Polynomial convexity and strong disk property

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Abstract

There exists an open set of $\mathbb{C}^n$ which is not polynomially convex and satisfies the strong disk property in $\mathbb{C}^n$ if $n \geq 2$.

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1. Introduction

Let $\Delta := \{ \zeta \in \mathbb{C} \mid |\zeta| < 1 \} \subset \mathbb{C}$. Let $X$ be a (reduced) complex space and $D$ an open set of $X$. We say that $D$ satisfies the strong disk property in $X$ if it satisfies the condition that if $\varphi : \bar{\Delta} \to X$ is a continuous map holomorphic on $\Delta$ such that $\varphi(\partial \Delta) \subset D$, then $\varphi(\bar{\Delta}) \subset D$ (see Abe et al. [3]).

Every polynomially convex open set of $\mathbb{C}^n$ satisfies the strong disk property in $\mathbb{C}^n$ (see Corollary 2). The purpose of this paper is to prove that the converse of this fact is not true if $n \geq 2$ (see Theorem 7). This also gives a negative answer to the problem of Bremermann [5, p. 182] (see also Ohsawa [10, p. 81]). The outline of the proof is as follows.

Every simply connected rationally convex open set of $\mathbb{C}^n$ satisfies the strong disk property in $\mathbb{C}^n$ (see Theorem 5 and Corollary 6). By Nishino [8,9] or by Duval [6] there exists a simply connected rationally convex open set $D$ of $\mathbb{C}^2$ which is not polynomially convex. Then $D$ is an open set of $\mathbb{C}^2$ which is not polynomially convex and satisfies the strong disk property in $\mathbb{C}^2$.
2. Preliminaries

An open set $D$ of a complex space $X$ is said to be $\mathcal{O}(X)$-convex if for every compact set $K$ of $D$ the set $\tilde{K}_X \cap D$ is also compact, where

$$\tilde{K}_X := \{x \in X \mid |f(x)| \leq \|f\|_K \text{ for every } f \in \mathcal{O}(X)\}.$$

**Proposition 1.** If $X$ is a Stein space, then every $\mathcal{O}(X)$-convex open set $D$ of $X$ satisfies the strong disk property in $X$.

**Proof.** Let $\varphi : \Delta \to X$ be a continuous map which is holomorphic on $\Delta$ such that $\varphi(\partial \Delta) \subset D$. Take a number $R \in (0, 1)$ such that $\varphi(\{ \zeta \in \mathbb{C} \mid R \leq |\zeta| \leq 1 \}) \subset D$. Let $K := \{ \zeta \in \mathbb{C} \mid |\zeta| = R \}$ and $L := \varphi(K)$. Then $\varphi(\Delta_R) = \varphi(\tilde{K}) \subset \tilde{L}_X$, where $\Delta_R := \{ \zeta \in \mathbb{C} \mid |\zeta| < R \} \subset \mathbb{C}$. Since $L$ is a compact set of $D$, we have that $\tilde{L}_X \subset D$ (see Stein [12, Satz 1.1]). It follows that $\varphi(\Delta) \subset D$. □

The open set $D$ of $\mathbb{C}^n$ is polynomially convex if and only if $D$ is $\mathcal{O}(\mathbb{C}^n)$-convex. We have the following corollary.

**Corollary 2.** Every polynomially convex open set $D$ of $\mathbb{C}^n$ satisfies the strong disk property in $\mathbb{C}^n$.

**Proposition 3.** An open set $D$ of $\mathbb{C}$ is polynomially convex if and only if $D$ satisfies the strong disk property in $\mathbb{C}$.

**Proof.** Assume that $D$ satisfies the strong disk property in $\mathbb{C}$. Let $D_0$ be an arbitrary connected component of $D$. Let $\gamma$ be an arbitrary closed Jordan path in $D_0$ and $E$ the bounded domain of $\mathbb{C}$ such that $\partial E = \gamma$. By the theorem of Carathéodory there exists a homeomorphism $\varphi : \Delta \to \tilde{E}$ such that $\varphi(\partial \Delta) = \gamma$ and that $\varphi|_{\Delta} : \Delta \to E$ is biholomorphic. Since $\varphi(\partial \Delta) = \gamma \subset D$, we have that $\tilde{E} = \varphi(\Delta) \subset D$. Since $\tilde{E}$ is connected and intersects with $D_0$, we have that $\tilde{E} \subset D_0$. It follows that $D_0$ is simply connected. Since we have proved that $D$ is simply connected, the open set $D$ is polynomially convex in $\mathbb{C}$ (see Remmert [11, p. 298]). The converse is true by Corollary 2. □

The converse of Proposition 1 is not true in general even if $\dim X = 1$. As an example, let $X := \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $D := \{ \zeta \in X \mid |\zeta| \neq 1 \}$. Then $D$ is not $\mathcal{O}(X)$-convex whereas $D$ satisfies the strong disk property in $X$.

If an open set $D$ of $\mathbb{C}^n$ satisfies the strong disk property in $\mathbb{C}^n$, then for any complex line $L$ in $\mathbb{C}^n$ the intersection $D \cap L$ satisfies the strong disk property in $L$. Therefore, we have the following corollary.

**Corollary 4.** If an open set $D$ of $\mathbb{C}^n$ satisfies the strong disk property in $\mathbb{C}^n$, then the intersection $D \cap L$ with an arbitrary complex line $L$ in $\mathbb{C}^n$ is simply connected.

3. Theorems

An open set $D$ of a complex space $X$ is said to be meromorphically $\mathcal{O}(X)$-convex if for every compact set $K$ of $D$ the set $\tilde{K}_X \cap D$ is also compact, where

$$\tilde{K}_X := \{x \in X \mid f(x) \in f(K) \text{ for every } f \in \mathcal{O}(X)\}.$$
(see Abe [1] and Lupacciolu [7]). An open set $D$ of $\mathbb{C}^n$ is rationally convex if and only if $D$ is meromorphically $\mathcal{O}^r(\mathbb{C}^n)$-convex (see Abe [1, Lemma 2]). We have the following theorem.

**Theorem 5.** Let $X$ be a Stein manifold and $D$ an open set of $X$. If $D$ is simply connected and meromorphically $\mathcal{O}^r(X)$-convex, then $D$ satisfies the strong disk property in $X$.

**Proof.** Let $\varphi: \tilde{\Delta} \to X$ be a continuous map which is holomorphic on $\Delta$ such that $\varphi(\partial \Delta) \subset D$. Take a number $R \in (0, 1)$ such that $\varphi([\zeta \in \mathbb{C} \mid |\zeta| \leq |\zeta| \leq 1]) \subset D$. Then $y: I \to X$, $y(t) := \varphi(Re^{2\pi it})$, is a closed real-analytic path in $D$, where $I := [0, 1]$. Since $D$ is simply connected, there exists a continuous map $\lambda: I \times I \to D$ such that $\lambda(0, t) = y(t)$ and $\lambda(1, t) = \lambda(s, 0) = \lambda(s, 1) = y(0)$ for every $s, t \in I$. Let $K := \lambda(I \times I)$. Take an arbitrary $f \in \mathcal{O}(X)$ such that $f \not\equiv 0$ on $K$. We have that

$$
\int_0^1 \frac{d f}{f} = \int_0^1 \frac{\bar{d}f(y(t))}{f(y(t))} dt = \int_0^1 \frac{d(f \circ \varphi)(Re^{2\pi it})}{(f \circ \varphi)(Re^{2\pi it})} dt
$$

$$
= \int_0^1 \frac{(f \circ \varphi)'(Re^{2\pi it}) \cdot 2\pi i Re^{2\pi it}}{(f \circ \varphi)(Re^{2\pi it})} dt = \int_{|\zeta|=R} \frac{(f \circ \varphi)'(\zeta)}{(f \circ \varphi)(\zeta)} d\zeta,
$$

which is equal to $2\pi i$ multiplied by the number of zero points with multiplicities of the function $f \circ \varphi$ in the open disk $\Delta_R := \{\zeta \in \mathbb{C} \mid |\zeta| < R\}$. Since $K$ is connected, there exists a connected open set $E$ of $X$ such that $K \subset E \subset D$ and $f \not\equiv 0$ on $E$. Let $\pi: \tilde{E} \to E$ be the universal covering of $E$. Then there exists a continuous map $\bar{\lambda}: I \times I \to \tilde{E}$ such that $\pi \circ \tilde{\lambda} = \lambda$. Since the set $\tilde{\lambda}((0 \times 1) \cup (I \times \{0, 1\}))$ is connected and contained in the discrete set $\pi^{-1}(y(0))$, we have that $\#\tilde{\lambda}((0 \times 1) \cup (I \times \{0, 1\})) = 1$. It follows that $\tilde{\lambda}(0, 0) = \tilde{\lambda}(0, 1)$ and therefore $\tilde{y}: I \to \tilde{E}$, $\tilde{y}(t) := \tilde{\lambda}(0, t)$, is a closed real-analytic path in $\tilde{E}$. Since $\tilde{E}$ is simply connected and $f \circ \pi \not\equiv 0$ on $\tilde{E}$, there exists $\tilde{g} \in \mathcal{O}(\tilde{E})$ such that $f \circ \pi = e^{\tilde{g}}$ on $\tilde{E}$. Then we have that

$$
\int_0^1 \frac{d f}{f} = \int_{\tilde{y}} \frac{d(f \circ \pi)}{f \circ \pi} = \int_{\tilde{y}} d\tilde{g} = \tilde{g}(\tilde{y}(1)) - \tilde{g}(\tilde{y}(0)) = 0.
$$

It follows that $f \circ \varphi \not\equiv 0$ on $\Delta_R$ for every $f \in \mathcal{O}(X)$ such that $f \not\equiv 0$ on $K$. Take an arbitrary $c \in \Delta_R$. Assume that $h(\varphi(c)) \not\in h(K)$ for some $h \in \mathcal{O}(X)$. Then for the function $f := h - h(\varphi(c))$ we have that $f(\varphi(c)) = 0$ and $f \not\equiv 0$ on $K$, which is a contradiction. It follows that $\varphi(c) \in \tilde{K}_X$. Since $D$ is meromorphically $\mathcal{O}^r(X)$-convex, we have that $\tilde{K}_X \subset D$ by Abe [1, Theorem 12]. Thus we proved that $\varphi(c) \in D$ for every $c \in \Delta_R$ and we have that $\varphi(\Delta) \subset D$. $\square$

**Corollary 6.** If an open set $D$ of $\mathbb{C}^n$ is simply connected and rationally convex, then $D$ satisfies the strong disk property in $\mathbb{C}^n$.

Not every rationally convex open set of $\mathbb{C}^n$ satisfies the strong disk property in $\mathbb{C}^n$. Let $D_1$ be an open set of $\mathbb{C}$ which is not simply connected. Then the open set $D := D_1 \times \mathbb{C}^{n-1}$ of $\mathbb{C}^n$ is rationally convex and does not satisfy the strong disk property in $\mathbb{C}^n$ (see Corollary 4). The Hartogs triangle $\{(z, w) \in \mathbb{C}^2 \mid |z| < |w| < 1\}$ is also a rationally convex open set of $\mathbb{C}^2$ which does not satisfy the strong disk property in $\mathbb{C}^2$. 
Theorem 7. There exists an open set of $\mathbb{C}^n$ which is not polynomially convex and satisfies the strong disk property in $\mathbb{C}^n$ if $n \geq 2$.

Proof. By Nishino [8,9] or by Duval [6] there exists a simply connected rationally convex open set $D_2$ of $\mathbb{C}^2$ such that $D_2$ is not polynomially convex. Then $D_2$ satisfies the strong disk property in $\mathbb{C}^2$ by Corollary 6. The open set $D := D_2 \times \mathbb{C}^{n-2}$ of $\mathbb{C}^n$, where $n \geq 2$, satisfies the strong disk property in $\mathbb{C}^n$ and is not polynomially convex. \hfill \Box

Let $D$ and $D^*$ be Stein open sets of $\mathbb{C}^n$ such that $D \subset D^*$. Bremermann [5, p. 182] proved that if $D$ is $\mathcal{O}(D^*)$-convex then the intersection $D \cap L$ with an arbitrary complex line $L$ in $\mathbb{C}^n$ is (homologically) simply connected relative to $D^* \cap L$ in the sense of Behnke–Stein [4, p. 444] and asks whether the converse of this fact is true or not (see also Ohsawa [10, p. 81]). By Theorem 7 and by Corollary 4, the answer to the problem of Bremermann [5, p. 182] is negative even if $D^* = \mathbb{C}^n$, where $n \geq 2$.

4. A remark on the Nishino domain

Let $S := \bigcup_{t \in [0,1]} \{ (z, w) \in \mathbb{C}^2 \mid (1 - t)z^2 - 2tz + w = 0 \}$ and let $M > 1$. Then the open set $D_M := \{ (z, w) \in \mathbb{C}^2 \mid 1 < |z| < M, |w| < 1 \} \setminus S$ of $\mathbb{C}^2$ is rationally convex, for it is clear that $D_M$ is convex with respect to the rational functions which are holomorphic on $D_M$ (see Abe [2, Theorem 4.2]). We call $D_M$ a Nishino domain. Nishino [8] proved that $D_M$ is simply connected for sufficiently large $M$ and that $D_M$ is not polynomially convex if $D_M$ is simply connected. We also refer to Nishino [9] for the revised proof of the latter fact. By Corollary 6 we see that $D_M$ satisfies the strong disk property in $\mathbb{C}^2$ for sufficiently large $M$. Here we give a direct proof of the following proposition.

Proposition 8. The Nishino domain $D_M$ satisfies the strong disk property in $\mathbb{C}^2$ for every $M > 1$.

Proof. Let $\varphi : \tilde{\Delta} \to \mathbb{C}^2$, $\zeta \mapsto (z(\zeta), w(\zeta))$, be a continuous map holomorphic on $\Delta$ such that $\varphi(\partial \Delta) \subset D_M$. By the maximum modulus principle we have that $|z| < M$ and $|w| < 1$ on $\tilde{\Delta}$. Take a number $R \in (0, 1)$ such that $|z| > 1$ and $(1 - t)z^2 - 2tz + w \neq 0$ on $\{ \zeta \in \mathbb{C} \mid R \leq |\zeta| \leq 1 \}$ for every $t \in I := [0, 1]$. Then $z^2 + sw \neq 0$, $-2z + sw \neq 0$ and $(1 - t)z^2 - 2tz + w \neq 0$ on the circle $\{ \zeta \in \mathbb{C} \mid |\zeta| = R \}$ for every $s, t \in I$. Let $N(s, t)$ be the number of zero points with multiplicities of the function $(1 - t)z^2 - 2tz + sw$ of $\zeta$ in the open disk $\Delta_R := \{ \zeta \in \mathbb{C} \mid |\zeta| < R \}$ for every $(s, t) \in L := (I \times \{ 0, 1 \}) \cup (\{ 1 \} \times I)$. We have that

$$N(s, t) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{2(1-t)zz'-2tz'+sw'}{(1-t)z^2-2tz+sw} \, d\zeta.$$  

Since $N(s, t)$ is a continuous function with discrete values on the connected set $L$, it must be constant on $L$. Since we have

$$N := N(0, 1) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{-2z'}{-2z} \, d\zeta = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{z'}{z} \, d\zeta$$  

and

$$N = N(0, 0) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{2zz'}{z^2} \, d\zeta = 2 \cdot \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{z'}{z} \, d\zeta = 2N,$$

we have

$$N = N(0, 1) = 2N(0, 0),$$

and

$$N(0, 1) = 2N(0, 0).$$
we obtain \( N = 0 \). Therefore \( N(1, t) = 0 \) for every \( t \in I \). It follows that \((1 - t)z^2 - 2tz + w \neq 0\) on \( \bar{\Delta} \) for every \( t \in I \). Take an arbitrary \( c \in \bar{\Delta} \). Then we have that \( z - tc \neq 0 \) on the circle \( \partial \Delta_R \) for every \( t \in I \) because \(|z| > 1\). Therefore the number of zero points with multiplicities of the function \( z - c \) of \( \zeta \) in \( \Delta_R \) is the same as that of the function \( z \) of \( \zeta \) in \( \Delta_R \) which is equal to

\[
\frac{1}{2\pi i} \int_{|\zeta|=R} \frac{z'}{z} d\zeta = N = 0.
\]

It follows that \(|z| > 1\) on \( \Delta_R \). Since we have proved that \( 1 < |z| < M \) and \(|w| < 1\) on \( \bar{\Delta} \) and that \((1 - t)z^2 - 2tz + w \neq 0\) on \( \bar{\Delta} \) for every \( t \in I \), we have that \( \varphi(\bar{\Delta}) \subset D_M \). \( \square \)

References