\infty} \beta_n\bigl(a^{-1}\lambda, bx\bigr) \frac{(at)^n}{n!}\right)\left(\sum_{n=0}^{\infty} \sigma_n\bigl(b^{-1}\lambda, a-1\bigr) \frac{(bt)^n}{n!}\right) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} \end{aligned} \tag{3.2}$$

where by (2.5) we have

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a^{k-1} b^{n-k} \beta_k\bigl(a^{-1}\lambda, bx\bigr)\sigma_{n-k}\bigl(b^{-1}\lambda, a-1\bigr). \tag{3.3}$$

We may also expand $F(t)$ as

$$\begin{aligned} F(t) &= \frac{t(1+\lambda t)^{ab\mu x}}{((1+\lambda t)^{b\mu}-1)} \cdot \frac{(1+\lambda t)^{ab\mu}-1}{((1+\lambda t)^{a\mu}-1)} \\ &= \left(b^{-1}\sum_{n=0}^{\infty} \beta_n\bigl(b^{-1}\lambda, ax\bigr)\frac{(bt)^n}{n!}\right)\left(\sum_{n=0}^{\infty} \sigma_n\bigl(a^{-1}\lambda, b-1\bigr)\frac{(at)^n}{n!}\right) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} \end{aligned} \tag{3.4}$$

where

$$c_n = \sum_{k=0}^{n} \binom{n}{k} b^{k-1} a^{n-k} \beta_k\bigl(b^{-1}\lambda, ax\bigr)\sigma_{n-k}\bigl(a^{-1}\lambda, b-1\bigr); \tag{3.5}$$

that is, since $F(t)$ is symmetric in a and b, so is c_n. Equating the expressions for c_n in (3.3) and (3.5) and replacing λ with $ab\lambda$ gives the identity of the theorem.

Putting $\lambda = x = 0$ in Theorem 3.1 gives the identity

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k-1} B_k \sigma_{n-k}(0, b-1) = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k-1} B_k \sigma_{n-k}(0, a-1) \tag{3.6}$$

which was proved by Tuenter [13]; the $b = 1$ case of (3.6) may be rearranged to give the recurrence

$$B_n = \frac{1}{a\bigl(1-a^n\bigr)} \sum_{k=0}^{n-1} \binom{n}{k} a^k B_k \sigma_{n-k}(0, a-1) \tag{3.7}$$

given in [5,6] and studied in [10].

Putting $b = 1$ in Theorem 3.1 and multiplying by a yields an identity

$$a\beta_n(a\lambda, ax) = \sum_{k=0}^{n} \binom{n}{k} a^k \beta_k(\lambda, x)\sigma_{n-k}(a\lambda, a-1) \tag{3.8}$$

which will be exploited in Section 5 to prove several congruences. This may be rewritten as

$$a\beta_n(a\lambda, ax) - a^{n+1}\beta_n(\lambda, x) = \sum_{k=0}^{n-1} \binom{n}{k} a^k \beta_k(\lambda, x)\sigma_{n-k}(a\lambda, a-1); \tag{3.9}$$

putting $x = 0$ in (3.9) gives Howard's recurrence [11, Theorem 4.1]. We remark that (3.8) may also be used to give another proof of the multiplication formula for degenerate Bernoulli polynomials [4, Eq. (5.5)]: We first observe that, since $(x|\lambda)_n$ is a homogeneous polynomial in λ and x of degree n, we have

$$a^{-m}\sigma_m(a\lambda, a-1) = \sum_{i=0}^{a-1} a^{-m}(i|a\lambda)_m = \sum_{i=0}^{a-1}\left(\frac{i}{a}|\lambda\right)_m. \tag{3.10}$$

Multiplying (3.8) by a^{-n} and applying (3.10) then yields

$$\begin{aligned} a^{1-n}\beta_n(a\lambda, ax) &= \sum_{k=0}^{n}\binom{n}{k}a^{k-n}\beta_k(\lambda, x)\sigma_{n-k}(a\lambda, a-1) \\ &= \sum_{k=0}^{n}\binom{n}{k}\beta_k(\lambda, x)\sum_{i=0}^{a-1}\left(\frac{i}{a}|\lambda\right)_{n-k} \\ &= \sum_{i=0}^{a-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k(\lambda, x)\left(\frac{i}{a}|\lambda\right)_{n-k} \\ &= \sum_{i=0}^{a-1}\beta_k\left(\lambda, x+\frac{i}{a}\right) \end{aligned} \tag{3.11}$$

via the identity (2.6) with $y = i/a$; putting $\lambda = 0$ in (3.11) gives the well-known multiplication theorem for the usual Bernoulli polynomials $B_n(x)$ (cf. [10, Eq. (3)]).

One may also observe by putting $x = 1$ in (1.1) that $\beta_n(\lambda, 1) = \beta_n(\lambda)$ for all $n \neq 1$, while $\beta_1(\lambda, 1) = \beta_1(\lambda) + 1$. Therefore putting both $x = 0$ and $x = 1$ in (3.8) yields

$$\beta_n(a\lambda, a) = \beta_n(a\lambda) + n\sigma_{n-1}(a\lambda, a-1); \tag{3.12}$$

replacing $a\lambda$ with λ and n with $m + 1$ gives the identity (1.2).

Since $\beta_n(\lambda)$ is an even (resp. odd) function when $n > 1$ is even (resp. odd) (cf. [4,11]), we have $\beta_0(-1) = 1$, $\beta_1(-1) = -1$, and $\beta_n(-1) = 0$ for $n > 1$. Thus putting $x = 0$ and $\lambda = -1$ in (3.8) yields the identity

$$a\beta_n(-a) = \sigma_n(-a, a-1) - an\sigma_{n-1}(-a, a-1). \tag{3.13}$$

The following special case of Theorem 3.1 illustrates that the values $\beta_n(\lambda)$ at rational arguments $\lambda = b/a$ are related to the connection coefficients for expressing certain factorial sums with increment b/a in terms of factorial sums with increment 1.

Corollary 3.2. *For all positive integers a and b and all nonnegative integers n we have*

$$\sigma_n(b/a, b-1) = \sum_{k=0}^{n}\binom{n}{k}(b/a)^{n+1-k}\beta_k(b/a)\sigma_{n-k}(1, a-1);$$

equivalently we may write this as

$$\sigma_n(b/a, b-1) = \sum_{k=0}^{n} \frac{n!}{k!}\binom{a}{n+1-k}(b/a)^{n+1-k}\beta_k(b/a).$$

Proof. From (1.1) we observe that $\beta_0(\lambda) = 1$ and $\beta_n(1) = 0$ for all $n > 0$. Putting $x = 0$ and $\lambda = 1/a$ in Theorem 3.1 gives the first result. Next we observe that

$$\sigma_m(1, a-1) = m!\sum_{j=m}^{a-1}\binom{j}{m} = m!\binom{a}{m+1} \tag{3.14}$$

from the familiar property $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$. Substituting (3.14) into the first expression gives the second statement.

If we put $a = 1$ and replace b with a in the above corollary we obtain the useful identity (cf. [3, Eq. (4.1)])

$$a\beta_n(a) = \sigma_n(a, a-1), \tag{3.15}$$

valid for integers $n \geqslant 0$ and $a > 0$; we give some applications of this identity in Section 4. It implies that $a\beta_n(a)$ is an integer for all nonnegative integers a and n, and $(-1)^{n+1}\beta_n(a) > 0$ for integers $a, n > 1$. It follows that, in the case where $b = ac$ with $c \in \mathbb{Z}$, all the coefficients $\binom{n}{k}c^{n+1-k}\beta_k(c)$ in the first form of the corollary are actually integers. Combining (3.15) with (3.13) also yields

$$(-1)^n\sigma_n(a, a-1) = \sigma_n(-a, a-1) - an\sigma_{n-1}(-a, a-1). \tag{3.16}$$

We conclude this section with a recurrence for the reciprocal polynomials of the degenerate Bernoulli numbers involving generalized factorial sums. Recall that for a polynomial $f(\lambda)$ of degree n, the *reciprocal polynomial* $\tilde{f}$ of f is the polynomial $\tilde{f}(\lambda) = \lambda^n f(1/\lambda)$.

Corollary 3.3. *For all positive integers a and n we have the recurrence*

$$0 = \sum_{k=0}^{n}\binom{n}{k}\tilde{\beta}_k(a)\sigma_{n-k}(1, a-1)$$

for the reciprocal polynomials $\tilde{\beta}_n$ of β_n, with $\tilde{\beta}_0(x) = 1$.

Proof. If we take $x = 0$ and $\lambda = 1/a$ in (3.8), we get, for $n > 0$,

$$a\beta_n(1) = 0 = \sum_{k=0}^{n}\binom{n}{k}a^k\beta_k(1/a)\sigma_{n-k}(1, a-1). \tag{3.17}$$

Since β_k is a polynomial of degree k, we have $\tilde{\beta}_k(a) = a^k\beta_k(1/a)$, and the statement is proved.

As in Corollary 3.2 we may use (3.14) to rewrite this recurrence as

$$0=\sum_{k=0}^{n}\binom{a}{n+1-k}\frac{\tilde{\beta}_k(a)}{k!}. \tag{3.18}$$

If we view the binomial coefficient $\binom{a}{n+1-k}=(a|1)_{n+1-k}/(n+1-k)!$ as a polynomial in a of degree $n+1-k$, then (3.18) says that $\sum_{k=0}^{n}\binom{a}{n+1-k}\tilde{\beta}_k(a)/k!$ is a polynomial in a which vanishes at all positive integers a, and therefore vanishes identically. Thus (3.18) may be viewed as holding for arbitrary a, although it becomes rather trivial for $a=0$. Putting $x=0$ and $\lambda=1/a$ in (1.1), and replacing t by at, yields

$$\frac{at}{(1+t)^a-1}=\sum_{m=0}^{\infty}\tilde{\beta}_m(a)\frac{t^m}{m!}. \tag{3.19}$$

Now the *Bernoulli numbers of the second kind* b_n may be defined by the generating function

$$\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}b_n t^n; \tag{3.20}$$

the numbers $n!b_n$ are also known as the *Cauchy numbers of the first type*. Since

$$\lim_{a\to 0}\frac{at}{(1+t)^a-1}=\frac{t}{\log(1+t)}, \tag{3.21}$$

we see that $\tilde{\beta}_n(0)=n!b_n$ for all n. Since

$$\lim_{a\to 0}\frac{\binom{a}{n+1-k}}{a}=\lim_{a\to 0}\frac{\binom{a-1}{n-k}}{n+1-k}=\frac{(-1)^{n-k}}{n+1-k}, \tag{3.22}$$

if we divide (3.18) by a and let $a\to 0$ we obtain the well-known recurrence

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}b_k}{n+1-k}=0 \tag{3.23}$$

for the Bernoulli numbers of the second kind (cf. [11, Eq. (2.2)]). Therefore Corollary 3.3 may be viewed as a degenerate generalization of the recurrence (3.23).

4. Congruences and identities for generalizations of Bernoulli numbers

In this section we derive some congruences for values of $\beta_n(\lambda)$ and the numbers b_n defined in (3.20). We also give several identities involving B_n, b_n, and $s(n,k)$. We begin with the following theorem concerning the factors of the integers $a\beta_n(a)$ for integers $a>1$.

Theorem 4.1. *Let n and a be positive integers, and let k,l be integers with $(k,an)=(l,an)=1$. If $d=(k^n-1,an)$, then the integer $\sigma_n(al,a-1)$ is divisible by an/d. Consequently the rational number $\beta_n(a)$ is an integer multiple of n/d.*

Proof. If n, a are positive integers and $(l, an) = 1$, then for any integer $i \in \{0, 1, \ldots, a-1\}$ the set of n factors of $(i|al)_n = i(i-al)(i-2al)\cdots(i-(n-1)al)$ comprises exactly the coset $i + (a)$ of the ideal (a) in the factor ring $R = \mathbb{Z}/an\mathbb{Z}$, and the set $\{0, 1, \ldots, a-1\}$ is a complete set of coset representatives. If $(k, an) = 1$, then k is a unit in R and therefore multiplication by k^n permutes both R and the cosets of (a) in R; that is, the factors of $k^n(i|al)_n = ki(ki-kal)(ki-2kal)\cdots(ki-(n-1)kal)$ comprise precisely a coset of (a) in R, and the set $\{k\cdot 0, k\cdot 1, \ldots, k(a-1)\}$ is a complete set of coset representatives. If we view the integer $S = \sigma_n(al, a-1) = \sum_{i=0}^{a-1}(i|al)_n$ as an element of R, then as a complete sum of products over cosets we have $k^nS = S$ in R; that is, $(k^n-1)S \equiv 0 \pmod{an}$ since multiplication by k^n merely permutes the terms in the sum for S modulo an. Since $(k^n-1)\sigma_n(al, a-1)$ is divisible by an it follows immediately that $\sigma_n(al, a-1)$ is divisible by an/d, where $d = (k^n-1, an)$. Taking $l = 1$ and using (3.15) gives the statement concerning $\beta_n(a)$.

This theorem gives information about both the numerator and denominator of the rational numbers $\beta_n(a)$. As one illustration of this theorem we give a new proof of a classical theorem of J.C. Adams [1] concerning the numerators of certain Bernoulli numbers. We will give a generalization of this result to degenerate Bernoulli polynomials in Section 5.

Corollary 4.2 *(Theorem of J.C. Adams, 1878). If $n > 0$ is an integer with p^r dividing n but $p-1$ does not divide n, then p^r divides the numerator of B_n.*

Proof. Take $a = p^s$ with $s > 0$ and write $n = mp^r$ with $(m, p) = 1$ and $p-1$ not dividing m. Since $p-1$ does not divide m we can choose k with $(k, p) = 1$ and $k^m \not\equiv 1 \pmod p$; this congruence depends only on the class of k modulo p. By the Chinese Remainder Theorem, k may be chosen so that $(k, an) = (k, m) = (k, p) = 1$ and $k^n = (k^m)^{p^r} \equiv k^m \not\equiv 1 \pmod p$. Since $k^n - 1$ is not divisible by p, the greatest common divisor $d = (k^n-1, mp^{r+s})$ must be a factor of m, say $m = dc$. Theorem 4.1 then implies that $a\beta_n(a)$ is divisible by cp^{r+s}. Since $a = p^s$ we see that $\beta_n(p^s)$ is an integer divisible by cp^r for all $s > 0$. Now we take the p-adic limit as $s \to \infty$; since $p^s \to 0$ and $\beta_n(p^s) \in p^r\mathbb{Z}_p$ for all $s > 0$, by the p-adic continuity of the polynomial $\beta_n(\lambda)$ we have $B_n = \beta_n(0) \in p^r\mathbb{Z}_p$ as well. Since B_n is a rational number in $p^r\mathbb{Z}_p$, p^r divides its numerator.

We may also derive the following important polynomial divisibility result from identity (3.15).

Theorem 4.3. *If $n > 1$ is odd then*

$$\beta_n\left(\frac{\pm 1}{n-2}\right) = 0.$$

Consequently $((n-2)^2\lambda^2 - 1)$ divides the polynomial $\beta_n(\lambda)$ in $\mathbb{Q}[\lambda]$ for all odd $n > 1$.

Proof. Let $n = 2m+1$ and suppose that $c \equiv 2m\lambda \pmod N$ for some integer N. Then by the definition (2.1) we have $(c-i|\lambda)_n \equiv (-1)^n(i|\lambda)_n = -(i|\lambda)_n \pmod N$ for all i, since their respective products contain the same factors (mod N) in reverse order. It follows from (1.6) that $\sigma_n(\lambda, c) \equiv -\sigma_n(\lambda, c) \pmod N$ and therefore $\sigma_n(\lambda, c) \equiv 0 \pmod N$ if N is odd, under the assumptions that $n = 2m+1$ and $c \equiv 2m\lambda \pmod N$.

Now suppose that p is any prime such that $p \equiv 1 \pmod{2m-1}$; it follows that $p^r \equiv 1 \pmod{2m-1}$ for all r, so there exists a sequence of positive integers $\{a_r\}$ such that $p^r - 1 =$

$(2m-1)a_r$ for all $r>0$. This implies that $a_r-1\equiv 2ma_r \pmod{p^r}$, so by the above result with $c=a_r-1$ and $\lambda=a_r$ we have $\sigma_n(a_r,a_r-1)\equiv 0 \pmod{p^r}$ for all r. This means that $\lim_{r\to\infty}\sigma_n(a_r,a_r-1)=0$ in $\mathbb{Z}_p$. But

$$\lim_{r\to\infty} a_r=\lim_{r\to\infty}\frac{p^r-1}{2m-1}=\frac{-1}{2m-1} \tag{4.1}$$

in $\mathbb{Z}_p$, so by the p-adic continuity of the polynomial $\beta_n(\lambda)$ we have

$$\beta_n\Big(\frac{-1}{2m-1}\Big)=\lim_{r\to\infty}\beta_n(a_r)=\lim_{r\to\infty}\frac{1}{a_r}\sigma_n(a_r,a_r-1)=0 \tag{4.2}$$

in $\mathbb{Z}_p$. Since $\beta_n(\lambda)$ is an odd function for odd $n>1$ [4,11] and it vanishes at $-1/(n-2)$ it must also vanish at $1/(n-2)$, completing the proof.

In [11] Howard gave an explicit formula for the coefficients of the polynomial $\beta_n(\lambda)$,

$$\beta_n(\lambda)=n!b_n\lambda^n+\sum_{j=1}^{[n/2]}\frac{n}{2j}B_{2j}s(n-1,2j-1)\lambda^{n-2j}, \tag{4.3}$$

where B_{2j} is the Bernoulli number defined in (1.3), $s(n,k)$ is the Stirling number of the first kind defined in (2.2), and b_n is the Bernoulli number of the second kind defined in (3.20). If we replace n with $n+2$ put $\lambda=1/n$ in (4.3), then applying Theorem 4.3 yields the identity

$$(n+1)!b_{n+2}=-\sum_{j=1}^{(n+1)/2}\frac{B_{2j}}{2j}s(n+1,2j-1)n^{2j} \tag{4.4}$$

which is valid for all odd positive integers n. This identity is quite useful for deriving various congruences for the numbers b_n.

Theorem 4.4. *For primes $p\geqslant 3$ we have $12b_{p^r+2}\equiv p^{2r} \pmod{p^{2r+1}\mathbb{Z}_p}$ for all positive integers r.*

Proof. For a polynomial $f(\lambda)=\sum_{i=0}^n a_i\lambda^i$ the *Newton polygon* of f at p is the upper convex hull of the set of points $\{(i,\operatorname{ord}_p a_i)\colon 0\leqslant i\leqslant n\}$. A basic property is that the Newton polygon of f at p has a side of slope m and horizontal run l if and only if f has l roots (counted with multiplicity) of p-adic ordinal $-m$ in an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We consider the polynomial

$$(1|\lambda)_{n+1}=1(1-\lambda)\cdots(1-n\lambda)=\sum_{k=1}^{n+1}s(n+1,k)\lambda^{n+1-k} \tag{4.5}$$

obtained by setting $x=1$ in (2.2). Clearly the roots are $\{1^{-1},2^{-1},\dots,n^{-1}\}$, so if $n=p^r$ then the Newton polygon of $(1|\lambda)_{n+1}$ at p has a side of slope r and horizontal run 1, and all other sides have slope at most $r-1$. The convexity of the Newton polygon implies that the largest slope occurs on the rightmost side, and it is easily seen that $s(n+1,1)=(-1)^n n!$. Therefore we have $\operatorname{ord}_p s(n+1,1)=\operatorname{ord}_p p^r!$, $\operatorname{ord}_p s(n+1,2)=\operatorname{ord}_p p^r!-r$, and $\operatorname{ord}_p s(n+1,k)\geqslant \operatorname{ord}_p p^r!-r-(k-2)(r-1)$ for all $k\geqslant 2$.

Multiplying (4.4) by 12 yields

$$12(n+1)!b_{n+2} = \sum_{j=1}^{(n+1)/2} \frac{-6B_{2j}}{j} s(n+1, 2j-1) n^{2j}. \tag{4.6}$$

Let c_j denote the jth term in the sum in (4.6) for $n = p^r$; then $c_1 = p^r!p^{2r}$ since $B_2 = 1/6$. We claim that $c_j \in p^r!p^{2r+1}\mathbb{Z}_p$ for all $j > 1$: We use the results of the preceding paragraph to compute for $j \geqslant 2$

$$\begin{aligned} \operatorname{ord}_p c_j &= \operatorname{ord}_p 6B_{2j} - \operatorname{ord}_p j + \operatorname{ord}_p s(n+1, 2j-1) + 2jr \\ &\geqslant \operatorname{ord}_p 6B_{2j} - \operatorname{ord}_p j + \operatorname{ord}_p p^r! - (2j-2)r + (2j-3) + 2jr \\ &= \operatorname{ord}_p p^r! + 2r + (2j-3) + \operatorname{ord}_p 6B_{2j} - \operatorname{ord}_p j. \end{aligned} \tag{4.7}$$

Observing that $\operatorname{ord}_p B_{2j} \geqslant -1$ for all j (by the well-known von Staudt–Clausen theorem, or Proposition 5.2 below), (4.7) shows that $\operatorname{ord}_p c_j \geqslant \operatorname{ord}_p p^r! + 2r + 1$ when $j \geqslant 3$, since $\operatorname{ord}_p j = 0$ for $j < p$ and $\operatorname{ord}_p j \leqslant \log_p j < j$ for all $j > 0$. Since $B_4 = -1/30$, (4.7) also shows $\operatorname{ord}_p c_2 \geqslant \operatorname{ord}_p p^r! + 2r + 1$ for all odd primes $p \neq 5$. To handle the $p = 5$ case we use the explicit formula

$$s(n+1, 3) = (-1)^n n! \sum_{1 \leqslant i < j \leqslant n} \frac{1}{ij} = (-1)^n \frac{n!}{2} \left(\left(\sum_{k=1}^{n} \frac{1}{k} \right)^2 - \sum_{k=1}^{n} \frac{1}{k^2} \right) \tag{4.8}$$

which follows from (4.5). For $n = p^r$ we observe that in (4.8) the difference of sums $S = (\sum_1^n 1/k)^2 - \sum_1^n (1/k)^2$ is congruent to $1/p^{2r} - 1/p^{2r} = 0$ modulo $p^{2-2r}\mathbb{Z}_p$, and therefore $\operatorname{ord}_p s(n+1, 3) \geqslant \operatorname{ord}_p p^r! - 2r + 2$. This proves that the inequality $\operatorname{ord}_p c_2 \geqslant \operatorname{ord}_p p^r! + 2r + 1$ is also valid for $p = 5$.

We have therefore shown that

$$12\big(p^r+1\big)!b_{p^r+2} \equiv p^r!p^{2r} \quad \big(\operatorname{mod} p^r!p^{2r+1}\mathbb{Z}_p\big) \tag{4.9}$$

for all odd primes p and positive integers r. Dividing (4.9) by $p^r!$ gives the statement of the theorem.

In [9] Howard observed that $B_n^{(n-1)} = -(n-1)n!b_n$, where $B_n^{(n-1)}$ is the *nth Bernoulli number of order* $n-1$, which can be defined by

$$\left(\frac{t}{e^t - 1}\right)^w = \sum_{n=0}^{\infty} B_n^{(w)} \frac{t^n}{n!}, \tag{4.10}$$

and he proved that if $n > 9$ is odd and composite, then the rational number $B_{n+2}^{(n+1)}$ has numerator divisible by n^4. An argument from (4.4) like that of the previous theorem can extend this divisibility result.

Theorem 4.5. *If $n > 15$ is odd and composite, then the rational number $B_{n+2}^{(n+1)}$ has numerator divisible by n^5.*

Proof. The case where $n = p^r > 15$ is odd and composite follows directly from (4.9), since $\operatorname{ord}_p p^r! \geqslant 3r$ when $p \geqslant 5$ and $r \geqslant 2$, and $\operatorname{ord}_3 3^r! \geqslant 3r + 1$ when $p = 3$ and $r \geqslant 3$. Therefore we now assume that n is of the form $n = mp^r$ with $p \geqslant 3$ prime, $m \geqslant 3$ odd, and $(m, p) = 1$, and we must show that $\operatorname{ord}_p (n+2)! b_{n+2} \geqslant 5r$. Let c_j denote the jth term in the sum (4.4) corresponding to such $n = mp^r$; we want to show that $\operatorname{ord}_p c_j \geqslant 5r$ for all j. We consider three cases:

For $j = 1$, we must show $\operatorname{ord}_p((mp^r)!/12) \geqslant 3r$. Observing that $\operatorname{ord}_p(mp^r)! \geqslant m \cdot \operatorname{ord}_p(p^r!) \geqslant mr$ accomplishes this, since $m \geqslant 3$ and $m \geqslant 5$ when $p = 3$.

For $j = 2$, we must show $\operatorname{ord}_p(s(mp^r + 1, 3)/120) \geqslant r$. Suppose that p^t is the highest power of p in the set $\{1, 2, \ldots, mp^r\}$; then $m \geqslant p^{t-r} + 2$ since m is odd and $(m, p) = 1$, so $\operatorname{ord}_p(mp^r!) \geqslant 2 + \operatorname{ord}_p(p^t!) \geqslant 3t$, with equality only if $m = 3, t = 1$ or $p = 3, m = 5, t = 2$. The case $p = 3, m = 5, t = 2$ is excluded since we assume $n > 15$. As in (4.5) we see that for $n = mp^r$ the Newton polygon of the polynomial $(1|\lambda)_{n+1}$ at p has maximum slope t, and therefore $\operatorname{ord}_p s(n + 1, 3) \geqslant \operatorname{ord}_p s(n + 1, 1) - 2t = \operatorname{ord}_p(mp^r!) - 2t \geqslant t \geqslant r$ with equality only if $m = 3, t = 1$. For $p = 3$ we have $m > p$ and thus $t > r$, so $\operatorname{ord}_3 s(n + 1, 3) \geqslant r + 1$; for $p = 5$ we likewise have $\operatorname{ord}_5 s(n + 1, 3) \geqslant r + 1$ in all cases except $m = 3$, $r = 1$. The required inequality is therefore proven for all odd composites greater than 15.

For $j \geqslant 3$, we make use of Howard's theorem [8, Theorem 2.1] that $s(n, k) \equiv 0 \pmod{\binom{n}{2}}$ if $n + k$ is odd. For our purposes it implies that $\operatorname{ord}_p s(mp^r + 1, 2j - 1) \geqslant r$ for all j. So we compute

$$\begin{aligned} \operatorname{ord}_p c_j &= \operatorname{ord}_p B_{2j} - \operatorname{ord}_p 2j + \operatorname{ord}_p s\big(mp^r + 1, 2j - 1\big) + 2jr \\ &\geqslant \operatorname{ord}_p B_{2j} - \operatorname{ord}_p j + (2j + 1)r \geqslant 5r \end{aligned} \tag{4.11}$$

for all $j \geqslant 3$, with equality if and only if $p = j = 3$ and $r = 1$.

We have shown that all terms in the sum (4.4) for $n = mp^r$ lie in $p^{5r}\mathbb{Z}_p$. Therefore p^{5r} divides the numerator of $B_{n+2}^{(n+1)}$ whenever p^r divides n, so the numerator of $B_{n+2}^{(n+1)}$ is divisible by n^5.

We remark that, from the perspective of identity (4.4), the odd composites 9 and 15 were excluded in this theorem due to the occurrence of the primes 3 and 5 in the denominators of B_2 and B_4. While it is true that for any given k, there are infinitely many odd composite integers n with the numerator of $B_{n+2}^{(n+1)}$ divisible by n^k, it seems that n^5 is the best possible modulus one can achieve in a result like Theorem 4.5 for odd composite n with only finitely many exceptions. Indeed for $n = 3p$ with $p > 5$ prime, the following theorem shows that $B_{n+2}^{(n+1)}$ has p-adic ordinal exactly 5.

Theorem 4.6. *For all primes $p > 3$, $40b_{3p^r+2} \equiv 3p^{2r} \pmod{p^{2r+1}\mathbb{Z}_p}$ for all positive integers r.*

Proof. The proof is similar to that of Theorem 4.4. Multiply both sides of (4.4) by 40, and let c_j denote the jth term in the resulting sum for $n = 3p^r$. We calculate $c_1 = 30p^{2r}(3p^r)!$ directly. By evaluating the difference of sums in (4.8) modulo $p^{2-2r}\mathbb{Z}_p$ we find that

$$\begin{aligned} \left(\sum_{k=1}^{3p^r} \frac{1}{k}\right)^2 - \sum_{k=1}^{3p^r} \frac{1}{k^2} &\equiv \left(\frac{1}{p^r} + \frac{1}{2p^r} + \frac{1}{3p^r}\right)^2 - \left(\frac{1}{p^{2r}} + \frac{1}{4p^{2r}} + \frac{1}{9p^{2r}}\right) \\ &= p^{-2r}\big((11/6)^2 - 49/36\big) = 2p^{-2r} \quad \big(\operatorname{mod} p^{2-2r}\mathbb{Z}_p\big), \end{aligned} \tag{4.12}$$

and therefore $c_2 \equiv -27p^{2r}(3p^r)!$ (mod $(3p^r)!p^{2r+2}\mathbb{Z}_p$). The Newton polygon of $(1|\lambda)_{n+1}$ at p has a side of slope r and horizontal run 3, and all other sides have slope at most $r-1$; thus $\operatorname{ord}_p s(3p^r+1,4) = \operatorname{ord}_p(3p^r!) - 3r$ and $\operatorname{ord}_p s(3p^r+1,k) \geqslant \operatorname{ord}_p(3p^r!) - 3r - (k-4)(r-1)$ for all $k \geqslant 4$. This shows that all c_j for $j \geqslant 3$ in the sum lie in $(3p^r)!p^{2r+1}\mathbb{Z}_p$, with the exception of c_3 when $p=7$ (since $B_6 = 1/42$).

To get the required congruence when $j=3$ and $p=7$ we must show that $\operatorname{ord}_p s(3p^r+1,5) \geqslant \operatorname{ord}_p(3p^r)! - 4r + 2$. Write the coefficient of λ^{n-4} in (4.5) as

$$s(n+1,5) = (-1)^n n! \sum_{1 \leqslant i<j<k<l \leqslant n} \frac{1}{ijkl} = (-1)^n n! S_n. \tag{4.13}$$

Since $s(n,k) \equiv 0 \pmod{\binom{n}{2}}$ if $n+k$ is odd [8, Theorem 2.1] we have $\operatorname{ord}_p s(3p+1,5) \geqslant 1$, giving the required result for $r=1$; via (4.13) it also implies that $\operatorname{ord}_p S_{3p} \geqslant -2$. Suppose $1 \leqslant i < j < k < l \leqslant 3p^r$. If not all of i, j, k, l are divisible by p^{r-1} then the term $1/(ijkl)$ of S_{3p^r} has p-adic ordinal at least $-4r+2$. Furthermore multiplication of i, j, k, l by p^{r-1} induces a bijection between {terms of S_{3p}} and {terms of S_{3p^r} in which all of i, j, k, l are divisible by p^{r-1}}. It follows that $\operatorname{ord}_p S_{3p^r} \geqslant -4(r-1) + \operatorname{ord}_p S_{3p} \geqslant -4r+2$ for all r; thus $\operatorname{ord}_p s(3p^r+1,5) \geqslant \operatorname{ord}_p(3p^r)! - 4r + 2$, so $c_3 \in (3p^r)!p^{2r+1}\mathbb{Z}_p$ when $p=7$.

In summary, we have shown that

$$40\big(3p^r+1\big)!b_{3p^r+2} \equiv 3p^{2r}\big(3p^r\big)! \quad \big(\operatorname{mod} \big(3p^r\big)!p^{2r+1}\mathbb{Z}_p\big) \tag{4.14}$$

for all primes $p > 3$; dividing (4.14) by $(3p^r)!$ gives the stated result.

The identity (4.4) should be compared to the identity

$$B_{n+2}^{(n+1)} = \binom{n+2}{2} \sum_{r=1}^{n+1} \frac{1}{r+1} s(n+1,r) n^{r+1} \tag{4.15}$$

[9, Eq. (4.2)], also valid for odd n, which Howard used to prove that the numerator of $B_{n+2}^{(n+1)}$ is divisible by n^4 for odd composite $n > 9$. Although the presence of Bernoulli numbers complicates the terms of (4.4) a bit relative to those of (4.15), it cuts the number of terms in half. If we observe that $B_{n+2}^{(n+1)} = -(n+1)(n+2)!b_{n+2}$ and equate the expressions in (4.4) and (4.15), we obtain

$$\sum_{r=1}^{n+1} \frac{1}{r+1} s(n+1,r) n^{r+1} = \sum_{j=1}^{(n+1)/2} \frac{B_{2j}}{j} s(n+1,2j-1) n^{2j}, \tag{4.16}$$

valid for all odd positive integers n.

We conclude this section with the following generalization of Theorem 4.3.

Theorem 4.7. *If n is odd, then for every divisor d of n,*

$$\beta_{n+2}(\pm 1/d) = 0.$$

Consequently $\prod_{d|n}(d^2\lambda^2 - 1)$ divides the polynomial $\beta_{n+2}(\lambda)$ in $\mathbb{Q}[\lambda]$ for all odd $n > 1$.

Proof. Suppose that $d = 2m + 1$ is a positive odd integer, and consider the polynomial $P(t) = ((1+t)^d - 1)/dt$. We see that it is a polynomial of degree $2m$ with rational coefficients and constant term 1, so we may write

$$P(t) = \prod_{i=1}^{2m} (1 - \alpha_i t) \tag{4.17}$$

where $\{\alpha_i\}_{i=1}^{2m}$ is the set of its reciprocal roots. If ζ is a fixed primitive dth root of 1, then we see that $P(\zeta^i - 1) = 0$ for $i = 1, 2, \ldots, 2m$; therefore the set of reciprocal roots is $\{\alpha_i\}_{i=1}^{2m} = \{(\zeta^i - 1)^{-1}\}_{i=1}^{2m}$. From (3.19), the generating function

$$\frac{1}{P(t)} = \frac{dt}{(1+t)^d - 1} = \sum_{n=0}^{\infty} \tilde{\beta}_n(d) \frac{t^n}{n!} \tag{4.18}$$

reveals that the sequence of rational numbers $\{\tilde{\beta}_n(d)/n!\}$ satisfies a linear recurrence of order $d - 1$.

We now consider the partial fraction decomposition

$$\frac{1}{P(t)} = \prod_{i=1}^{2m} \left(\frac{1}{1 - \alpha_i t} \right) = \sum_{i=1}^{2m} \frac{A_i}{1 - \alpha_i t} \tag{4.19}$$

where $A_i = \prod_{k \neq i} (1 - \alpha_k/\alpha_i)^{-1} = \alpha_i^{2m-1} \prod_{k \neq i} (\alpha_i - \alpha_k)^{-1} = \alpha_i^{2m-1} C_i$ for $1 \leqslant i \leqslant 2m$. Since $P(t)$ has all rational coefficients but all the reciprocal roots α_i are nonreal, the reciprocal roots must occur as a union of m complex conjugate pairs; in fact $\overline{\alpha_i} = \alpha_j$ when $i + j = d$, because $\overline{\zeta^i} = \zeta^j$. So when $i + j = d$ we see that $\overline{A}_i = A_j$ and also $\overline{C}_i = C_j$. Furthermore, since

$$\alpha_i + \overline{\alpha_i} = \frac{1}{\zeta^i - 1} + \frac{1}{\zeta^j - 1} = \frac{\zeta^i + \zeta^j - 2}{2 - \zeta^i - \zeta^j} = -1 \tag{4.20}$$

we find that each α_i has real part $-1/2$; therefore the product $C_i = \prod_{k \neq i} (\alpha_i - \alpha_k)^{-1}$ has zero real part, since each of its $2m - 1$ factors has zero real part. Since C_i is purely imaginary and $\overline{C}_i = C_j$, we have $C_i = -C_j$ when $i + j = d$. Finally, since $(\zeta^i - 1)(-\zeta^j) = \zeta^j - 1$, we note that $\alpha_i/\alpha_j = -\zeta^j$ is a $2d$th root of unity when $i + j = d$. It follows that $\alpha_i^n = \alpha_j^n$ whenever $i + j = d$ and n is a multiple of $2d$.

By pairing each α_i with its conjugate α_{d-i} and using the above observations, we may write the partial fraction decomposition as

$$\frac{1}{P(t)} = \sum_{i=1}^{m} C_i \left(\frac{\alpha_i^{d-2}}{1 - \alpha_i t} - \frac{\bar{\alpha}_i^{d-2}}{1 - \bar{\alpha}_i t} \right). \tag{4.21}$$

Expanding both sides as power series yields

$$\sum_{n=0}^{\infty} \tilde{\beta}_n(d) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{i=1}^{m} C_i \left(\alpha_i^{n+d-2} - \bar{\alpha}_i^{n+d-2} \right) t^n. \tag{4.22}$$

Since $\alpha_i^n = \bar{\alpha}_i^n$ whenever n is a multiple of $2d$, we see that $\tilde{\beta}_{n+2}(d) = 0$ whenever $(n+2) + d - 2 = 2kd$, that is, whenever $n = (2k-1)d$, which is precisely when the odd integer d is a divisor of the odd integer n. Since $\tilde{\beta}_{n+2}$ is an odd function we also have $\tilde{\beta}_{n+2}(-d) = 0$ under the same condition, completing the proof.

Remark. A referee has noted that a result similar to Theorem 4.7 but for $\beta_n(\lambda, 1-\lambda)$ is given in [7].

As in (4.4) we may substitute $\lambda = 1/d$ for any divisor d of an odd integer n into (4.3) and apply this theorem to get new identities of the form

$$(n+1)!b_{n+2} = -\sum_{j=1}^{(n+1)/2} \frac{B_{2j}}{2j} s(n+1, 2j-1)d^{2j}; \tag{4.23}$$

the identity (1.7) follows directly since $s(n+1, 0) = 0$ and $B_k = 0$ for odd $k > 1$, as we now record.

Corollary 4.8. *If c and d are any two divisors of the odd integer n, then*

$$\sum_{k=1}^{n+1} B_k s(n+1, k-1)\left(\frac{c^k - d^k}{k}\right) = 0.$$

Alternately, we may equate the expressions in (4.23) and (4.15) to give

$$\sum_{r=1}^{n+1} \frac{1}{r+1} s(n+1, r)n^{r+1} = \sum_{j=1}^{(n+1)/2} \frac{B_{2j}}{j} s(n+1, 2j-1)d^{2j}, \tag{4.24}$$

which we can rewrite by putting $k = r+1$ in the left sum and $k = 2j$ in the right, to give the identity (1.8).

Corollary 4.9. *If d is any divisor of the odd integer n, then*

$$\sum_{k=1}^{n+2} s(n+1, k-1)\left(\frac{n^k - 2B_k d^k}{k}\right) = 0.$$

5. Congruences for degenerate Bernoulli polynomials

Many of our results for the polynomials $\beta_n(\lambda, x)$ are generalizations of properties of the usual Bernoulli polynomials $B_n(x)$ which may be obtained by putting $\lambda = 0$. An exception is our first result in this section, which follows directly from the generating function (1.1) and has no analogue for the usual Bernoulli polynomials.

Proposition 5.1. *If $\lambda \in \mathbb{Z}_p^\times$ and $x \in \mathbb{Z}_p$ then $\beta_n(\lambda, x) \in n!\mathbb{Z}_p$ for all nonnegative integers n.*

Proof. If $\lambda \in \mathbb{Z}_p^\times$ then $((1+\lambda t)^\mu - 1)/t \in 1 + t\mathbb{Z}_p[\![t]\!]$ and therefore $((1+\lambda t)^\mu - 1)/t$ is a unit in the power series ring $\mathbb{Z}_p[\![t]\!]$. The generating function in (1.1) therefore lies in $\mathbb{Z}_p[\![t]\!]$, so $\beta_n(\lambda, x)/n!$ lies in $\mathbb{Z}_p$ for all n.

As polynomials in $\mathbb{Q}[\lambda, x]$ the coefficients of the degenerate Bernoulli polynomials do not in general have squarefree denominators; for example $\beta_3(\lambda) = \frac{1}{4}\lambda^3 - \frac{1}{4}\lambda$ has denominators divisible by 2^2, and other examples may be found in [11, §6]. However, as the next proposition shows, if λ, x are integers then $\beta_n(\lambda, x)$ is always a rational number with squarefree denominator.

Proposition 5.2. *For any prime p we have $p\beta_n(\lambda, x) \in \mathbb{Z}_p$ for all $\lambda, x \in \mathbb{Z}_p$ and all nonnegative integers n.*

Proof. We first consider the case where $x = 0$ and use induction on $\operatorname{ord}_p \lambda$. For the case $\operatorname{ord}_p \lambda = 0$ the proposition follows from Proposition 5.1. If we put $a = p$ and $x = 0$ in (3.8) we get

$$p\beta_n(p\lambda) = \sum_{k=0}^{n} \binom{n}{k} p^k \beta_k(\lambda)\sigma_{n-k}(p\lambda, p-1). \tag{5.1}$$

Assume that the proposition holds for $\operatorname{ord}_p \lambda = j$, and let $\lambda' = p\lambda$ have ordinal $j+1$, so $\operatorname{ord}_p \lambda = j$. By the induction hypothesis all terms in the sum on the right in (5.1) lie in $\mathbb{Z}_p$, so $p\beta_n(p\lambda) \in \mathbb{Z}_p$, proving the proposition for λ' with $\operatorname{ord}_p \lambda' = j+1$. By induction the proposition is true for all nonzero $\lambda \in \mathbb{Z}_p$ when $x = 0$; it follows for $\lambda = 0$ from p-adic continuity of the polynomial $\beta_n(\lambda)$. Then for any $x \in \mathbb{Z}_p$, putting $x = 0$ and replacing y with x in (2.6) yields

$$p\beta_n(\lambda, x) = \sum_{k=0}^{n} \binom{n}{k} p\beta_k(\lambda)(x|\lambda)_{n-k}; \tag{5.2}$$

all terms in the sum on the right lie in $\mathbb{Z}_p$, so $p\beta_n(\lambda, x) \in \mathbb{Z}_p$ as well, completing the proof.

We can now prove one of our primary congruence results between the degenerate Bernoulli polynomials and generalized factorial sums.

Theorem 5.3. *For $\lambda, x \in \mathbb{Z}_p$ and positive integers a and n, the congruence*

$$a\beta_n(a\lambda, ax) \equiv \sigma_n(a\lambda, a-1) \pmod{an\mathbb{Z}_p}$$

holds if $\lambda \in \mathbb{Z}_p^\times$; or if p is odd and p divides a; or if $p = 2$ and 4 divides a. Furthermore this congruence holds modulo $\frac{1}{2}an\mathbb{Z}_p$ if $p = 2$ and 2 divides a.

Proof. From Theorem 3.1 with $b = 1$ we have

$$a\beta_n(a\lambda, ax) = \sigma_n(a\lambda, a-1) + \sum_{k=1}^{n} \binom{n}{k} a^k \beta_k(\lambda, x)\sigma_{n-k}(a\lambda, a-1), \tag{5.3}$$

so we must show that all terms in the sum in (5.3) lie in $an\mathbb{Z}_p$. By Proposition 5.1 if $\lambda \in \mathbb{Z}_p^\times$ then $\beta_k(\lambda, x) \in k!\mathbb{Z}_p$, so $\binom{n}{k}\beta_k(\lambda, x) \in (n|1)_k\mathbb{Z}_p$, completing the proof in that case. Now suppose p divides a; we calculate

$$\begin{aligned} \operatorname{ord}_p \binom{n}{k} a^k &= \operatorname{ord}_p (n|1)_k + k \operatorname{ord}_p a - \operatorname{ord}_p k! \\ &= \operatorname{ord}_p (n|1)_k + k \operatorname{ord}_p a - \frac{k - S(k)}{p-1} \\ &= \operatorname{ord}_p (n|1)_k + \operatorname{ord}_p a + (k-1)\left(\operatorname{ord}_p a - \frac{1}{p-1}\right) + \frac{S(k)-1}{p-1}, \end{aligned} \tag{5.4}$$

where $S(k)$ denotes the sum of the digits in the base p expansion of the nonnegative integer k.

If $p|a$, (5.4) implies that $\operatorname{ord}_p \binom{n}{k} a^k \geqslant \operatorname{ord}_p n + \operatorname{ord}_p a$ for all $k > 0$, and $\operatorname{ord}_p \binom{n}{k} a^k \geqslant \operatorname{ord}_p n + \operatorname{ord}_p a + 1$ if $k > 1$ and $p > 2$. Since $\beta_1(\lambda, x) = x + \frac{\lambda-1}{2} \in \mathbb{Z}_p$ for $p > 2$, we find that all terms in the sum in (5.3) lie in $\frac{1}{2}an\mathbb{Z}_p$, proving the stated result except for the case where $p = 2$ and $4|a$.

Finally, if $p = 2$ and $4|a$, (5.4) implies that $\operatorname{ord}_p \binom{n}{k} a^k \geqslant \operatorname{ord}_p n + \operatorname{ord}_p a + 1$ if $k > 1$, so all terms in the sum in (5.3) with $k > 1$ lie in $an\mathbb{Z}_2$; thus

$$a\beta_n(a\lambda, ax) \equiv \sigma_n(a\lambda, a-1) + na\beta_1(\lambda, x)\sigma_{n-1}(a\lambda, a-1) \pmod{an\mathbb{Z}_2}. \tag{5.5}$$

In this case we observe that $\sigma_0(a\lambda, a-1) = a$ and

$$\sigma_{n-1}(a\lambda, a-1) = \sum_{i=0}^{a-1} (i|a\lambda)_{n-1} \equiv \sum_{i=0}^{a-1} i = \frac{a(a-1)}{2} \equiv 0 \pmod{2\mathbb{Z}_2} \tag{5.6}$$

if $n > 1$. Therefore since $\beta_1(\lambda, x) = x + \frac{\lambda-1}{2}$ we have $na\beta_1(\lambda, x)\sigma_{n-1}(a\lambda, a-1) \in an\mathbb{Z}_2$, completing the proof.

The next proposition will allow us to restate Theorem 5.3 in terms of power sum polynomials, which allows us to generalize and extend some important congruences for the usual Bernoulli numbers.

Proposition 5.4. *Suppose p divides a. Then the polynomial congruence*

$$(x|a\lambda)_n \equiv x^n \quad \left(\operatorname{mod} \frac{1}{2}an\lambda\mathbb{Z}_p[x]\right)$$

holds for all $\lambda \in \mathbb{Z}_p$ and all nonnegative integers n. Consequently we have

$$\sigma_n(a\lambda, c) \equiv \sigma_n(0, c) \quad \left(\operatorname{mod} \frac{1}{2}an\lambda\mathbb{Z}_p\right)$$

for all $\lambda \in \mathbb{Z}_p$ and all nonnegative integers n and c.

Proof. It is clear that

$$(x|a\lambda)_n = x(x - a\lambda)\cdots\bigl(x - (n-1)a\lambda\bigr) \equiv x^n \quad \bigl(\operatorname{mod} a\lambda\mathbb{Z}_p[x]\bigr) \tag{5.7}$$

for any $\lambda \in \mathbb{Z}_p$, which proves the theorem in the case where p does not divide n or where $p = 2$ and $\operatorname{ord}_2 n = 1$. Now suppose p divides n; write $n = mp^r$ with $r > 0$ and $(m, p) = 1$. In [14, Eq. (2.15)] we proved the polynomial identity

$$(x|\lambda)_{p^r} \equiv x^{p^{r-1}}\bigl(x^{p-1} - \lambda^{p-1}\bigr)^{p^{r-1}} \quad \left(\operatorname{mod} \frac{1}{2}p^r\lambda\mathbb{Z}[\lambda, x]\right). \tag{5.8}$$

Applying an evaluation homomorphism $\lambda \mapsto a\lambda$ with $\lambda \in \mathbb{Z}_p$ yields

$$(x|a\lambda)_{p^r} \equiv x^{p^{r-1}}\bigl(x^{p-1} - (a\lambda)^{p-1}\bigr)^{p^{r-1}} \quad \left(\operatorname{mod} \frac{1}{2}ap^r\lambda\mathbb{Z}_p[x]\right). \tag{5.9}$$

Since $x^{p-1} - (a\lambda)^{p-1} \equiv x^{p-1}$ (mod $\frac{1}{2}ap\lambda\mathbb{Z}_p[x]$), it follows by induction on r that

$$\bigl(x^{p-1} - (a\lambda)^{p-1}\bigr)^{p^{r-1}} \equiv x^{p^{r-1}(p-1)} \quad \left(\operatorname{mod} \frac{1}{2}ap^r\lambda\mathbb{Z}_p[x]\right), \tag{5.10}$$

so that

$$(x|a\lambda)_{p^r} \equiv x^{p^r} \quad \left(\operatorname{mod} \frac{1}{2}ap^r\lambda\mathbb{Z}_p[x]\right). \tag{5.11}$$

Then

$$\begin{aligned}(x|a\lambda)_{mp^r} &= \prod_{j=0}^{m-1}\bigl(x - jp^r a\lambda\big|a\lambda\bigr)_{p^r} \equiv \prod_{j=0}^{m-1}\bigl(x - jp^r a\lambda\bigr)^{p^r} \\ &\equiv x^{mp^r} \quad \left(\operatorname{mod} \frac{1}{2}ap^r\lambda\mathbb{Z}_p[x]\right),\end{aligned} \tag{5.12}$$

proving the proposition.

Combining Proposition 5.4 with Theorem 5.3 yields the following useful result:

Corollary 5.5. *If p divides a, then the congruence*

$$a\beta_n(a\lambda, ax) \equiv \sigma_n(0, a-1) \quad \left(\operatorname{mod} \frac{1}{2}an\mathbb{Z}_p\right)$$

holds for all $\lambda, x \in \mathbb{Z}_p$ and all nonnegative integers n. For $p = 2$, the above congruence also holds modulo $an\mathbb{Z}_p$ if $\lambda \in 2\mathbb{Z}_2$ and $4|a$.

Evaluating the power sum polynomial occurring in Corollary 5.5 gives a polynomial generalization of some well-known congruences for the usual Bernoulli numbers.

Corollary 5.6 *(Degenerate polynomial Carlitz and Adams theorems). Suppose $n > 2$ is an integer and p^r divides n for some integer $r \geqslant 0$. Then we have*

$$p\beta_n(\lambda, x) \equiv \begin{cases} 0, & \text{if } p-1 \text{ does not divide } n, \\ p-1, & \text{if } p-1 \text{ divides } n \end{cases} \quad \left(\operatorname{mod} p^{r+1}\mathbb{Z}_p\right)$$

for all $\lambda, x \in p\mathbb{Z}_p$ if p is odd; for $p = 2$, this congruence holds modulo $2^r\mathbb{Z}_2$ if $\lambda, x \in 4\mathbb{Z}_2$ and it holds modulo $2^{r+1}\mathbb{Z}_2$ if $r > 0$, $x \in 4\mathbb{Z}_2$ and $\lambda \in 8\mathbb{Z}_2$.

Proof. For any $t \in \mathbb{Z}_p$, the *Teichmüller representative* $\hat{t}$ of t is defined by the p-adic limit $\lim_{r\to\infty} t^{p^r}$; it satisfies $\hat{t}^p = \hat{t}$ and $\hat{t} \equiv t \pmod{p\mathbb{Z}_p}$. By induction on r we have $t^{p^r} \equiv \hat{t} \pmod{p^{r+1}\mathbb{Z}_p}$ for all positive integers r. We observe that the set $\{\hat{1}, \hat{2}, \ldots, \widehat{p-1}\}$ is precisely the set of $(p-1)$-st roots of unity in $\mathbb{Z}_p$. Therefore since

$$\sum_{\zeta^{p-1}=1} \zeta^m = \begin{cases} 0, & \text{if } p-1 \text{ does not divide } m, \\ p-1, & \text{if } p-1 \text{ divides } m, \end{cases} \tag{5.13}$$

for $n = mp^r$ we have

$$\begin{aligned} \sigma_n(0, p-1) &= 0^{mp^r} + 1^{mp^r} + 2^{mp^r} + \cdots + (p-1)^{mp^r} \\ &\equiv \hat{1}^m + \hat{2}^m + \cdots + \widehat{p-1}^m \\ &= \begin{cases} 0, & \text{if } p-1 \text{ does not divide } m, \\ p-1, & \text{if } p-1 \text{ divides } m \end{cases} \quad \left(\operatorname{mod} p^{r+1}\mathbb{Z}_p\right). \end{aligned} \tag{5.14}$$

Combining this with the $a = p$ case of Corollary 5.5 yields the stated congruence for odd p.

For $p = 2$, we use Corollary 5.5 with $a = 4$, yielding

$$4\beta_n(4\lambda, 4x) \equiv \sigma_n(0, 3) \pmod{2n\mathbb{Z}_2} \tag{5.15}$$

for $\lambda, x \in \mathbb{Z}_2$, and this holds modulo $4n\mathbb{Z}_2$ if $\lambda \in 2\mathbb{Z}_2$. If $n = 2^r m$ with m odd, then $2^n \equiv 0 \pmod{2^{r+2}}$ since we assume $n > 2$, and by induction on r we have $3^n \equiv 1 \pmod{2^{r+2}}$ for all positive integers r. It follows that $\sigma_n(0, 3) \equiv 2 \pmod{4n\mathbb{Z}_2}$; therefore from (5.15) we have

$$4\beta_n(4\lambda, 4x) \equiv 2 \quad \left(\operatorname{mod} 2^{r+1}\mathbb{Z}_2\right) \tag{5.16}$$

for $\lambda, x \in \mathbb{Z}_2$, and this holds modulo $2^{r+2}\mathbb{Z}_2$ if $\lambda \in 2\mathbb{Z}_2$. Dividing (5.16) by 2 yields the stated result for $p = 2$.

Putting $\lambda = x = 0$ in Corollary 5.6 gives the theorem of Carlitz [2] that if n is even and $(p-1)p^r$ divides n then p^r divides the numerator of $B_n + \frac{1}{p} - 1$, and also the theorem of J.C. Adams [1] that we restated in Corollary 4.2. Carlitz also proved a version of the above corollary for $p > 2$ and $x = 0$ in [3].

Our final result is a congruence for the divided degenerate Bernoulli polynomials $\beta_n(\lambda, x)/n$.

Theorem 5.7. *If $\lambda \in \mathbb{Z}_p^\times$, $x \in \mathbb{Z}_p$, and $p-1$ does not divide n then for all positive integers a we have*

$$\frac{\beta_{n+1}(a\lambda, ax)}{n+1} \equiv \frac{\sigma_{n+1}(a\lambda, a-1)}{a(n+1)} \pmod{an\mathbb{Z}_p}.$$

Proof. Replacing n with $n+1$ in (3.8) yields the identity

$$\beta_{n+1}(a\lambda, ax) = a^{-1}\sigma_{n+1}(a\lambda, a-1) + \sum_{k=1}^{n+1} \binom{n+1}{k} a^{k-1} \beta_k(\lambda, x)\sigma_{n+1-k}(a\lambda, a-1), \tag{5.17}$$

so we must show that all terms in the sum on the right lie in $a(n+1)n\mathbb{Z}_p$. In Theorem 4.1 we showed that for any integers k and l relatively prime to an we have $(k^n-1)\sigma_n(al, a-1) \equiv 0 \pmod{an}$. If $p-1$ does not divide n then k may be chosen so that $k^n \not\equiv 1 \pmod p$ and therefore $\sigma_n(al, a-1) \in an\mathbb{Z}_p$ for all integers $l \in \mathbb{Z}_p^\times$; by continuity $\sigma_n(a\lambda, a-1) \in an\mathbb{Z}_p$ for all $\lambda \in \mathbb{Z}_p^\times$ when $p-1$ does not divide n. Since $\beta_k(\lambda, x) \in k!\mathbb{Z}_p$ for $\lambda \in \mathbb{Z}_p^\times$, it follows that all terms in the sum on the right lie in $a(n+1)n\mathbb{Z}_p$ when $p-1$ does not divide n, proving the theorem.

In conclusion we observe that the degenerate Staudt–Clausen theorem, which was proved by Carlitz [3] and later by Howard [11], also follows directly from the results of this section. This theorem may be stated as follows:

(i) If $\lambda \in \mathbb{Z}_p^\times$, then for all integers $n \geqslant 0$ we have $\beta_n(\lambda) \in \mathbb{Z}_p$.
(ii) If p is odd and $\lambda \in \mathbb{Z}_p$, then for $n > 0$,

$$p\beta_n(p\lambda) \equiv \begin{cases} 0, & \text{if } p-1 \text{ does not divide } n \\ -1, & \text{if } p-1 \text{ divides } n \end{cases} \pmod{p\mathbb{Z}_p}. \tag{5.18}$$

(iii) If $p=2$ and $\lambda \in \mathbb{Z}_2$, then for $n > 0$,

$$2\beta_n(2\lambda) \equiv \begin{cases} 1, & \text{if } n \text{ is even} \\ \lambda, & \text{if } n \text{ is odd} \end{cases} \pmod{2\mathbb{Z}_2}. \tag{5.19}$$

It will be seen that Proposition 5.1 implies (i), and the case $a=p$, $x=0$ of Theorem 5.3 implies (ii) and the even n case of (iii). To get the odd n case of (iii), put $a=2$ and $x=0$ in (5.3) to get

$$2\beta_n(2\lambda) = \sigma_n(2\lambda, 1) + 2n\beta_1(\lambda)\sigma_{n-1}(2\lambda, 1) + \sum_{k=2}^{n} \binom{n}{k} 2^k \beta_k(\lambda)\sigma_{n-k}(2\lambda, 1). \tag{5.20}$$

Then observe that all terms in the sum in (5.20) lie in $2\mathbb{Z}_2$, that $\sigma_m(2\lambda, 1) \equiv 1 \pmod{2\mathbb{Z}_2}$ for $m > 0$, and $\beta_1(\lambda) = (\lambda-1)/2$ to get (5.19).

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