Supplement submodules and a generalization of projective modules

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Let \( R \) be a ring with Jacobson radical \( J(R) \). Given a left \( R \)-module \( M \), a supplement submodule of \( M \) is a submodule \( K \subseteq M \) for which there exists \( K' \subseteq M \) such \( M = K + K' \) and \( K \) is minimal with respect to this property. In general, supplement submodules are strict generalizations of direct summands. Those rings for which every supplement submodule of a finitely generated projective module is a direct summand have been widely treated in the literature (see \([6–8,11,13,16]\)). However, it is an open problem to give an internal characterization of these rings (see \([9]\)). The purpose of this note is to give more results about supplement submodules in projective modules and determine their relationship with a generalization of projective modules (radical-projective modules). These results will allow us to generalize some characterizations of rings in which every supplement of a finitely generated projective module is a direct summand, to those in which every supplement of a (non-necessarily finitely generated) projective module is a direct summand.

Throughout this paper, \( R \) will denote an associative ring with 1, module will mean left \( R \)-module and morphisms will operate on the right. \( J(R) \) will denote the Jacobson radical of \( R \) and \( R\text{-Mod} \) will be the category whose objects are left \( R \)-modules. Recall that a submodule \( K \) of a module \( M \) is said to be superfluous (written \( K \ll M \)) if for every \( L \subseteq M \) it is verified \( K + L \neq M \). A morphism \( f : M \to N \) in \( R\text{-Mod} \) is called a superfluous epimorphism if it is an epimorphism with \( \text{Ker} \ f \ll M \). It is trivial that a submodule \( K \) of a module \( M \) is a supplement of \( K' \subseteq M \) if and only if \( K + K' = M \) and \( K \cap K' \ll K \). For every module \( M \), \( J(M) \) will denote its Jacobson radical and given \( x \in R \), \( (x : 0) \) will be the left ideal of \( R \) \( \{r \in R : rx = 0\} \).
1. Supplements in projective modules

First of all, we shall describe supplement submodules in terms of a special class of endomorphisms.

**Definition 1.1.** Let \( x \in R \). We shall say that:

(i) \( x \) is a weak left (CE) element if:
   (a) There exists \( r_0 \in R \) such that \( r_0 x^2 = x \).
   (b) For every \( r \in R \) with \( rx^2 = x \), there exists \( t \in R \) with \( tx = x \).

(ii) \( x \) is a left (CE) element if:
   (a) There exists \( r_0 \in R \) such that \( r_0 x^2 = x \).
   (b) \( x - x^2 \in J(R) \).

**Remarks 1.2.**

(i) It is easy to prove that \( x \in R \) is left (CE) if and only if there exists \( j \in J(R) \) with \( (1 - j)^{-1} x^2 = x \).

(ii) It is not difficult to see that each left (CE) element is a weak left (CE) element. However, the converse is not true. For example, let \( F \) be a field with \( \text{char } F \neq 2 \) and \( R \) the upper triangular matrix ring over \( F \). Take \( b \in F \) and \( x = \begin{pmatrix} -1 & b \\ 0 & 0 \end{pmatrix} \). Then \( x \) is weak left (CE) but it is not left (CE).

Left (CE) elements were used by Zöschinger in [16] and by I. Sakhajev in [12]. The relationship with supplements in projective modules is:

**Proposition 1.3.** Let \( P \) be a projective module and \( p \in \text{End}_R(P) \). The following assertions are equivalent:

(i) \( p \) is a weak left (CE) element in \( \text{End}_R(P) \).

(ii) \( \text{Im } p \) is a supplement of \( \text{Ker } p \) in \( P \).

**Proof.** (i) \( \Rightarrow \) (ii). The existence of \( g_0 \in \text{End}_R(P) \) such that \( g_0 p^2 = p \) implies that \( \text{Im } p p = \text{Im } p \) and thus \( \text{Im } p + \text{Ker } p = P \). Let \( K \subseteq \text{Im } p \) with \( K + \text{Ker } p = P \). Firstly, suppose that \( P \) is free with basis \( \{ x_\alpha : \alpha \in \Gamma \} \). Since \( (K)p = \text{Im } p \) we can find \( \{ k_\alpha : \alpha \in \Gamma \} \subseteq K \) and \( \{ y_\alpha : \alpha \in \Gamma \} \subseteq P \) with \( (x_\alpha)p = (k_\alpha)p \) and \( k_\alpha = (y_\alpha)p \) for each \( \alpha \in \Gamma \). Consider \( h \) the unique endomorphism of \( P \) such that \( (x_\alpha)h = y_\alpha \) for each \( \alpha \in \Gamma \). Then \( hp^2 = p \) and, applying the hypotheses, there exists \( f \in \text{End}_R(P) \) with \( fhp = p \). Given \( x \in P \) and taking \( \{ r_\alpha : \alpha \in \Gamma \} \subseteq R \) with \( \{ \alpha \in \Gamma : r_\alpha \neq 0 \} \) finite and \( (x)f = \sum_{\alpha \in \Gamma} r_\alpha x_\alpha \), we obtain

\[
(x)p = (x)fhp = \sum_{\alpha \in \Gamma} r_\alpha (x_\alpha)hp = \sum_{\alpha \in \Gamma} r_\alpha k_\alpha \in K
\]

and so \( K = \text{Im } p \).
If $P$ is projective, $P'$ is a projective module such that $P \oplus P'$ is free and $\pi_P$, $\iota_P$ are the corresponding canonical projection and injection respectively, it suffices to apply the latter fact to the morphism $\pi_P \iota_P$ to obtain that $K = \text{Im}(\pi_P \iota_P)$. But this implies that $K = \text{Im} \ p$.

(ii) $\Rightarrow$ (i). If $\text{Im} \ p + \ker \ p = P$ then $(P)p^2 = \text{Im} \ p$ and we can get a commutative diagram

![Diagram](image_url)

because $P$ is projective. Thus $g_0p^2 = p$.

Let $g \in \text{End}_R(P)$ such that $gp^2 = p$. Then $(P)gP + \ker \ p = P$; since $\text{Im} \ p$ is minimal with respect to this property we get that $(P)gp = \text{Im} \ p$; by projectivity of $P$ we obtain a commutative diagram with exact row

![Diagram](image_url)

But this means $hgp = p$. □

For left (CE) morphisms we have:

**Proposition 1.4.** Let $P$ be a projective module and $p \in \text{End}_R(P)$. The following assertions are equivalent:

(i) $p$ is a left (CE) element in $\text{End}_R(P)$.

(ii) $\text{Im} \ p$ is a supplement of $\text{Im}(1_P - p)$ in $P$.

**Proof.** (i) $\Rightarrow$ (ii). Clearly, $\text{Im} \ p + \text{Im}(1_P - p) = P$. Given $g_0 \in \text{End}_R(P)$ such that $g_0p^2 = p$ we have $p - p^2 = g_0(p - p^2)p$; since $\text{Im}(g_0(p - p^2)) \ll P$ by [1, 17.11] and $(\text{Im}(g_0(p - p^2)))p \ll \text{Im} \ p$ by [1, 5.18] we get

$$\text{Im} \ p \cap \text{Im}(1_P - p) = \text{Im}(p - p^2) = \text{Im}(g_0(p - p^2)p) \ll \text{Im} \ p.$$ 

(ii) $\Rightarrow$ (i). Since $\text{Im}(p - p^2) = \text{Im} \ p \cap \text{Im}(1_P - p) \ll P$, $p - p^2 \in J(\text{End}_R(P))$ by [1, 17.11]. Now $\text{Im} p^2 + \text{Im}(1_P - p) + \text{Im}(p - p^2) = P$ implies that $\text{Im} p^2 + \text{Im}(1_P - p) = P$ and from the minimality of $\text{Im} \ p$ follows that $\text{Im} p^2 = \text{Im} \ p$. Since $P$ is projective we can find $g_0 \in \text{End}_R(P)$ with $g_0p^2 = p$ (see proof of Proposition 1.3). □
Remark 1.5. If $M$ is a module and $p$ is an endomorphism of $M$, then $\text{Im} \ p \oplus \text{Im} \ (1_M - p) = M$ if and only if $p$ is idempotent. The above result generalizes this one in the case of projective modules, supplements and left (CE) morphisms.

The following result says that, when the module is projective, every supplement is of the form $\text{Im} \ p$ for a left (CE) morphism $p$. This is a generalization of the well known result concerning direct summands (see, for example, [1, 5.7]).

Theorem 1.6. Let $P$ be a projective module and $K, K' \subseteq P$. The following assertions are equivalent:

(i) $K$ is a supplement of $K'$.
(ii) There exists $p \in \text{End}_R(P)$ left (CE) such that:
   (a) $\text{Im} \, p = K$.
   (b) $\text{Im} \,(1_P - p) \subseteq K'$.
   (c) $(K')p \ll K$.
(iii) There exists $p \in \text{End}_R(P)$ weak left (CE) such that:
   (a) $\text{Im} \, p = K$.
   (b) $\ker p \subseteq K'$.
   (c) $(K')p \ll K$.

Proof. (i) $\Rightarrow$ (ii). Using the decomposition
\[ \frac{P}{K \cap K'} = \frac{K}{K \cap K'} \oplus \frac{K'}{K \cap K'} \]
and the projectivity of $P$ we can construct a commutative diagram
\[
\begin{array}{ccc}
K & \xrightarrow{\pi_1} & K \\
\downarrow{p} & & \downarrow{\pi_2} \\
K & \xrightarrow{\pi_2} & P \\
\end{array}
\]
with canonical morphisms $\pi_1$ and $\pi_2$. Since $(K)p\pi_1 = \frac{K}{K \cap K'}$, $p|_K$ is epic by [1, 5.15]; then (a) holds and, by projectivity, there exists $g_0 \in \text{End}_R(P)$ with $g_0p^2 = p$. Now, for every $x \in P$ and $k \in K, k' \in K'$ with $x = k + k'$ we have that $(x)p + K \cap K' = k + K \cap K'$ and so $x - (x)p \in K'$ for each $x \in P$. Thus (b) holds and $(P)(p - p^2)\pi_1 = 0$; this implies that $(P)(p - p^2) \subseteq K \cap K' \ll K$ and $p - p^2 \in J(\text{End}_R(P))$ by [1, 17.11]. This proves that $p$ is a left (CE) morphism. Finally, (c) follows from the equation $(K')p\pi_1 = 0$.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Since, by Proposition 1.3, $\text{Im} \, p$ is a supplement of $\ker p$ in $P$ we have $K + K' = P$. Let $L \subseteq K$ such that $L + K' = P$. By (c) $(L)p = \text{Im} \, p$ and so $L + \ker p = P$. Since $\text{Im} \, p$ is a supplement of $\ker p$, $L = K$ and the result is proved. $\square$
The above result can be rewritten as follows in the case of the ring:

**Corollary 1.7.** Let $K$ and $K'$ be left ideals of the ring $R$. Then, the following statements are equivalent:

(i) $K$ is a supplement of $K'$.

(ii) There exists a left (CE) element $x$ of $R$ such that

(a) $Rx = K$.

(b) $R(1 - x) \subseteq K'$.

(c) $K'x \ll K$.

(iii) There exists a weak left (CE) element $x$ of $R$ such that

(a) $Rx = K$.

(b) $(x : 0) \subseteq K'$.

(c) $K'x \ll K$.

**Remark 1.8.** The left (CE) endomorphism obtained in Theorem 1.6 is not uniquely determined by $K$ and $K'$. For example, let $F$ be a field and consider $R$ the ring of upper triangular matrices over $F$. Let $K = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, $K' = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$, and $a, b \in F$ distinct and nonzero. Then $K$ is a supplement of $K'$ in $R R$ and $p = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ are distinct left (CE) morphisms (they are idempotent) of $R$ verifying the conditions of the mentioned theorem.

If $M$ is a module denote by $S(M)$ the set of supplement submodules of $M$.

**Corollary 1.9.** For every projective module $P$ with endomorphism ring $E = \text{End}_R(P)$ there is a bijection between the sets $S(P)$ and $S(E) E$.

**Proof.** Given $K$ a supplement submodule of $P$ take, using Theorem 1.6, $p_K$ a left (CE) endomorphism of $P$ such that $\text{Im} p_K = K$. Denote by $\Phi(K)$ the left ideal $E \cdot p_K$ of $E$. By Corollary 1.7 the ideal $\Phi(K)$ is a supplement submodule of $E E$ that does not depend on the election of the morphism $p_K$ because, for every left (CE) morphism $p'$ with $\text{Im} p' = K$, we can construct a commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{p_K} & K \\
\downarrow{h'} & & \downarrow{p'} \\
& P & \\
\end{array}
\]

from which we infer that $E \cdot p_K = E \cdot p'$.

Analogously, if $L$ is a supplement submodule of $E E$ denote, using Corollary 1.7, by $x_L$ a left (CE) element in $E$ such that $L = E \cdot x_L$. By Proposition 1.4, $\Psi(L) = \text{Im} x_L$ is a
supplement submodule of $P$; reasoning as above, $\Psi(L)$ does not depend on the election of $x_L$ and we have maps

$$S(P) \xrightarrow{\phi} S(E),$$

which are easily verified to be mutually inverse. □

2. Radical-projective modules

Projective modules are exactly the direct summands of free modules. We shall describe supplement submodules of free modules in terms of radical-projective modules using left (CE) morphisms.

**Definition 2.1.** We shall say that a module $M$ is radical-projective if for every epimorphism $g: A \to B$ in $R\text{-Mod}$ and every morphism $f: M \to B$, there exists $h: M \to A$ such that $(M)(f - hg) \ll B$.

In the following section we shall give an example of a non-projective radical-projective module. The relationship of radical-projective modules with supplements is:

**Lemma 2.2.** Let $P$ be a projective module and $K$ a supplement submodule of $P$. Then $K$ is radical-projective.

**Proof.** By Theorem 1.6 there exists $p \in \operatorname{End}_R(P)$ a left (CE) morphism such that $\operatorname{Im} p = K$. Take $g: A \to B$ an epimorphism in $R\text{-Mod}$ and a morphism $f: K \to B$. Since $P$ is projective there exists a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow{h'} & & \downarrow{pf} \\
K & = & \text{Im } p
\end{array}
$$

Taking $h = h'|_K$ we get

$$(K)(f - hg) = (K)p(f - hg) = (K)(1 - p)hg = (K)(p - p^2)hg \ll B$$

since $(K)(p - p^2) \ll K$. Thus $K$ is radical-projective. □

**Remark 2.3.** Note that the morphism $h$ obtained verifies $(K)(f - hg) \ll J(R)B$ since, by [16, 2.1], $J(K) = J(R)K$ and $(K)(p - p^2) \ll J(R)K$.

The following lemma is a slight generalization of [8, 2.3].
Lemma 2.4. Let \( P \) be a projective module and \( K \) a submodule of \( P \). The following statements are equivalent:

(i) \( K \) is a supplement in \( P \).
(ii) There exist a superfluous epimorphism \( q : M \to K \) and a morphism \( p : P \to M \) in \( R\text{-Mod} \) such that \( qp \) is a superfluous epimorphism.

Moreover, when these conditions are satisfied, for every superfluous epimorphism \( q' : M \to K \) in \( R\text{-Mod} \) there exists \( p' : P \to M \) such that \( q'p' \) is a superfluous epimorphism. In addition, the morphism \( p' \) can be obtained so that \( p'q' \) is a left \((CE)\) endomorphism of \( P \) with image \( K \).

Proof. (i) \( \Rightarrow \) (ii) Applying Theorem 1.6 let \( t \in \text{End}_R(P) \) be a left \((CE)\) morphism with \( \text{Im} t = K \). Then \( q = t|_K \) is a superfluous epimorphism because \( (K)q = K \) and \( \text{Ker}q = K \cap \text{Ker} t \ll K \) by Proposition 1.3. Now take \( q' : M \to K \) a superfluous epimorphism in \( R\text{-Mod} \). By projectivity of \( P \) we can get a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{q'} & K \\
\downarrow t \ & \ & \downarrow \ \\
P & \xleftarrow{p'} & \\
\end{array}
\]

Since \( (K)p'q' = K \) and \( q' \) is a superfluous epimorphism, \( p'|_K \) is epic by [1, 5.15]; moreover, it is a superfluous epimorphism because \( t|_K \) is. Thus \( q'p' = q'(p'|_K) \) is a superfluous epimorphism because it is the composition of two such epimorphism. Note, in addition, that \( p'q' = t \) is a left \((CE)\) morphism with image \( K \).

(ii) \( \Rightarrow \) (i). [8, 2.3]. \( \square \)

Theorem 2.5. The following statements are equivalent for a left \( R \)-module \( M \):

(i) \( M \) is radical projective.
(ii) There exist a free module \( F \), a supplement submodule \( K \) of \( F \) and a superfluous epimorphism \( \varphi : M \to K \).

Proof. (i) \( \Rightarrow \) (ii). Let \( F \) be a convenient free module such that there exists an epimorphism \( \psi : F \to M \). Since \( M \) is radical projective, we can find \( h \in \text{End}_R(M,F) \) such that \( (M)(1_M - h\psi) \ll M \). Then \( h\psi \) is a superfluous epimorphism since the equality \( M = (M)h\psi + (M)(1_M - h\psi) \) implies that \( M = (M)h\psi \), and \( \text{Ker} h\psi \leq \text{Im}(1_M - h\psi) \ll M \). By Lemma 2.4 \( \text{Im} h \) is a supplement submodule of \( F \); moreover, \( h : M \to \text{Im} h \) is a superfluous epimorphism since \( \text{Ker} h \leq \text{Ker} h\psi \). Thus the result follows taking \( K = \text{Im} h \) and \( \varphi = h \).

(ii) \( \Rightarrow \) (i). By Lemma 2.4 there exists \( p : F \to M \) such that \( \varphi p \) is a superfluous epimorphism and \( p\varphi \) is a left \((CE)\) morphism of \( F \) with \( \text{Im}(p\varphi) = K \). Take \( g : A \to B \).
an epimorphism in $R$-Mod and $f : M \to B$ a morphism. By Lemma 2.2 $K$ is radical-projective and we can construct a diagram

$$
\begin{array}{c}
A \\
\downarrow g \\
K \downarrow h' \\
\end{array}
\quad
\begin{array}{c}
B \\
\downarrow (p|_K)f \\
\end{array}
$$

with $(K)(pf - h'g) \ll B$. Denote $h = \phi h'$. Since $p\phi$ is left (CE) with image $K$, $(p|_K)\phi$ is epic and so is $p|_K$ by [1, 5.15]; then

$$
(M)(f - hg) = (K)p(f - \phi h'g) \\
= (K)(pf - h'g + h'g - p\phi h'g) \\
\leq (K)(pf - h'g) + (K)(1_F - p\phi)h'g.
$$

Taking $t \in \text{End}_R(F)$ with $t(p\phi)^2 = p\phi$ (which exists because $p\phi$ is left (CE)) and using that $(K)p\phi = K$ we get

$$
(K)(1_F - p\phi)h'g = (K)(p\phi - (p\phi)^2)h'g \\
= (K)t(p\phi - (p\phi)^2)p\phi h'g \ll B
$$

as $t(p\phi - (p\phi)^2) \in J(\text{End}_R(F))$. Since the sum of two superfluous submodules is again superfluous, Eqs. (1) and (2) say that

$$
(M)(f - hg) \leq (K)(pf - h'g) + (K)(1_F - p\phi)h'g \ll B
$$

which concludes the proof. \qed

**Remark 2.6.** Remark 2.3 implies that the morphism $h$ obtained in the proof of (i) verifies $(M)(f - hg) \leq J(R)B$.

With this result we can extend properties of supplement submodules to radical projective modules:

**Corollary 2.7.** Let $M$ be a radical-projective module. Then:

(i) $J(M) = J(R)M$.

(ii) Supplement submodules of $M$ are radical-projective.

(iii) If $N$ is another radical projective module then $M \oplus N$ is radical-projective.

**Proof.** (i) Let $K$ be a supplement submodule of a free module and $\phi : M \to K$ a superfluous epimorphism. By Lemma 2.4 there exists $p : K \to M$ such that $\phi p$ is a
superfluous epimorphism. Since \( \varphi \) is a superfluous epimorphism \( p \) is too by [8, 2.1]. Now applying [1, 9.15] and [16, 2.1] we obtain

\[
J(M) = (J(K)p = (J(R)K)p = J(R)M
\]
as required.

(ii) Let \( L \) be a supplement submodule of \( M \). Applying Theorem 2.5, let \( F \) be a free module, \( K \) a supplement submodule of \( F \) and \( \varphi : M \to K \) a superfluous epimorphism; then \( (L)\varphi \) is a supplement submodule of \( K \) since \( \frac{L + \text{Ker}\varphi}{\text{Ker}\varphi} \) is a supplement submodule of \( \frac{M}{\text{Ker}\varphi} \) by [15, 41.1.(4) and 41.1.(7)]. As supplements of supplements are again supplements, \( (L)\varphi \) is a supplement submodule of \( F \). But \( \varphi|_L \) is a superfluous epimorphism from \( L \) to \( (L)\varphi \) and by Theorem 2.5, \( L \) is radical projective.

(iii) By standard arguments.  

Radical projective modules are related with \( J(R) \)-projective modules defined in [8]. Recall that a module \( M \) is said to be \( J(R) \)-projective if for every epimorphism \( g : A \to B \) in \( R \)-Mod with \( J(R)B = 0 \) and every morphism \( f : M \to B \), there exists \( h : M \to A \) such that \( hg = f \). From Remark 2.6 follows that every radical-projective module is \( J(R) \)-projective. The converse is true when the module is finitely generated:

**Proposition 2.8.** Let \( M \) be a finitely generated module. Then \( M \) is radical projective if and only if \( M \) is \( J(R) \)-projective.

**Proof.** Suppose that \( M \) is \( J(R) \)-projective. Let \( g : A \to B \) be an epimorphism in \( R \)-Mod and \( f : M \to B \) a morphism. If \( \pi \) is the canonical projection from \( B \) to \( \frac{B}{J(R)B} \), we can obtain a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{s\pi} & B \\
\downarrow{b} & & \downarrow{f\pi} \\
M & & \\
\end{array}
\]

which implies that \( (M)(f - hg) \leq J(R)B \). But \( M \) is finitely generated and so is \( (M)(f - hg) \); thus \( (M)(f - hg) \ll B \) because finitely generated submodules of the radical are superfluous. \( \square \)

In the next section we shall give an example of a module which is \( J(R) \)-projective but not radical-projective.

3. Supplements as summands. Examples

As we have mentioned in the introduction, rings for which every supplement submodule of a finitely generated projective module is a direct summand have been widely studied
The aim of this section is to extend some of the characterizations of these rings to those in which every supplement submodule of a (non-necessarily finitely generated) projective module is a direct summand. We start with an easy observation:

**Lemma 3.1.** Let $P$ be a projective module and $K \subseteq P$ a supplement submodule of $P$. The following statements are equivalent:

(i) $K$ is a direct summand.

(ii) $K$ is projective.

(iii) For every $K' \subseteq P$ such that $K$ is a supplement of $K'$, the quotient module $\frac{P}{K}$ has a projective cover.

(iv) $K$ has a projective cover.

**Proof.** (ii) $\Rightarrow$ (i) Let $p \in \text{End}_R(P)$ be a left (CE) morphism with $\text{Im } p = K$ (Theorem 1.6).

Since $K \cong \frac{P}{\text{Ker } p} \cong \frac{K}{K \cap \text{Ker } p}$

$K \cap \text{Ker } p$ is a direct summand of $K$; but $K \cap \text{Ker } p$ is superfluous in $K$ by Proposition 1.3 and it has to be zero. From Proposition 1.3 follows that $P = K \oplus \text{Ker } p$.

(ii) $\Rightarrow$ (iii) Given $K' \subseteq P$ as in (iii) we have $\frac{P}{K'} \cong \frac{K}{K \cap K'}$. Then the canonical projection from $K$ to $\frac{K}{K \cap K'}$ induces a projective cover.

(iii) $\Rightarrow$ (iv) If $K$ is a supplement of $K'$, then $\frac{P}{K} \cong \frac{K}{K \cap K'}$ has a projective cover $\varphi : Q \to \frac{K}{K \cap K'}$. Taking $\pi$ the canonical projection from $K$ to $\frac{K}{K \cap K'}$, there exists $\psi : Q \to K$ such that $\psi \pi = \varphi$. Since $\pi$ has superfluous kernel, $\psi$ is the required projective cover.

(iv) $\Rightarrow$ (ii) Let $\varphi : Q \to K$ be a projective cover of $K$. By Lemma 2.2 there exists $h : K \to Q$ such that $(K)(1_K - h\varphi) \ll K$. Since $K = (K)(1_K - h\varphi) + (K)h\varphi$, $h\varphi$ is an epimorphism and so is $h$ by [1, 5.15]. By projectivity of $Q$, $h$ is split and $\text{Ker } h$ is a direct summand of $K$; but $\text{Ker } h$ is superfluous $(\text{Ker } h \leq (K)(1_K - h\varphi))$ and it has to be zero. That is, $h$ is an isomorphism and $K$ is projective.

The following result reduces the study of the property to the case the ring. Recall that, for every $x \in R$, $Rx$ is a direct summand of $R$ if and only if $x$ is (von Neumann) regular (i.e., there exists $t \in R$ such that $xtx = x$). For modules $M$ and $N$, $N$ is said to be $M$-cyclic if there exists an epimorphism $\varphi : M \to N$.

**Corollary 3.2.** Let $P$ be a projective module with endomorphism ring $E = \text{End}_R(P)$. The following statements are equivalent:

(i) Every supplement submodule of $P$ is a direct summand.

(ii) Every supplement submodule of $E$ is a direct summand.

(iii) Every weak left (CE) element of $E$ is (von-Neumann) regular.

(iv) Every left (CE) element of $E$ is (von-Neumann) regular.

(v) If $x \in E$ is a left (CE) element and $t \in E$ is such that $tx^2 = x$, then $xtx = x$. 

(vi) Every $P$-cyclic radical-projective $R$-module is projective.
(vii) Every $P$-cyclic radical-projective $R$-module has a projective cover.

**Proof.** The equivalence of (ii)--(iv) is clear by virtue of the previous remark and Theorem 1.6. For (v) see [16, 1.2].

(i) $\Rightarrow$ (ii) Let $L \leq E$ be a left ideal and suppose that it is a supplement. Following the notation of Corollary 1.9, $\Psi(L)$ is a direct summand of $P$, that is, there exists $e \in E$ idempotent with $\Psi(L) = \text{Im } e$; thus $L = \Phi\Psi(L) = E \cdot e$ is a direct summand of $E$.

(ii) $\Rightarrow$ (i) Is proved similarly.

(i) $\Rightarrow$ (vi) Given a $P$-cyclic radical-projective module $M$ we can find, reasoning as in Theorem 2.5, a supplement submodule $K$ of $P$ and a superfluous epimorphism $\varphi: M \rightarrow K$. By hypothesis, $\varphi$ is a split epimorphism; since $\text{Ker } \varphi$ is superfluous, it is an isomorphism. That is, $M$ is projective.

(vi) $\Rightarrow$ (vii) Trivial.

(vii) $\Rightarrow$ (i) (vii) implies that every supplement submodule of $P$ has a projective cover; apply now Lemma 3.1. $\square$

**Remarks 3.3.**

(i) The equivalence of (i) and (ii) was proved by H. Zöschinger in [16] in the finitely generated case and by A. Mohammed and F.L. Sandomierski in [8]. We give a different proof using (CE) morphisms.

(ii) Condition (v) appears in [11].

The following result gives a characterization of rings for which every supplement submodule of a (non-necessarily finitely generated) projective module is a direct summand. It is a generalization of [8, 4.1] and [12, 3]. For a set $\Gamma$, $\mathcal{RFM}_\Gamma(R)$ will denote the ring of row finite $\Gamma$-matrices with entries in $R$.

**Corollary 3.4.** For the ring $R$ the following assertions are equivalent:

(i) Every supplement submodule of a projective module is a direct summand.

(ii) For any set $\Gamma$, every weak left (CE) matrix $A \in \mathcal{RFM}_\Gamma(R)$ is (von-Neumann) regular.

(iii) For any set $\Gamma$, every left (CE) matrix $A \in \mathcal{RFM}_\Gamma(R)$ is (von-Neumann) regular.

(iv) For every set $\Gamma$, left (CE) matrix $A \in \mathcal{RFM}_\Gamma(R)$ and $T \in \mathcal{RFM}_\Gamma(R)$ with $TA^2 = A$ it is verified $ATA = A$.

(v) Every radical-projective module has a projective cover.

(vi) Every radical-projective module is projective.

**Proof.** Follows from Corollary 3.2. $\square$

We conclude the paper giving some examples. If we want to give a left (CE) element which is not idempotent, we have to find a ring $S$ that does not verify Corollary 3.4. Such a ring was found by Gerasimov and Sakhajev in [4]. Let $x \in R$ be a left (CE) element of
$R$ and $j \in J(R)$ such that $(1 - j)^{-1}x^2 = x$. Reasoning as in [3, 4.3], set $z = 1 - j$, for every $n \in \mathbb{N}^*$ ($= \mathbb{N} - \{0\}$) let $a_n = z^{-n-1}x^n$ and denote $I_x = \sum_{n=1}^{\infty}a_n R$. $I_x$ is a projective right ideal of $R$ such that $\frac{R}{I_x}$ is a flat $R$-module and $\frac{I_x}{J(R)} \cong (x + J(R))\frac{R}{J(R)}$. This ideal determines if $Rx$ is a direct summand:

**Lemma 3.5.** Let $x \in R$ be a left (CE) element. Let $z$, $\{a_n; n \in \mathbb{N}^*\}$ and $I_x$ as above. The following assertions are equivalent:

1. $Rx$ is a direct summand of $R R$.
2. $I_x$ is finitely generated.

**Proof.** (i) $\Rightarrow$ (ii). If $Rx$ is a direct summand then so is $x R$. Using the same argument as in [3, 4.2] we get that $I_x \cong x R$ and $I_x$ is finitely generated.

(ii) $\Rightarrow$ (i). If $I_x$ is finitely generated, there exists $m \in \mathbb{N}^*$ such that $a_m^2 = a_m$ by [3, 3.1]. But $a_m + J(R) = x + J(R)$ and following the argument of [16, 2] we deduce that $Rx$ is a direct summand $\square$

**Example 3.6.** In [4] the authors construct a semilocal ring $S$ and $x, y \in S$ such that $yx = 0$ and $1 - (x + y) \in J(S)$. Take $z = x + y$; then $x$ is left (CE) (because $z^{-1}x^2 = x$) and, by the previous lemma, it is not (von-Neumann) regular since $I_x$ is not finitely generated (see [3, 5.2]). Consequently, $Rx$ is a supplement submodule that is not a direct summand; moreover, by Theorem 2.5, the module $Rx$ is a non-projective radical-projective module.

Another example of a non-projective radical-projective module is given by a submodule of the endomorphism ring of an uniserial module which is not quasi-small. Let $U$ be a module. Recall that $U$ is quasi-small if given a family of modules $\{U_\alpha; \alpha \in \Gamma\}$ such that $U$ is isomorphic to a direct summand of $\bigoplus_{\alpha \in \Gamma} U_\alpha$, there exists a finite subset $\Lambda \subseteq \Gamma$ such that $U$ is isomorphic to a direct summand of $\bigoplus_{\alpha \in \Lambda} U_\alpha$. Moreover, recall that $U$ is uniserial if its lattice of submodules is linearly ordered under set inclusion.

Suppose that $U$ is an uniserial module. Then the endomorphism ring $E = \text{End}_R(U)$ has two (two sided) ideals $L$ and $K$ ($L = \{f \in E; f$ is not injective and $K = \{f \in E; f$ is not surjective\}$) such that every proper left ideal of $E$ is contained either in $L$ or in $K$ (see [2, 1.2]). If, in addition, $U$ is quasi-small, there exists $\{f_n; n \in \mathbb{N}^*\} \subseteq E$ such that $f_n f_{n+1} = f_n \forall n \in \mathbb{N}^*$ and $K = \sum_{n=1}^{\infty} Rf_n$. Moreover, the ideal $K$ is an indecomposable non-finitely generated projective left $E$-module with $\frac{K}{J(E)K}$ simple (see proof of [3, 5.3]). Using the argument of [8, 4.1] we get:

**Proposition 3.7.** Let $U$ be an uniserial module which is not quasi-small with endomorphism ring $E = \text{End}_R(U)$. Let $K, L$ and $\{f_n; n \in \mathbb{N}^*\}$ be as above. Fix $m \in \mathbb{N}^*$ such that $f_m \notin J(E)K$. Then the module $\frac{K}{J(E)K}$ is a non-projective radical-projective left $E$-module.

**Proof.** First note that since $\frac{K}{J(E)K}$ is simple, $Ef_n + J(E)K = K$ for every nonzero natural number $n$ such that $f_a \notin J(E)K$; thus $J(E) \frac{K}{Ef_n} = \frac{K}{Ef_n}$. 


Let \( \varphi : E \to E \) be the morphism given by \((f)\varphi = f(1 - f_{m+1}) \forall f \in E \). Since \((f_m)\varphi = 0 \) we get a morphism \( \overline{\varphi} : \frac{E}{f_m} \to E \). Set \( \pi \) the canonical projection from \( E \) to \( \frac{E}{f_m} \). Given \( f \in E \) there exist \( g_1 \in \frac{E}{f_m} \) and \( g_2 \in J(E)K \) with \( ff_{m+1} = g_1 + g_2 \). This implies that \((f + Ef_m)\overline{\varphi}\pi = (f - g_2) + Ef_m \) and that \( \text{Im} \overline{\varphi}\pi + J(E)\frac{E}{f_m} = \frac{E}{f_m} \); since \( \frac{E}{f_m} \) is superfluous \((\frac{E}{f_m} \) is cyclic), \( \overline{\varphi}\pi \) is epic. Now, from \( \text{Ker} \overline{\varphi}\pi \leq \frac{K}{E} \leq J(E) \cdot \frac{E}{f_m} \) it follows that \( \overline{\varphi}\pi \) is a superfluous epimorphism. By Lemma 2.4 \( \text{Im} \overline{\varphi} \) is a supplement submodule of \( E \) and by Theorem 2.5 \( \frac{E}{f_m} \) is a radical-projective module.

Note that \( \frac{E}{f_m} \) is not projective because \( K \) is indecomposable. \( \square \)

**Remark 3.8.** Notice that, in the previous result, it is proven that \( \text{Im} \overline{\varphi} \) is a supplement submodule of \( R \) which is not a direct summand. That is, \( R(1 - f_n) \) is a supplement of \( R \) that is not a direct summand for every non-zero natural number \( n \) such that \( f_n \notin J(E)K \).

**Examples 3.9.** (1) In \([3, 5, 4]\) the authors give an uniserial module \( U \) over a semilocal ring \( S \) that is not quasi-small. In particular, the ideal \( K = \{ f \in E : f \) is not surjective of \( E = \text{End}_S(U) \) is a projective left \( S \)-module of the form \( \sum_{n=1}^{\infty} Ef_n \) for a convenient set \( \{f_n : n \in \mathbb{N}^*\} \subseteq E \) with \( f_n f_{n+1} = f_n \) \( \forall n \in \mathbb{N}^* \). By the previous lemma, \( \frac{E}{f_m} \) is a non-projective radical projective-module for every non-zero natural number \( n \) with \( f_n \notin J(E)K \).

(2) In the same example the authors give an indecomposable projective module \( P \) over a semilocal ring \( T \) such that \( \text{End}_T(P) \) is isomorphic to \( \text{End}_S(U) \). Take \( F \) an isomorphism from \( \text{End}_S(U) \) to \( \text{End}_T(P) \) and \( p \in \text{End}_S(U) \) a left (CE) element which is not (von-Neumann) regular (which exists by virtue of the previous example and Corollary 3.2). Then \( (p)F \) is a left (CE) element of \( \text{End}_T(P) \) that is not regular. In particular, \( \text{Im}[(p)F] \) is a supplement submodule of \( P \) that is not a direct summand. Moreover, \( \text{Im}[(p)F] \) is a non-projective radical-projective left \( T \)-module.

It is known that rings with polynomial identity, semihereditary rings and rings whose prime quotients rings are left Goldie rings verify the condition (i) of Corollary 3.4 for finitely generated projective modules (see \([7, 11]\) for more examples). Moreover, left noetherian rings and commutative domains verify the conditions of Corollary 3.4. More examples of this type of rings are:

**Examples 3.10.** (i) Every (respectively finitely generated) supplement submodule of an hereditary (respectively semihereditary) projective module is a direct summand by Lemma 3.1. Moreover, if \( R \) is a hereditary (respectively semihereditary) ring, every supplement submodule of a (respectively finitely generated) projective module is a direct summand.

(ii) In \([10]\) U. Oberst and H.J. Schneider calls a ring left (PH) if every finitely generated submodule of a free module has a projective cover. By Lemma 3.1 every left (PH) ring verifies (i) of Corollary 3.4 for finitely generated projective modules.

(iii) Let \( P \) be a projective module such that \( E = \text{End}_R(P) \) is an exchange ring. Then supplements submodules of \( P \) are direct summands, since idempotents lift modulo \( J(E) \) (see \([16, 2]\) and \([5]\)). Moreover, if \( R \) is an exchange ring, every supplement in a finitely generated
generated projective module is a direct summand because $\mathcal{M}_n(R)$ is an exchange ring for every non-zero natural number $n$, by [14, 2].

Finally, we give a $J(R)$-projective module that is not radical-projective.

**Example 3.11.** Let $F$ be a field and $R = F[t]$ the ring of formal power series in $t$ with coefficients in $F$. Then $R$ is noetherian and $J(R)$ is not left T-nilpotent. By [1, 28.3] there exists a left $R$-module $M$ such that $J(R)M = M$. Then $M$ is trivially $J(R)$-projective; but it is not radical-projective since $R$ verifies the conditions of Theorem 3.4 and $M$ is not projective.

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**References**