The spectrum of the Laplacian matrix of a balanced binary tree

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Abstract

Let $L(B_k)$ be the Laplacian matrix of an unweighted balanced binary tree $B_k$ of $k$ levels. We prove that spectrum of $L(B_k)$ is

$$\sigma \left( L(B_k) \right) = \bigcup_{j=1}^{k-1} \sigma(T_j) \cup \sigma(S_k),$$

where, for $1 \leq j \leq k - 1$, $T_j$ is the $j \times j$ principal submatrix of the tridiagonal $k \times k$ matrix $S_k$,

$$S_k = \begin{bmatrix}
1 & \sqrt{2} & 0 & \cdots & 0 \\
\sqrt{2} & 3 & \sqrt{2} & \ddots & \vdots \\
0 & \sqrt{2} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 3 & \sqrt{2} \\
0 & \cdots & 0 & \sqrt{2} & 2
\end{bmatrix}.$$  

We derive that the multiplicity of each eigenvalue of $T_j$, $1 \leq j \leq k - 1$, as an eigenvalue of $L(B_k)$, is at least $2^{k-j-1}$. Finally, for each $T_j$, using some results in [Electron. J. Linear Algebra 6 (2000) 62], we obtain lower and upper bounds for its smallest eigenvalue and an upper bound for its largest eigenvalue. In particular, we give upper bounds for the second largest eigenvalue and for the largest eigenvalue of $L(B_k)$. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let $G$ be a graph with vertices $1, 2, \ldots, n$. Let $d_i$ be the degree of the vertex $i$. Let $A(G)$ be the adjacency matrix of $G$ and let $D(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of $G$ is $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin’s theorem, it follows that its eigenvalues are nonnegative real numbers. Moreover, since its rows sum to 0, 0 is the smallest eigenvalue of $L(G)$. In [1], some of the many results known for Laplacian matrices are given. Fiedler [2] proved that $G$ is a connected graph if and only if the second smallest eigenvalue of $L(G)$ is positive. This eigenvalue is called the algebraic connectivity of $G$ and it is denoted by $\alpha(G)$. This concept has been studied by many authors. In [1, Section 3], some results concerning $\alpha(G)$ and some of its many applications are presented.

Denote by $B_k$ an unweighted balanced binary tree of $k$ levels and then $n = 2^k - 1$ vertices. As usual, we denote by $\sigma(A)$ the spectrum of an $n \times n$ matrix $A$. For $k = 2$, the spectrum of $L(B_2)$ is $\sigma(L(B_2)) = \{0, 1, 3\}$. Henceforth, we assume $k \geq 3$.

This paper is organized as follows. In Section 2, we label the vertices of $B_k$ in such a way that its Laplacian matrix $L(B_k)$ becomes a symmetric persymmetric matrix. In [3, Proposition 3.2(d)], it is proved that each eigenvalue of the tridiagonal $(k - 1) \times (k - 1)$ matrix

$$
\begin{bmatrix}
3 & -2 & 0 & \cdots & 0 \\
-1 & \ddots & -2 & \ddots & \vdots \\
0 & -1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 3 & -2 \\
0 & \cdots & 0 & -1 & 1
\end{bmatrix}
$$

(1)

is an eigenvalue of $L(B_k)$ and that its smallest eigenvalue is the algebraic connectivity of $B_k$. In this paper, we characterize completely the eigenvalues of $L(B_k)$. In fact, in Section 3, using the fact that $L(B_k)$ is a symmetric persymmetric matrix, we obtain that

$$
\sigma(L(B_k)) = \bigcup_{j=1}^{k-1} \sigma(T_j) \cup \sigma(S_k),
$$

where $T_j$ is the nonsingular tridiagonal $j \times j$ matrix given by
and $S_k$ is the singular tridiagonal $k \times k$ matrix given by

$$S_k = \begin{bmatrix}
1 & \sqrt{2} & 0 & \cdots & 0 \\
\sqrt{2} & 3 & \sqrt{2} & \cdots & 0 \\
0 & \sqrt{2} & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \sqrt{2} & 2
\end{bmatrix}.$$ 

Observe that $T_{k-1}$ is similar to the matrix in (1). In fact,

$$RD^{-1} = T_{k-1},$$

where $D = \text{diag}(1, -\sqrt{2}, (-\sqrt{2})^2, \ldots, (-\sqrt{2})^{k-2})$ and $R$ is the matrix of order $(k - 1) \times (k - 1)$ with ones along the secondary diagonal and zeros elsewhere. Moreover, we show that the multiplicity of each eigenvalue of $T_j$, $1 \leq j \leq k - 1$, as an eigenvalue of $L(B_k)$, is at least $2^{k-j-1}$. Finally, in Section 4, for each $T_j$, using some results in [3], we obtain lower and upper bounds for its smallest eigenvalue, and an upper bound for its largest eigenvalue. In particular, we give upper bounds for the second largest eigenvalue and for the largest eigenvalue of $L(B_k)$.

2. $L(B_k)$ as a symmetric persymmetric matrix

In order to illustrate our labeling for the vertices of $B_k$, let us take $k = 4$. Our labeling for $B_4$ is shown below.
We see that 8 is the root vertex and the vertices on the $i$-level, $i = 2, 3, 4$, from left to right are

\begin{align*}
&i = 2: \quad 7 \\
&i = 3: \quad 5 \quad 6 \quad 10 \quad 11 \\
&i = 4: \quad 1 \quad 2 \quad 3 \quad 4 \quad 12 \quad 13 \quad 14 \quad 15
\end{align*}

We observe that in each level symmetrical vertices, with respect to a vertical separating the two branches of the tree, sum to $2^i$. This labeling for the vertices of $B_4$ yields to

\[ L(B_4) = \begin{bmatrix} U_{22} & b_{22} & 0 \\ b_{22}^T & 2 & b_{22}^T R \\ 0 & Rb_{22} & R U_{22} R \end{bmatrix}, \quad (2) \]

where $0$ is the zero matrix of order $7 \times 7$,

\[ U_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 3 \end{bmatrix}, \quad (3) \]

\[ b_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}^T \quad (4) \]

and

\[ R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]
The matrix $R$ reverses the order of the rows (columns) of any matrix under premultiplication (postmultiplication) and it is called the reversal matrix. In particular, $R^2 = I$.

**Definition 1.** A square matrix $A$ is said to be a persymmetric matrix if and only if $RAR = A^T$.

We have

$$RL(\mathcal{B}_4)R = \begin{bmatrix} 0 & 0 & Rb_{22} & RU_{22}R \\ 0 & b_{22} & 2 & b_{22}^TR \\ R & 0 & 0 & 0 \\ 0 & Ru_{22} & 0 & Rb_{22} \end{bmatrix} = L(\mathcal{B}_4).$$

Therefore, $L(\mathcal{B}_4)$ is a symmetric persymmetric matrix. We can see that $U_{22}$ is a block tridiagonal matrix,

$$U_{22} = \begin{bmatrix} I_{22} & -C_{22} & 0 \\ -C_{22}^T & 3I_2 & -C_2 \\ 0^T & -C_2^T & 3 \end{bmatrix},$$

where $I_{22}$ and $I_2$ denote the identity matrices of order $2^2 \times 2^2$ and $2 \times 2$, respectively, and

$$C_{22} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In this point, it is convenient to introduce the following notation: If $l$ is a power of 2, $I_l$ denotes the identity matrix of order $l \times l$ and $C_l$ is the 0–1 matrix of order $l \times l/2$ given by
In general, we label the vertices of $B_k$ so that $2^{k-1}$ is the root vertex and the vertices on the $i$th level for $i = 2, 3, \ldots, k$, from left to right, are

$$2^{k-1} - 2^{i-1} + 1, 2^{k-1} - 2^{i-1} + 2, 2^{k-1} - 2^{i-1} + 3, \ldots, 2^{k-1} - 2^{i-1} + 2^{i-2}$$

for the vertices on the left branch and

$$2^{k-1} + 2^{i-1} - 2^{i-2}, 2^{k-1} + 2^{i-1} - 2^{i-2} + 1, \ldots, 2^{k-1} + 2^{i-1} - 2, 2^{k-1} + 2^{i-1} - 1$$

for the vertices on the right branch. Thus, in each level symmetrical vertices, with respect to a vertical separating the two branches of the tree, sum to $2^k$.

It is not difficult to note that

$$L(B_k) = \begin{bmatrix} U_{2^{k-2}} & b_{2^{k-2}}^T & 0 \\ b_{2^{k-2}}^T & 2 & b_{2^{k-2}}^T R \\ 0 & Rb_{2^{k-2}} & RU_{2^{k-2}} R \end{bmatrix} ,$$

where

$$U_{2^{k-2}} = \begin{bmatrix} I_{2^{k-2}} & -C_{2^{k-2}} & 0 & \cdots & \cdots & \cdots & 0 \\ -C_{2^{k-2}}^T & 3I_{2^{k-3}} & -C_{2^{k-3}} & \ddots & \ddots & \ddots & \vdots \\ 0 & -C_{2^{k-3}}^T & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 3I_2 \\ 0 & \cdots & \cdots & 0 & -C_2^T & 3 \end{bmatrix}$$

of order $(2^{k-1} - 1) \times (2^{k-1} - 1)$,
\[ b_{2k-2} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & -1 \end{bmatrix}^T \]  
\[ (7) \]
of order \((2^{k-1}-1) \times 1\) and \(R\) is the reversal matrix of order \((2^{k-1}-1) \times (2^{k-1}-1)\).  
As in the case \(k = 4\), one can easily prove that \(RL(B_k)R = L(B_k)\). Therefore, \(L(B_k)\) is a symmetric persymmetric matrix.

3. The eigenvalues of \(L(B_k)\)

If all the eigenvalues of an \(n \times n\) matrix \(A\) are real numbers, we assume that  
\[ \lambda_n(A) \leq \lambda_{n-1}(A) \leq \cdots \leq \lambda_2(A) \leq \lambda_1(A). \]
We recall a basic fact on symmetric persymmetric matrices of order \((2p+1) \times (2p+1)\) [4].

Lemma 1. Let \(A\) be a complex symmetric persymmetric \((2p+1) \times (2p+1)\) matrix. Then \(A\) has the form
\[ A = \begin{bmatrix} U & b & VR \\ b^T & a & b^T R \\ RV & Rb & RUR \end{bmatrix}, \]  
\[ (8) \]
where \(U\) and \(V\) are complex symmetric matrices of order \(p \times p\), \(b\) is a \(p\)-dimensional complex column vector and \(a = a_{p+1,p+1}\) is a complex scalar. Moreover, if
\[ F = \begin{bmatrix} U + V & \sqrt{2}b \\ \sqrt{2}b^T & a \end{bmatrix}, \]  
then
\[ \sigma(A) = \sigma(F) \cup \sigma(U - V). \]  
\[ (9) \]

Theorem 2. For the spectrum of \(L(B_k)\), we have
\[ \sigma(L(B_k)) = \sigma(F_{2k-2}) \cup \sigma(U_{2k-2}), \]  
\[ (10) \]
where
\[ F_{2k-2} = \begin{bmatrix} U_{2k-2} & c_{2k-2} \\ c_{2k-2}^T & 2 \end{bmatrix}, \]  
\[ (11) \]
and
\[ c_{2k-2} = \sqrt{2}b_{2k-2} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & -\sqrt{2} \end{bmatrix}^T. \]  
\[ (12) \]
Moreover, the smallest eigenvalue and the largest eigenvalue of \(U_{2k-2}\) are, respectively, the algebraic connectivity of \(B_k\) and the second largest eigenvalue of \(L(B_k)\). Also, the largest eigenvalue of \(F_{2k-2}\) is the largest eigenvalue of \(L(B_k)\).
Proof. We know that \( L(B_k) \) is a symmetric persymmetric matrix of order \((2^k - 1) \times (2^k - 1)\). From (5) and (8), we see that \( U = U_{2^k - 2}, V = 0 \) and \( b = b_{2^k - 2} \). Thus, \( F = F_{2^k - 2} \) is given by (11) with \( c_{2^k - 2} \) as in (12). Now, it is clear that (10) is an immediate consequence of (9). The rest of the proof follows from (10) together with the fact that the eigenvalues of \( U_{2^k - 2} \) interlace the eigenvalues of \( F_{2^k - 2} \). □

Now, we search for the eigenvalues of \( U_{2^k - 2} \) and \( F_{2^k - 2} \).

Lemma 3. Let \( a, b \) and \( c \) be real numbers. Let

\[
\beta_1 = a,
\]

\[
\beta_j = b - \frac{2}{\beta_{j-1}}, \quad j = 2, 3, \ldots, k - 1, \quad \beta_{k-1} \neq 0,
\]

and

\[
\beta_k = c - \frac{2}{\beta_{k-1}}.
\]

Let

\[
M = 
\begin{bmatrix}
 aI_{2^k-2} & C_{2^k-2} & 0 & \cdots & \cdots & \cdots & 0 \\
 C_{2^k-2}^T & bI_{2^k-3} & C_{2^k-3} & \cdots & \cdots & \cdots & \cdots \\
 0 & C_{2^k-3}^T & bI_{2^k-4} & \cdots & \cdots & \cdots & \cdots \\
 \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \\
 \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \\
 \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & bI_2 \\
 0 & \cdots & \cdots & \cdots & \cdots & \cdots & C_2^T \\
 0 & \cdots & \cdots & \cdots & \cdots & \cdots & b \\
 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \sqrt{2} \\
 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 2
\end{bmatrix}
\]

and

\[
N = 
\begin{bmatrix}
 aI_{2^k-2} & C_{2^k-2} & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
 C_{2^k-2}^T & bI_{2^k-3} & C_{2^k-3} & \cdots & \cdots & \cdots & \cdots & \cdots \\
 0 & C_{2^k-3}^T & bI_{2^k-4} & \cdots & \cdots & \cdots & \cdots & \cdots \\
 \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots \\
 \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & bI_2 & C_2 \\
 0 & \cdots & \cdots & \cdots & \cdots & \cdots & C_2^T & b & \sqrt{2} \\
 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \sqrt{2} & 2
\end{bmatrix}
\]
Then:
(a) If $\beta_j \neq 0$ for $j = 1, 2, \ldots, k - 2$,
$$\det M = \beta_1^{2k-2} \beta_2^{2k-3} \cdots \beta_{k-3}^2 \beta_{k-2} \beta_{k-1}. \quad (13)$$
(b) $\det M \neq 0$ if and only if $\beta_j \neq 0$ for $j = 1, 2, \ldots, k - 1$.
(c) If $\beta_j \neq 0$ for $j = 1, 2, \ldots, k - 1$,
$$\det N = \beta_1^{2k-2} \beta_2^{2k-3} \cdots \beta_{k-3}^2 \beta_{k-2} \beta_k. \quad (14)$$
(d) If $\beta_j \neq 0$ for $j = 1, 2, \ldots, k$ then $\det N \neq 0$. If $\beta_j = 0$ for some $j$, $1 \leq j \leq k - 2$, then $\det N = 0$. Also, if $\beta_{k-1} \neq 0$ and $\beta_k = 0$ then $\det N = 0$.

Proof. (a) Suppose $\beta_j \neq 0$ for $j = 1, 2, \ldots, k - 2$. The Gaussian elimination procedure, without row interchanges, applied to $M$ yields to the upper triangular matrix

$$\begin{bmatrix}
\beta_1 I_{2k-2} & C_{2k-2} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \beta_2 I_{2k-3} & C_{2k-3} & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & \beta_3 I_{2k-4} & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \beta_{k-2} I_2 & C_2 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & \beta_{k-1}
\end{bmatrix}. \quad (15)$$

Therefore, $\det M$ is given by (13) if $\beta_j \neq 0$ for $j = 1, 2, \ldots, k - 2$.

(b) From (13), we have that if $\beta_j \neq 0$ for $j = 1, 2, \ldots, k - 1$ then $\det M \neq 0$. Conversely, if $\beta_j = 0$ for some $j$, $1 \leq j \leq k - 2$, then the corresponding intermediate matrix in the Gaussian elimination procedure has two rows which are equal or if $\beta_{k-1} = 0$ then the last row in the upper triangular matrix in (15) is a row of zeros. In both cases, $\det M = 0$.

(c) Suppose $\beta_j \neq 0$ for $j = 1, 2, \ldots, k - 1$. We apply the Gaussian elimination procedure, without row interchanges, to reduce the matrix $N$ to an upper triangular matrix. Just before the last step, we have the matrix

$$\begin{bmatrix}
\beta_1 I_{2k-2} & C_{2k-2} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \beta_2 I_{2k-3} & C_{2k-3} & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & \beta_3 I_{2k-4} & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \beta_{k-2} I_2 & C_2 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & \beta_{k-1} \sqrt{2} \\
0 & \cdots & \cdots & \cdots & \cdots & \sqrt{2} & c
\end{bmatrix}.$$
Finally, the Gaussian elimination gives
\[
\begin{bmatrix}
\beta_1 I_{2^k-2} & C_{2^k-2} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \beta_2 I_{2^k-3} & C_{2^k-3} & \ddots & \ddots & \cdots & \cdots \\
0 & 0 & \beta_3 I_{2^k-4} & \ddots & \ddots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\
0 & \cdots & \cdots & 0 & \beta_{k-1} & \sqrt{2} & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & 0 & 0 & c - \frac{2}{\beta_{k-1}} & \cdots \\
\end{bmatrix}
\]
(16)

Thus, (14) is proved.

(d) From (14), we see that if \( \beta_j \neq 0 \) for \( j = 1, 2, \ldots, k \) then \( \det N \neq 0 \). If \( \beta_j = 0 \) for some \( j \), \( 1 \leq j \leq k-2 \), the corresponding intermediate matrix in the Gaussian elimination procedure has two rows which are equal or if \( \beta_k = 0 \), \( \beta_{k-1} \neq 0 \), all the entries of the last row in the upper triangular matrix in (16) are zeros. In both cases, \( \det N = 0 \).

Theorem 4. Let
\[
P_0(\lambda) = 1, \quad P_1(\lambda) = \lambda - 1,
\]
(17)
and
\[
P_j(\lambda) = (\lambda - 3)P_{j-1}(\lambda) - 2P_{j-2}(\lambda) \quad \text{for } j = 2, 3, \ldots, k-1
\]
(18)
\[
S_k(\lambda) = (\lambda - 2)P_{k-1}(\lambda) - 2P_{k-2}(\lambda).
\]
(19)

Hence:

(a) If \( \lambda \in \mathbb{R} \) is such that \( P_j(\lambda) \neq 0 \) for \( j = 1, 2, \ldots, k-2 \), then
\[
\det (\lambda I - U_{2^k-2}) = P_1^{2^k-3}(\lambda)P_2^{2^k-4}(\lambda) \cdots P_{k-3}^{2^k-4}(\lambda)P_{k-2}(\lambda)P_{k-1}(\lambda).
\]
(20)

(b) \( \det(\lambda I - U_{2^k-2}) \neq 0 \) if and only if \( \lambda \in \mathbb{R} \) is such that \( P_j(\lambda) \neq 0 \) for \( j = 1, 2, \ldots, k-1 \).

(c) If \( \lambda \in \mathbb{R} \) is such that \( P_j(\lambda) \neq 0 \) for \( j = 1, 2, \ldots, k-1 \), then
\[
\det (\lambda I - F_{2^k-2}) = P_1^{2^k-3}(\lambda)P_2^{2^k-4}(\lambda) \cdots P_{k-3}^{2^k-4}(\lambda)P_{k-2}(\lambda)S_k(\lambda).
\]
(21)

(d) If \( \lambda \in \mathbb{R} \) is such that \( P_j(\lambda) \neq 0 \) for \( j = 1, 2, \ldots, k-1 \) and \( S_k(\lambda) \neq 0 \), then \( \det(\lambda I - F_{2^k-2}) \neq 0 \). If \( \lambda \in \mathbb{R} \) is such that \( P_j(\lambda) = 0 \) for some \( j \), \( 1 \leq j \leq k-2 \), or if \( \lambda \in \mathbb{R} \) is such that \( S_k(\lambda) = 0 \), then \( \det(\lambda I - F_{2^k-2}) = 0 \).

Proof. (a) Let \( \lambda \in \mathbb{R} \). We apply Lemma 3, part (a), with \( a = \lambda - 1 \) and \( b = \lambda - 3 \). Then from (6), we have that the matrix \( M \) of Lemma 3 is \( M = \lambda I - U_{2^k-2} \). Suppose that \( P_j(\lambda) \neq 0 \) for \( j = 1, 2, \ldots, k-2 \). Then
\[ \beta_1 = \lambda - 1 = \frac{P_1(\lambda)}{P_0(\lambda)} \neq 0 \]
and
\[ \beta_j = (\lambda - 3) - \frac{2P_{j-2}(\lambda)}{P_{j-1}(\lambda)} = \frac{P_j(\lambda)}{P_{j-1}(\lambda)} \neq 0. \]

From (13),
\[
\det (\lambda I - U_{2^k-2}) = P_{2^k-2}^1(\lambda) P_{2^k-4}^2(\lambda) P_{2^k-6}^3(\lambda) \cdots P_{k-3}^{2^2}(\lambda) P_{k-4}^{2^4}(\lambda) P_{k-5}^{2^6}(\lambda) \cdots P_{k-1}^{2^k}(\lambda) \]
\[ = P_{2^k-2}^1(\lambda) P_{2^k-4}^2(\lambda) \cdots P_{k-3}^{2^2}(\lambda) P_{k-2}^{2^4}(\lambda) \cdots P_{k-1}^{2^2}(\lambda). \]

Thus, (20) is proved.

(b) We apply part (b) of Lemma 3. Suppose that \( \lambda \in \mathbb{R} \) is such that \( P_j(\lambda) \neq 0 \) for \( j = 1, 2, \ldots, k - 1 \). Then \( \beta_j = P_j(\lambda)/P_{j-1}(\lambda) \neq 0 \) for \( j = 1, 2, \ldots, k - 1 \). It follows that \( \det(\lambda I - U_{2^k-2}) \neq 0 \). Conversely, suppose that \( \det(\lambda I - U_{2^k-2}) \neq 0 \) and that \( \lambda \in \mathbb{R} \) is such that \( P_j(\lambda) = 0 \) for some \( j, 1 \leq j \leq k - 1 \). Since \( P_0(\lambda) = 1 \neq 0 \), we may assume \( P_{j-1}(\lambda) \neq 0 \) and \( P_j(\lambda) = 0 \). Then \( \beta_j = P_j(\lambda)/P_{j-1}(\lambda) = 0 \) and thus \( \det(\lambda I - U_{2^k-2}) = 0 \), which is a contradiction.

(c) Let \( \lambda \in \mathbb{R} \). We apply part (c) of Lemma 3 with \( a = \lambda - 1 \), \( b = \lambda - 3 \) and \( c = \lambda - 2 \). Then, from (11) and (12), we have that the matrix \( N \) of Lemma 3 is \( N = \lambda I - F_{2^k-2} \). Suppose that \( P_j(\lambda) \neq 0 \) for \( j = 1, 2, \ldots, k - 1 \). Then, as in part (a), \( \beta_j = P_j(\lambda)/P_{j-1}(\lambda) \neq 0 \) for \( j = 1, 2, \ldots, k - 1 \). Moreover,
\[ \beta_k = (\lambda - 2) - \frac{2}{\beta_{k-1}} = (\lambda - 2) - \frac{2P_{k-2}(\lambda)}{P_{k-1}(\lambda)} = \frac{S_k(\lambda)}{P_{k-1}(\lambda)}. \]

From (14),
\[
\det (\lambda I - F_{2^k-2}) = P_{2^k-2}^1(\lambda) P_{2^k-4}^2(\lambda) P_{2^k-6}^3(\lambda) \cdots P_{k-3}^{2^2}(\lambda) P_{k-4}^{2^4}(\lambda) P_{k-5}^{2^6}(\lambda) \cdots P_{k-1}^{2^k}(\lambda) S_k(\lambda)
\]
\[ = P_{2^k-2}^1(\lambda) P_{2^k-4}^2(\lambda) \cdots P_{k-3}^{2^2}(\lambda) P_{k-2}^{2^4}(\lambda) P_{k-1}^{2^2}(\lambda) S_k(\lambda). \]

Thus, (21) is proved.

(d) Now, we apply part (d) of Lemma 3. Suppose that \( \lambda \in \mathbb{R} \) is such that \( P_j(\lambda) \neq 0 \) for \( j = 1, 2, \ldots, k - 1 \) and \( S_k(\lambda) \neq 0 \). Then \( \beta_j = P_j(\lambda)/P_{j-1}(\lambda) \neq 0 \) for \( j = 1, 2, \ldots, k - 1 \) and \( \beta_k = S_k(\lambda)/P_{k-1}(\lambda) \neq 0 \). Hence, \( \det(\lambda I - F_{2^k-2}) \neq 0 \). Suppose that \( \lambda \in \mathbb{R} \) is such that \( P_j(\lambda) = 0 \) for some \( j, 1 \leq j \leq k - 2 \). Since \( P_0(\lambda) = 1 \neq 0 \), we may suppose \( P_{j-1}(\lambda) \neq 0 \) and \( P_j(\lambda) = 0 \). Thus, \( \beta_j = P_j(\lambda)/P_{j-1}(\lambda) = 0 \) and therefore \( \det(\lambda I - F_{2^k-2}) = 0 \). Finally, if \( \lambda \in \mathbb{R} \) is such that \( P_k(\lambda) \neq 0 \) and \( S_k(\lambda) = 0 \) then \( \beta_k = S_k(\lambda)/P_{k-1}(\lambda) = 0 \) and hence \( \det(\lambda I - F_{2^k-2}) = 0 \). \( \square \)
An immediate consequence of Theorems 2 and 4 is:

**Corollary 5.**

\[
\sigma(U_{2k-2}) = \bigcup_{j=1}^{k-1} \{ \lambda \in \mathbb{R} : P_j(\lambda) = 0 \},
\]

\[
\sigma(F_{2k-2}) = \bigcup_{j=1}^{k-2} \{ \lambda \in \mathbb{R} : P_j(\lambda) = 0 \} \cup \{ \lambda \in \mathbb{R} : S_k(\lambda) = 0 \}
\]

and

\[
\det(\lambda I - L(\mathcal{B}_k)) = P_1^{2k-2}(\lambda)P_2^{2k-3}(\lambda)\cdots P_{k-3}^{2}(\lambda)P_{k-2}^{2}(\lambda)P_{k-1}(\lambda)S_k(\lambda).
\]

**Lemma 6.** Let \( T_1 = [1] \) and, for \( j = 2, 3, \ldots, k-1 \), let \( T_j \) be the tridiagonal \( j \times j \) matrix given by

\[
T_j = \begin{bmatrix}
1 & \sqrt{2} & 0 & \cdots & 0 \\
\sqrt{2} & 3 & \sqrt{2} & \ddots & \\
0 & \sqrt{2} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \sqrt{2} & 3
\end{bmatrix}.
\] (22)

Let \( S_k \) be the tridiagonal \( k \times k \) matrix given by

\[
S_k = \begin{bmatrix}
1 & \sqrt{2} & 0 & \cdots & 0 \\
\sqrt{2} & 3 & \sqrt{2} & \ddots & \\
0 & \sqrt{2} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 3 & \sqrt{2} \\
0 & \cdots & 0 & 2 & \sqrt{2}
\end{bmatrix}.
\] (23)

Then

\[
\det(\lambda I - T_j) = P_j(\lambda), \quad j = 1, 2, \ldots, k-1,
\]

and

\[
\det(\lambda I - S_k) = S_k(\lambda).
\]

**Proof.** It is well known (see for instance [5, p. 229]) that the characteristic polynomials, \( p_j \), of the \( j \times j \) principal submatrix of the symmetric tridiagonal \( T \),
satisfy the three-term recursion formula

\[ p_j(\lambda) = (\lambda - a_j) p_{j-1}(\lambda) - b_{j-1}^2 p_{j-2}(\lambda) \]

with

\[ p_0(\lambda) = 1 \quad \text{and} \quad p_1(\lambda) = \lambda - a_1. \]

In our case, \( a_1 = 1, \) \( a_j = 3 \) for \( j = 2, 3, \ldots, k - 1, \) \( a_k = 2 \) and \( b_j = \sqrt{2} \) for \( j = 1, 2, \ldots, k - 1. \) For these values, the above recursion formula gives the polynomials \( P_j, j = 0, 1, 2, \ldots, k - 1, \) and the polynomial \( S_k. \)

**Theorem 7.** Let \( T_j, j = 1, 2, \ldots, k - 1, \) and \( S_k \) be the symmetric tridiagonal matrices defined in Lemma 6. Then:

(a) \( \sigma(U_{2k-2}) = \bigcup_{j=1}^{k-1} \sigma(T_j). \)

(b) \( \sigma(F_{2k-2}) = \bigcup_{j=1}^{k-2} \sigma(T_j) \cup \sigma(S_k). \)

(c) \( \sigma(L(B_k)) = \bigcup_{j=1}^{k-1} \sigma(T_j) \cup \sigma(S_k). \)

(d) The multiplicity of each eigenvalue of the matrix \( T_j, \) as an eigenvalue of \( L(B_k), \) is at least \( 2^{k-j-1}. \)

**Proof.** (a), (b) and (c) are immediate consequences of Corollary 5 and Lemma 6. Moreover, since

\[
\det(\lambda I - L(B_k)) = P_1^{2k-2}(\lambda) P_2^{2k-3}(\lambda) \cdots P_{k-3}^{2k-2}(\lambda) P_{k-2}^2(\lambda) P_{k-1}(\lambda) S_k(\lambda)
\]

and

\[
\det(\lambda I - T_j) = P_j(\lambda), \quad j = 1, 2, \ldots, k - 1,
\]

we have that the multiplicity of each eigenvalue of the matrix \( T_j, \) as an eigenvalue of \( L(B_k), \) is at least \( 2^{k-j-1}. \)

We recall the following interlacing property [6]:

*Let \( T \) be a symmetric tridiagonal matrix with nonzero codiagonal entries and \( \lambda_i^{(j)} \) be the \( j \)th smallest eigenvalue of its \( j \times j \) principal submatrix. Then*
\[ \frac{1}{\lambda(j+1)} \frac{1}{\lambda(j+1)} \frac{1}{\lambda(j+1)} < \frac{1}{\lambda(j+1)} \quad \lambda(j+1) < \lambda(j+1) \]

From this interlacing property and Theorem 7, we have:

**Theorem 8.** Let \( L(B_k) \) be the Laplacian matrix of \( B_k \). Then:

(a) \( \sigma(T_{j-1}) \cap \sigma(T_j) = \emptyset \) for \( j = 2, 3, \ldots, k-1 \) and \( \sigma(T_{k-1}) \cap \sigma(S_k) = \emptyset \).

(b) The largest eigenvalue of \( S_k \) is the largest eigenvalue of \( L(B_k) \).

(c) The smallest eigenvalue of \( T_{k-1} \) is the algebraic connectivity of \( B_k \).

(d) The largest eigenvalue of \( T_{k-1} \) is the second largest eigenvalue of \( L(B_k) \).

**Example 1.** Let \( k = 3 \). Then \( n = 7 \) and \( \sigma(L(B_3)) = \sigma(T_1) \cup \sigma(T_2) \cup \sigma(S_3) \),

\[
T_1 = [1], \quad T_2 = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & \sqrt{2} \\ 0 & \sqrt{2} & 2 \end{bmatrix}.
\]

Hence, the eigenvalues of \( L(B_3) \) are 1 with multiplicity equal to 2, and \( 2 - \sqrt{3}, 2 + \sqrt{3}, 3 + \sqrt{2}, 3 - \sqrt{2} \) and 0, with multiplicity equal to 1.

**Example 2.** Let \( k = 10 \). Then \( n = 1023 \). The eigenvalues of \( L(B_{10}) \), rounded to three decimal places, are given in the following table:

<table>
<thead>
<tr>
<th>( T_1 )</th>
<th>( T_2 )</th>
<th>( T_3 )</th>
<th>( T_4 )</th>
<th>( T_5 )</th>
<th>( T_6 )</th>
<th>( T_7 )</th>
<th>( T_8 )</th>
<th>( T_9 )</th>
<th>( S_{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>0.268</td>
<td>0.097</td>
<td>0.040</td>
<td>0.018</td>
<td>0.009</td>
<td>0.004</td>
<td>0.002</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>0.373</td>
<td>2.194</td>
<td>3.399</td>
<td>5.141</td>
<td>4.122</td>
<td>4.574</td>
<td>5.496</td>
<td>4.871</td>
<td>5.578</td>
<td>5.634</td>
</tr>
<tr>
<td>4.709</td>
<td>4.709</td>
<td>3.739</td>
<td>5.141</td>
<td>5.365</td>
<td>5.496</td>
<td>5.578</td>
<td>5.634</td>
<td>5.672</td>
<td>5.690</td>
</tr>
<tr>
<td>4.709</td>
<td>4.709</td>
<td>3.739</td>
<td>5.141</td>
<td>5.365</td>
<td>5.496</td>
<td>5.578</td>
<td>5.634</td>
<td>5.672</td>
<td>5.690</td>
</tr>
</tbody>
</table>

**4. Bounds for some eigenvalues**

In [3] quite tight lower and upper bounds for the algebraic connectivity of \( B_k \), which is the smallest eigenvalue of \( T_{k-1} \), are given. These bounds are immediate consequences of Lemmas 3.4 and 3.6 in the above mentioned paper:
Let $G_m$, of order $m \times m$, $m \geq 3$, be the matrix

$$
G_m = \begin{bmatrix}
3 & -\sqrt{2} & 0 & \cdots & 0 \\
-\sqrt{2} & 3 & -\sqrt{2} & \ddots & \vdots \\
0 & -\sqrt{2} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 3 & -\sqrt{2} \\
0 & \cdots & 0 & -\sqrt{2} & 1
\end{bmatrix}.
$$

Then

\begin{equation}
\lambda_m(G_m) \leq \frac{1}{2^{m+1} - 2m + 1 - \frac{2m}{2^{m-1}}} \tag{24}
\end{equation}

and

\begin{equation}
\lambda_m(G_m) \geq \frac{1}{2^{m+1} - 2m - \frac{2m+2-\sqrt{2}(2m+1-2m)}{2^{m+1-1} - \sqrt{2}(2m-1)}} + \frac{1}{3-2\sqrt{2}\cos\left(\frac{\pi}{2^{m+1}}\right)}. \tag{25}
\end{equation}

Since $T_j$ for $j = 3, 4, \ldots, k - 1$ is similar to $G_j$, we can apply (24) and (25) to obtain the following upper and lower bounds for the smallest eigenvalue of each $T_j$:

\begin{equation}
\lambda_j(T_j) \leq \frac{1}{2^{j+1} - 2j + 1 - \frac{2j}{2^{j-1}}}, \tag{26}
\end{equation}

\begin{equation}
\frac{1}{2^{j+1} - 2j - \frac{2j+2-\sqrt{2}(2j+1-2j)}{2^{j+1-1} - \sqrt{2}(2j-1)}} + \frac{1}{3-2\sqrt{2}\cos\left(\frac{\pi}{2^{j+1}}\right)} \leq \lambda_j(T_j). \tag{27}
\end{equation}

In [3], the authors obtained (25) using the fact that the eigenvalues of the $m \times m$ matrix

$$
H_m = \begin{bmatrix}
3 & -\sqrt{2} & 0 & \cdots & 0 \\
-\sqrt{2} & 3 & -\sqrt{2} & \ddots & \vdots \\
0 & -\sqrt{2} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 3 & -\sqrt{2} \\
0 & \cdots & 0 & -\sqrt{2} & 3 - \sqrt{2}
\end{bmatrix}
$$

are given by

\begin{equation}
\lambda_i(H_m) = 3 - 2\sqrt{2}\cos\left(\frac{(2m-2i+1)\pi}{2m+1}\right), \quad i = 1, 2, \ldots, m, \tag{28}
\end{equation}

together, among others results, with the fact that

\begin{equation}
\lambda_i(G_m) \leq \lambda_i(H_m), \quad i = m, m - 1, \ldots, 2, 1. \tag{29}
\end{equation}
Since $H_m$ is similar to
\[
K_m = \begin{bmatrix}
3 - \sqrt{2} & \sqrt{2} & 0 & \cdots & 0 \\
\sqrt{2} & 3 & \sqrt{2} & \ddots & \\
0 & \sqrt{2} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 3 & \sqrt{2} \\
0 & \cdots & 0 & \sqrt{2} & 3
\end{bmatrix},
\]
we have that
\[
\lambda_1(T_j) \leq \lambda_1(K_j) = 3 - 2\sqrt{2} \cos \left( \frac{(2j - 1)\pi}{2j + 1} \right), \quad j = 3, \ldots, k - 1. \quad (30)
\]

The second right-hand side in (30) gives, in particular, an upper bound for the largest eigenvalue of $T_j$:
\[
\lambda_2(L(B_k)) = \lambda_1(T_{k-1}) \\
\leq \lambda_1(K_{k-1}) \leq 3 - 2\sqrt{2} \cos \left( \frac{(2k - 3)\pi}{2k - 1} \right). \quad (31)
\]

Thus, we have an upper bound for the second largest eigenvalue $\lambda_2(L(B_k))$. The inequalities given in (29) follow from the fact that:

*The eigenvalues of a Hermitian matrix increase if a positive semidefinite matrix is added to it* [7, Corollary 4.3.3].

We have
\[
K_k = \begin{bmatrix}
1 & \sqrt{2} & 0 & \cdots & \cdots & 0 \\
\sqrt{2} & 3 & \sqrt{2} & \ddots & \ddots & \\
0 & \sqrt{2} & \ddots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 3 & \sqrt{2} \\
0 & \cdots & \cdots & 0 & \sqrt{2} & 2
\end{bmatrix}
+ \begin{bmatrix}
2 - \sqrt{2} & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \cdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & 0 \\
0 & \cdots & \cdots & 0 & 0 & 1
\end{bmatrix}
\]
Therefore,
\[
\lambda_1(L(B_k)) = \lambda_1(S_k) \\
\leq \lambda_1(K_k) = 3 - 2\sqrt{2}\cos\left(\frac{(2k-1)\pi}{2k+1}\right).
\]

(32)

Thus, we have an upper bound for the largest eigenvalue of the Laplacian matrix of \( B_k \).

**Example 3.** Let \( k = 10 \). With the results rounded to four decimal places, the use of the above bounds gives

\[
\begin{array}{cccc}
  j & \lambda_j(T_j) & \lambda_1(T_j) & \lambda_1(L(B_{10})) \\
  3 & 0.0964 & 0.0968 & 4.7093 & 4.7635 \\
  4 & 0.0402 & 0.0403 & 5.1407 & 5.1667 \\
  5 & 0.0181 & 0.0181 & 5.3651 & 5.3794 \\
  6 & 0.0085 & 0.0085 & 5.4957 & 5.5044 \\
  7 & 0.0041 & 0.0041 & 5.5782 & 5.5839 \\
  8 & 0.0020 & 0.0020 & 5.6335 & 5.6374 \\
  9 & 0.0010 & 0.0010 & 5.6724 & 5.6752 \\
\end{array}
\]

and (32) gives 5.7028 as an upper bound for \( \lambda_1(L(B_{10})) = 5.6900 \).

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**References**