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The quasineutral limit of compressible Navier–Stokes–Poisson system with heat conductivity and general initial data

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ABSTRACT

The quasineutral limit of compressible Navier–Stokes–Poisson system with heat conductivity and general (ill-prepared) initial data is rigorously proved in this paper. It is proved that, as the Debye length tends to zero, the solution of the compressible Navier–Stokes–Poisson system converges strongly to the strong solution of the incompressible Navier–Stokes equations plus a term of fast singular oscillating gradient vector fields. Moreover, if the Debye length, the viscosity coefficients and the heat conductivity coefficient independently go to zero, we obtain the incompressible Euler equations. In both cases the convergence rates are obtained.

1. Introduction

In the present paper we study the quasineutral limit of compressible Navier–Stokes–Poisson system

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\rho\{\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\} + \nabla P(\rho,\theta) + \rho\nabla\Phi = \mu\Delta\mathbf{u} + (\mu+\nu)\nabla \operatorname{div}\mathbf{u}, \tag{1.2}$$

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$$c_V\rho\{\partial_t\theta+(\mathbf{u}\cdot\nabla)\theta\}+P(\rho,\theta)\operatorname{div}\mathbf{u}=\kappa\Delta\theta+\nu(\operatorname{div}\mathbf{u})^2+2\mu\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{u}),\tag{1.3}$$

$$-\lambda^2\Delta\Phi=\rho-1,\tag{1.4}$$

for $x\in\mathbb{T}^N\subset\mathbb{R}^N$ $(N=2,3)$, the N-dimensional torus, where ρ, $\mathbf{u}=(u_1,\dots,u_N)$, θ, and Φ denote the electron density, velocity, temperature, and the electrostatic potential, respectively. $\mathbb{D}(\mathbf{u})=(d_{ij})_{i,j=1}^N$, $d_{ij}=\frac{1}{2}(\partial_iu_j+\partial_ju_i)$. The constants ν and μ are the viscosity coefficients with $\mu>0$ and $2\mu+N\nu>0$. $c_V>0$ is the specific heat constant, $\kappa>0$ the heat conductivity coefficient, and $\lambda>0$ the scaled Debye length. The pressure function $P(\rho,\theta)$ takes the form

$$P(\rho,\theta)=R\rho\theta,\quad R>0.\tag{1.5}$$

Without loss of generality, we assume $c_V=R\equiv1$ for notational simplicity. The Navier–Stokes–Poisson system (1.1)–(1.4) can be used to describe the dynamics of plasma, where the compressible fluid of electron interacts with its own electric field against a charged ion background, see Degond [3].

The purpose of the present paper is to investigate the quasineutral limit of the compressible Navier–Stokes–Poisson system (1.1)–(1.4). We shall prove rigorously that, as the Debye length $\lambda\to0$, the solution of the compressible Navier–Stokes–Poisson system converges strongly to the strong solution of the incompressible Navier–Stokes equations plus a term of fast singular oscillating gradient vector fields as long as the strong solution of the latter exists. Moreover, we also consider the convergence of the compressible Navier–Stokes–Poisson system (1.1)–(1.4) to the incompressible Euler equations by performing the combined quasineutral, vanishing viscosity and vanishing heat conductivity limit, i.e. $\lambda\to0$ and $\mu,\nu,\kappa\to0$.

We first give some formal analysis. We use the subscript λ to indicate that the unknowns are dependent on λ and set $\phi_\lambda=\lambda\Phi_\lambda$. Thus, we can rewrite the system (1.1)–(1.4) as

$$\partial_t\rho_\lambda+\operatorname{div}(\rho_\lambda\mathbf{u}_\lambda)=0,\tag{1.6}$$

$$\rho_\lambda\{\partial_t\mathbf{u}_\lambda+(\mathbf{u}_\lambda\cdot\nabla)\mathbf{u}_\lambda\}+\nabla(\rho_\lambda\theta_\lambda)+\frac{1}{\lambda}\rho_\lambda\nabla\phi_\lambda=\mu\Delta\mathbf{u}_\lambda+(\nu+\mu)\nabla\operatorname{div}\mathbf{u}_\lambda,\tag{1.7}$$

$$\rho_\lambda\{\partial_t\theta_\lambda+(\mathbf{u}_\lambda\cdot\nabla)\theta_\lambda\}+\rho_\lambda\theta_\lambda\operatorname{div}\mathbf{u}_\lambda=\kappa\Delta\theta_\lambda+\nu(\operatorname{div}\mathbf{u}_\lambda)^2+2\mu\mathbb{D}(\mathbf{u}_\lambda):\mathbb{D}(\mathbf{u}_\lambda),\tag{1.8}$$

$$-\lambda\Delta\phi_\lambda=\rho_\lambda-1.\tag{1.9}$$

The system (1.6)–(1.9) is equipped with the initial data

$$\rho_\lambda(x,0)=\rho_{0\lambda}(x),\qquad\mathbf{u}_\lambda(x,0)=\mathbf{u}_{0\lambda}(x),\qquad\theta_\lambda(x,0)=\theta_{0\lambda}(x).\tag{1.10}$$

Letting $\lambda\to0$ formally in the Poisson equation (1.9), we have $\rho_\lambda=1$. Moreover, if we assume that

$$\mathbf{u}_\lambda\to\mathbf{v},\qquad\theta_\lambda\to\theta$$

as $\lambda\to0$, we may expect that the compressible Navier–Stokes–Poisson system (1.6)–(1.9) converges to the incompressible Navier–Stokes equations (see [17])

$$\begin{cases}\nabla\cdot\mathbf{v}=0,\\ \partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla\Pi=\mu\Delta\mathbf{v},\\ \partial_t\theta+(\mathbf{v}\cdot\nabla)\theta=\kappa\Delta\theta+\dfrac{\mu}{2}\displaystyle\sum_{i,j=1}^N(\partial_iv_j+\partial_jv_i)^2,\end{cases}\tag{1.11}$$

as the Debye length goes to zero, where $\nabla\Pi$ is expected to be taken as the limit of the singular electric field and the gradient of pressure together. Furthermore, if we let $\mu \to 0$ and $\kappa \to 0$ in (1.11), it yields the incompressible Euler equations

$$\begin{cases} \nabla \cdot \mathbf{v} = 0, \\ \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla\Pi = 0, \\ \partial_t \theta + (\mathbf{v} \cdot \nabla)\theta = 0. \end{cases} \tag{1.12}$$

Recently, there are many progresses on the quasineutral limit of the compressible isentropic Navier–Stokes–Poisson system (i.e. the system (1.6), (1.7) and (1.9) with the pressure $P_\lambda = a\rho_\lambda^\gamma$, $\gamma > 1$, $a > 0$), Wang [23] studied the quasineutral limit for the smooth solution with well-prepared initial data. Wang and Jiang [24] studied the combined quasineutral and inviscid limit of the compressible Navier–Stokes–Poisson system for weak solution and obtained the convergence of Navier–Stokes–Poisson system to the incompressible Euler equations with general initial data. In [24], the vanishing of viscosity coefficients was required in order to take the quasineutral limit and no convergence rate was derived therein. Ju, Li and Wang [11] improved the arguments in [24] and obtained the convergence rate. Donatelli and Marcati [4] investigated the quasineutral limit of the isentropic Navier–Stokes–Poisson system in the whole space $\mathbb{R}^3$ and obtained the convergence of weak solution of the Navier–Stokes–Poisson system to the weak solution of the incompressible Navier–Stokes equations by means of dispersive estimates of Strichartz's type under the assumption that the Mach number is related to the Debye length. Notice that their arguments cannot be applied to the periodic case since the dispersive phenomenon disappears in this situation. Ju, Li and Wang [10] studied the quasineutral limit of the isentropic Navier–Stokes–Poisson system both in the whole space and in the torus without restriction on the viscosity coefficients.

However, there is no analysis on the quasineutral limit of the compressible non-isentropic Navier–Stokes–Poisson system yet. In the present paper, we shall consider the *general ill-prepared initial data* for the system (1.6)–(1.9), so the fast oscillating singular term will be produced by the non-divergence free part of initial momentum, and has to be described carefully in order to pass into the quasineutral limit.

In order to describe the oscillations in time, we introduce the following group $\mathcal{L} = e^{\tau L}$, $\tau \in \mathbb{R}$, where L is the operator defined on the space $\mathcal{H} = (L^2(\mathbb{T}^N))^N \times \{\nabla\psi,\ \psi \in H^1(\mathbb{T}^N)\}$ by

$$L\begin{pmatrix} \mathbf{w} \\ 0 \end{pmatrix} = 0, \quad \text{if } \operatorname{div}\mathbf{w} = 0,$$
$$L\begin{pmatrix} \nabla q \\ \nabla\psi \end{pmatrix} = \begin{pmatrix} -\nabla\psi \\ \nabla q \end{pmatrix}. \tag{1.13}$$

Then it is easy to check that $e^{\tau L}$ is an isometry on space $H^s(\mathbb{T}^N) \times H^s(\mathbb{T}^N)$. Let us consider the evolution of velocity and electric field. From (1.7) and (1.9), it is easy to obtain the following equation

$$\partial_t \nabla\phi_\lambda - \frac{1}{\lambda}\mathcal{Q}\mathbf{u}_\lambda = -\mathcal{Q}\big(\mathbf{u}_\lambda \nabla \cdot (\nabla\phi_\lambda)\big), \tag{1.14}$$

where the operator $\mathcal{Q}\mathbf{v} = \nabla\Delta^{-1}\nabla \cdot \mathbf{v}$ is the Leray's projector on the space of gradient of vector field $\mathbf{v} \in (L^2(\mathbb{T}^N))^N$, which is defined as follows

$$\mathcal{Q}\mathbf{v} = \nabla\Delta^{-1}\nabla \cdot \mathbf{v}, \qquad \mathcal{P}\mathbf{v} = (I - \mathcal{Q})\mathbf{v}, \qquad \nabla \cdot \mathcal{P}\mathbf{v} = 0.$$

We project the momentum equation (1.7) on the "gradient vector fields" to obtain

$$\partial_t \mathcal{Q}\mathbf{u}_\lambda + \frac{1}{\lambda}\nabla\phi_\lambda = -\mathcal{Q}\big((\mathbf{u}_\lambda\cdot\nabla)\mathbf{u}_\lambda\big) - \mathcal{Q}\left(\frac{1}{\rho_\lambda}\nabla P_\lambda\right) + \mu\mathcal{Q}(\Delta\mathbf{u}_\lambda) + (\nu+\mu)\mathcal{Q}(\nabla\operatorname{div}\mathbf{u}_\lambda) + \mu\mathcal{Q}\left(\left(\frac{1}{\rho_\lambda}-1\right)\Delta\mathbf{u}_\lambda\right) + (\nu+\mu)\mathcal{Q}\left(\left(\frac{1}{\rho_\lambda}-1\right)\nabla\operatorname{div}\mathbf{u}_\lambda\right). \tag{1.15}$$

Define

$$U_\lambda = \begin{pmatrix}\mathcal{Q}\mathbf{u}_\lambda\\ \nabla\phi_\lambda\end{pmatrix}, \qquad V_\lambda = \mathcal{L}\left(-\frac{t}{\lambda}\right)U_\lambda.$$

Then we can rewrite the system (1.14)–(1.15) as

$$\partial_t V_\lambda = \mathcal{L}\left(-\frac{t}{\lambda}\right)\begin{pmatrix}k_0\\ k_1\end{pmatrix} \tag{1.16}$$

with

$$k_0 = -\mathcal{Q}\big((\mathbf{u}_\lambda\cdot\nabla)\mathbf{u}_\lambda\big) - \mathcal{Q}\left(\frac{1}{\rho_\lambda}\nabla P_\lambda\right) + \mu\mathcal{Q}(\Delta\mathbf{u}_\lambda) + (\nu+\mu)\mathcal{Q}(\nabla\operatorname{div}\mathbf{u}_\lambda) + \mu\mathcal{Q}\left(\left(\frac{1}{\rho_\lambda}-1\right)\Delta\mathbf{u}_\lambda\right) + (\nu+\mu)\mathcal{Q}\left(\left(\frac{1}{\rho_\lambda}-1\right)\nabla\operatorname{div}\mathbf{u}_\lambda\right), \tag{1.17}$$

$$k_1 = \mathcal{Q}\big(\mathbf{u}_\lambda\nabla\cdot(\nabla\phi_\lambda)\big). \tag{1.18}$$

Now we can construct the oscillating terms as follows. Let $\mathbf{v}\in C([0,T];\ H^s(\mathbb{T}^N))$ be a divergence free function. Consider the following linear system

$$\begin{cases} \partial_t\nabla q + \frac{1}{2}\mathcal{Q}\big((\mathbf{v}\cdot\nabla)\nabla q + (\nabla q\cdot\nabla)\mathbf{v} + \mathbf{v}\Delta q\big) - (\mu+\nu/2)\nabla\operatorname{div}(\nabla q) = 0,\\ \partial_t\nabla p + \frac{1}{2}\mathcal{Q}\big((\mathbf{v}\cdot\nabla)\nabla p + (\nabla p\cdot\nabla)\mathbf{v} + \mathbf{v}\Delta p\big) - (\mu+\nu/2)\nabla\operatorname{div}(\nabla p) = 0 \end{cases} \tag{1.19}$$

with initial data

$$\big(\nabla q(x,0), \nabla p(x,0)\big) = \big(\mathcal{Q}\mathbf{u}_0(x), \nabla\phi_0(x)\big).$$

It is direct to prove that there exists a unique global smooth solution $(\nabla q, \nabla p)$ to the oscillating system (1.19) satisfying

$$\big\|(\nabla q, \nabla p)(t)\big\|_{H^s(\mathbb{T}^N)} \leqslant C(T)\big\|(\mathcal{Q}\mathbf{u}_0, \nabla\phi_0)\big\|_{H^s(\mathbb{T}^N)}, \tag{1.20}$$

where $C(T)>0$ is a constant depending only on T.

Define

$$\begin{pmatrix}\mathbf{u}_{\mathrm{osc}}(x,t)\\ \nabla\phi_{\mathrm{osc}}(x,t)\end{pmatrix} = \mathcal{L}\left(\frac{t}{\lambda}\right)\begin{pmatrix}\nabla q(x,t)\\ \nabla p(x,t)\end{pmatrix}. \tag{1.21}$$

Before stating our results rigorously, we first recall the local well-posedness result on the initial value problem for the incompressible Navier–Stokes system (1.11) in multi-dimension. One can refer to [17] for the proof.

Proposition 1.1. *Assume that* $s \geqslant N/2 + 1$ *and*

$$\begin{cases} \mathbf{v}(x,0) = \mathbf{v}_0(x) \in H^{s+3}, & \operatorname{div} \mathbf{v}_0 = 0, \\ \theta(x,0) = \theta_0(x) \in H^{s+3}, & \inf_{x \in \mathbb{T}^N} \theta_0(x) > 0. \end{cases} \tag{1.22}$$

Then there exists some time T^* $(0 < T^* \leqslant +\infty)$ *such that the initial problem* (1.11) *and* (1.22) *admits a unique strong solution* $(\mathbf{v}, \theta)$ *satisfying, for any* $T < T^*$,

$$\mathbf{v} \in C^i([0,T], H^{s+3-i}), \quad i = 0, 1, \qquad \|\mathbf{v}(t)\|_{H^{s+3}} \leqslant C_0 \|\mathbf{v}_0\|_{H^{s+3}}, \tag{1.23}$$

$$\theta \in C^i([0,T], H^{s+3-i}), \quad i = 0, 1, \qquad \|\theta(t)\|_{H^{s+3}} \leqslant C_0 \|\mathbf{v}_0\|_{H^{s+3}} \tag{1.24}$$

with $C_0 > 0$ *a constant. Moreover, if* $N = 2$, *the initial problem* (1.11) *and* (1.22) *admits a global unique strong solution* $(\mathbf{v}, \theta) \in C^i([0,\infty), H^{s+3-i})$, $i = 0, 1$.

Our main results of this paper read as follows.

Theorem 1.2. *Let* $0 < T < T^*$ *defined in Proposition* 1.1 *and suppose that* $(\mathbf{v}, \theta) \in C^i([0,T], H^{s+3-i})$, $i = 0, 1$, $s > N/2 + 2$, *be the unique strong solution of the initial problem* (1.11) *and* (1.22). *Assume that the initial data* $(\rho_{0\lambda}(x), \mathbf{u}_{0\lambda}(x), \theta_{0\lambda}(x))$ *satisfies*

$$\rho_{0\lambda}(x) = 1 - \lambda \Delta \phi_{0\lambda}(x), \qquad \inf_{x \in \mathbb{T}^N} \rho_{0\lambda}(x) > 0, \qquad \nabla \phi_{0\lambda} \in H^{s+1}(\mathbb{T}^N), \tag{1.25}$$

$$\mathbf{u}_{0\lambda} \in H^s(\mathbb{T}^N), \qquad \theta_{0\lambda}(x) \in H^s(\mathbb{T}^N), \qquad \inf_{x \in \mathbb{T}^N} \theta_{0\lambda}(x) > 0, \tag{1.26}$$

and

$$\|\mathcal{P}\mathbf{u}_{0\lambda} - \mathbf{v}_0\|_{H^s} + \|\mathcal{Q}\mathbf{u}_{0\lambda} - \mathcal{Q}\mathbf{u}_0\|_{H^s} \leqslant \tilde{C}\lambda, \tag{1.27}$$

$$\|\rho_{0\lambda}(x) - 1 + \lambda \Delta \phi_0(x)\|_{H^s} \leqslant \tilde{C}\lambda^2, \qquad \|\theta_{0\lambda} - \theta_0\|_{H^s} \leqslant \tilde{C}\lambda \tag{1.28}$$

for some constant $\tilde{C} > 0$, *where* ϕ_0 *and* $\mathbf{u}_0$ *are defined by* (2.1). *Then there is a small constant* $\delta_T > 0$ *such that, for any* $\lambda \in (0, \delta_T]$, *the initial value problem for Navier–Stokes–Poisson system* (1.6)–(1.9) *admits a unique classical solution* $(\rho_\lambda, \mathbf{u}_\lambda, \theta_\lambda, \phi_\lambda)$ *on* $[0,T]$ *satisfying*

$$\sup_{0 \leqslant t \leqslant T} \|(\rho_\lambda, \mathbf{u}_\lambda, \theta_\lambda)(t)\|_{H^s} + \sup_{0 \leqslant t \leqslant T} \|\nabla \phi_\lambda(t)\|_{H^{s+1}} \leqslant C_1 \tag{1.29}$$

uniformly with respect to λ. *Moreover, it holds that*

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant T} \left\{ \|(\rho_\lambda - 1)(t)\|_{H^s} + \|(\mathbf{u}_\lambda - \mathbf{v} - \mathbf{u}_{\mathrm{osc}})(t)\|_{H^s} + \|(\theta_\lambda - \theta)(t)\|_{H^s} \right\} \\ &\quad + \sup_{0 \leqslant t \leqslant T} \|(\nabla \phi_\lambda - \nabla \phi_{\mathrm{osc}})(t)\|_{H^{s+1}} \leqslant C_2 \lambda \end{aligned} \tag{1.30}$$

with $C_2 > 0$ *independent of* λ.

If we further perform the combined quasineutral, vanishing viscosity and vanishing heat conductivity limit, i.e. $\lambda \to 0$ and $\mu, \nu, \kappa \to 0$, we obtain the convergence of the Navier–Stokes–Poisson system (1.1)–(1.4) to the incompressible Euler equations (1.12). Namely,

Theorem 1.3. *Let $0 < T < T^{**}$ and suppose that $(\mathbf{v}, \theta) \in C^i([0, T], H^{s+3-i})$, $i = 0, 1$, $s > N/2 + 2$, be the unique strong solution of the initial problem* (1.12) *and* (1.22)*, where T^{**} is the maximal existing time of $(\mathbf{v}, \theta)$. Assume that the initial data $(\rho_{0\lambda}(x), \mathbf{u}_{0\lambda}(x), \theta_{0\lambda}(x))$ satisfies the conditions* (1.25)–(1.28)*. Then, there is a small constant $\bar{\delta}_T > 0$ such that, for any $\lambda \in (0, \bar{\delta}_T]$, the initial value problem for Navier–Stokes–Poisson system* (1.6)–(1.9) *admits a unique classical solution $(\rho_\lambda, \mathbf{u}_\lambda, \theta_\lambda, \phi_\lambda)$ on $[0, T]$ satisfying*

$$\sup_{0\leqslant t\leqslant T} \big\|(\rho_\lambda, \mathbf{u}_\lambda, \theta_\lambda)(t)\big\|_{H^s} + \sup_{0\leqslant t\leqslant T} \big\|\nabla\phi_\lambda(t)\big\|_{H^{s+1}} \leqslant C_3 \tag{1.31}$$

uniformly with respect to λ as $\mu, \nu, \kappa \to 0$. Moreover, it holds that

$$\begin{aligned}\sup_{0\leqslant t\leqslant T} \big\{\big\|(\rho_\lambda - 1)(t)\big\|_{H^s} + \big\|(\mathbf{u}_\lambda - \mathbf{v} - \mathbf{u}_{\mathrm{osc}})(t)\big\|_{H^s} + \big\|(\theta_\lambda - \theta)(t)\big\|_{H^s}\big\}& \\ + \sup_{0\leqslant t\leqslant T} \big\|(\nabla\phi_\lambda - \nabla\phi_{\mathrm{osc}})(t)\big\|_{H^{s+1}} \leqslant C_4\lambda&\end{aligned} \tag{1.32}$$

with $C_4 > 0$ independent of λ. Here $(\mathbf{v}, \theta)$ is the unique strong solution of the initial problem (1.12) *and* (1.22)*, and $(\mathbf{u}_{\mathrm{osc}}, \phi_{\mathrm{osc}})$ is the fast singular oscillating gradient velocity vector field and electric field defined by* (1.19) *and* (1.21) *with $\mu = \nu \equiv 0$.*

Remark 1.1. The method developed in this paper can be applied to the situation when the doping function is a perturbation of a constant state

$$\mathcal{C}(x) = 1 + \lambda g(x)$$

with $g(x) \in C^2(\mathbb{T}^N)$, a given function, satisfying $\int_{\mathbb{T}^N} g\,dx = 0$.

Remark 1.2. We believe that the method developed in this paper can be also applied to investigate the quasineutral limit problem to more complex model such as the full Navier–Stokes–Poisson system with more general pressure, which will be studied in a forthcoming paper.

The proofs of Theorems 1.2 and 1.3 mainly consist of three steps. First, we apply the homogenization technique to construct the approximate solution to the classical solution (if exists) of the system (1.6)–(1.9). Then by using the theories of symmetric quasilinear hyperbolic system and the estimates of second order elliptic equations, we show that the remainder term exists in the same time interval as the approximate term for fixed small $\lambda > 0$. Moreover, we obtain the uniform estimates with respect to λ (the uniform estimates with respect to μ, ν and κ can also be obtained by further analysis). These facts are sufficient for us to complete the proofs of Theorems 1.2 and 1.3.

It should be noted that the quasineutral limit is a well-known challenging and modelling problem in fluid dynamics and kinetic models for semiconductors and plasmas. In both cases there exist only partial results. In particular, the quasineutral limit has been performed in Vlasov–Poisson system by Brenier [1], Grenier [5], and Masmoudi [18], in Vlasov–Poisson–Fokker–Planck system by Hsiao, Li and Wang [7,8], in Schrödinger–Poisson system by Puel [21], Jüngel and Wang [13], and Ju et al. [9], in drift-diffusion-Poisson system by Gasser et al. [6], Jüngel and Peng [12], Wang et al. [25]. For the hydrodynamic model, besides the results mentioned above for the Navier–Stokes–Poisson system, there are also many results on Euler–Poisson system, for example, for the isentropic Euler–Poisson system [2,19,22,23] and for non-isentropic Euler–Poisson system [16,20]. Li and Lin [14] considered the quasineutral limit to the isentropic quantum hydrodynamical model with the help of modulated energy method for general initial data.

Before ending this section, we recall the following Moser-type calculus inequalities which will be used frequently in the sequel.

Proposition 1.4 *(Moser-type inequalities). (See [15].)*

(1) *For $f, g \in H^s \cap L^\infty$ and $|\alpha| \leqslant s$, it holds that*

$$\left\|D^\alpha(fg)\right\|_{L^2} \leqslant C_s\left(\|f\|_{L^\infty}\left\|D^s g\right\|_{L^2} + \|g\|_{L^\infty}\left\|D^s\right\|_{L^2}\right). \tag{1.33}$$

(2) *For $f \in H^s$, $Df \in L^\infty$, $g \in H^{s-1} \cap L^\infty$ and $|\alpha| \leqslant s$, it holds that*

$$\left\|D^\alpha(fg) - fD^\alpha(g)\right\|_{L^2} \leqslant C_s\left(\|Df\|_{L^\infty}\left\|D^{s-1}g\right\|_{L^2} + \|g\|_{L^\infty}\left\|D^s f\right\|_{L^2}\right). \tag{1.34}$$

Notations. In this paper, C and C_i $(i = 1, 2, \ldots)$ denote the generic positive constants, which may change from line to line and are independent of λ. $C(T)$ and $C_i(T)$ denote the constant depending on the time T. H^s denotes the standard Sobolev space $W^{s,2}(\mathbb{T}^N)$. For the multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$, we denote $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}$ and $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$.

The rest of this paper is arranged as follows. In Section 2, we construct the approximate solutions to the problem (1.6)–(1.10). In Section 3, we establish the local existence of solution to the remainder system and obtain the uniform estimates. The proofs of our main results are given in Section 4.

2. Construction of approximate solutions

In this section we shall construct the approximation to the system (1.6)–(1.9). Noticing the fast singular oscillating vector fields $(\mathbf{u}_{\text{osc}}, \nabla\phi_{\text{osc}})$ obtained by (1.21), we find that the fast singular oscillating vector fields $(\mathbf{u}_{\text{osc}}, \nabla\phi_{\text{osc}})$ satisfy

$$\begin{cases} \partial_t \mathbf{u}_{\text{osc}} + \dfrac{1}{2}\mathcal{Q}\big((\mathbf{v}\cdot\nabla)\mathbf{u}_{\text{osc}} + (\mathbf{u}_{\text{osc}}\cdot\nabla)\mathbf{v} + \mathbf{v}\nabla\cdot\mathbf{u}_{\text{osc}}\big) \\ \qquad - (\mu+\nu/2)\nabla\operatorname{div}\mathbf{u}_{\text{osc}} + \dfrac{1}{\lambda}\nabla\phi_{\text{osc}} = 0, \\ \partial_t \nabla\phi_{\text{osc}} + \dfrac{1}{2}\mathcal{Q}\big((\mathbf{v}\cdot\nabla)\nabla\phi_{\text{osc}} + (\nabla\phi_{\text{osc}}\cdot\nabla)\mathbf{v} + \mathbf{v}\Delta\phi_{\text{osc}}\big) \\ \qquad - (\mu+\nu/2)\nabla\Delta\phi_{\text{osc}} - \dfrac{1}{\lambda}\mathbf{u}_{\text{osc}} = 0, \\ \big(\mathbf{u}_{\text{osc}}(x,0), \nabla\phi_{\text{osc}}(x,0)\big) = \big(\mathcal{Q}\mathbf{u}_0(x), \nabla\phi_0(x)\big). \end{cases} \tag{2.1}$$

Thus it is natural to define

$$\rho_{\text{osc}} = -\Delta\phi_{\text{osc}}.$$

We conclude that the fast oscillating part $(\rho_{\text{osc}}, \mathbf{u}_{\text{osc}}, \phi_{\text{osc}})$ satisfies the following initial value problem

$$\begin{cases} \partial_t \rho_{\text{osc}} + [\mathbf{v}+\mathbf{u}_{\text{osc}}]\cdot\nabla\rho_{\text{osc}} + \dfrac{1}{\lambda}(1+\lambda\rho_{\text{osc}})\nabla\cdot\mathbf{u}_{\text{osc}} = k_2, \\ \partial_t \mathbf{u}_{\text{osc}} + \big([\mathbf{v}+\mathbf{u}_{\text{osc}}]\cdot\nabla\big)\mathbf{u}_{\text{osc}} + (\mathbf{u}_{\text{osc}}\cdot\nabla)\mathbf{v} + \dfrac{1}{\lambda}\nabla\phi_{\text{osc}} = k_3, \\ -\Delta\phi_{\text{osc}} = \rho_{\text{osc}}, \\ \rho_{\text{osc}}(x,0) = -\Delta\phi_0(x), \quad \mathbf{u}_{\text{osc}}(x,0) = \mathcal{Q}\mathbf{u}_0(x), \end{cases} \tag{2.2}$$

where

$$k_2 = \nabla\cdot\big(\rho_{\rm osc}[\mathbf{v}+\mathbf{u}_{\rm osc}]\big) + \frac{1}{2}\nabla\cdot\big((\mathbf{v}\cdot\nabla)\nabla\phi_{\rm osc} + (\nabla\phi_{\rm osc}\cdot\nabla)\mathbf{v} + \mathbf{v}\Delta\phi_{\rm osc}\big) - (\mu+\nu/2)\Delta^2\phi_{\rm osc}, \quad (2.3)$$

$$k_3 = \frac{1}{2}\mathcal{Q}\big((\mathbf{v}\cdot\nabla)\mathbf{u}_{\rm osc} + (\mathbf{u}_{\rm osc}\cdot\nabla)\mathbf{v} - \mathbf{v}\nabla\cdot\mathbf{u}_{\rm osc}\big) + (\mathbf{u}_{\rm osc}\cdot\nabla)\mathbf{u}_{\rm osc}$$
$$+ \mathcal{P}\big((\mathbf{v}\cdot\nabla)\mathbf{u}_{\rm osc} + (\mathbf{u}_{\rm osc}\cdot\nabla)\mathbf{v}\big) + (\mu+\nu/2)\nabla\,\mathrm{div}\,\mathbf{u}_{\rm osc}. \quad (2.4)$$

Moreover, by virtue of (1.20) and (1.21), we obtain that

$$\|k_2\|_{H^{s-2}(\mathbb{T}^N)} + \|k_3\|_{H^{s-2}(\mathbb{T}^N)} \leqslant C\big\|(\nabla\phi_0, \mathcal{Q}\mathbf{u}_0, \mathbf{v}_0)\big\|_{H^s(\mathbb{T}^N)}, \quad (2.5)$$

where the constant $C > 0$ is independent of λ. To approximate the classical solution $W = (\rho_\lambda, \mathbf{u}_\lambda, \theta_\lambda, \phi_\lambda)^{\rm T}$ of the initial value problem (1.6)–(1.10) for small λ, we still need to introduce an additional correction term

$$W_{\rm cor} = (\lambda\rho_{\rm cor}, \mathbf{u}_{\rm cor}, \theta_{\rm cor}, \phi_{\rm cor})^{\rm T}.$$

By utilizing the fast singular oscillating part and the given functions k_2 and k_3, we can construct $(\rho_{\rm cor}, \mathbf{u}_{\rm cor}, \theta_{\rm cor}, \phi_{\rm cor})$ by solving the following linear initial value problem

$$\begin{cases} \partial_\tau\mathbf{u}_{\rm cor} + \nabla\phi_{\rm cor} = k_4, \\ \partial_\tau\nabla\phi_{\rm cor} - \mathbf{u}_{\rm cor} = \nabla(-\Delta)^{-1}k_2, \\ \rho_{\rm cor} = -\Delta\phi_{\rm cor}, \\ \partial_\tau\theta_{\rm cor} = k_5, \\ (\mathbf{u}_{\rm cor}, \nabla\phi_{\rm cor}, \theta_{\rm cor})(x,0) = (\mathbf{0}, \mathbf{0}, 0), \end{cases} \quad (2.6)$$

where

$$k_4 = -k_3 - \nabla\theta + \mu\Delta\mathbf{u}_{\rm osc} + (\mu+\nu)\nabla\,\mathrm{div}\,\mathbf{u}_{\rm osc},$$

$$k_5 = -\mathbf{u}_{\rm osc}\cdot\nabla\theta - \theta\nabla\cdot\mathbf{u}_{\rm osc} + \nu(\mathrm{div}\,\mathbf{u}_{\rm osc})^2 + \frac{\mu}{2}\sum_{i,j=1}^{N}\big(\partial_i v_j + \partial_j v_i + \partial_i u^j_{\rm osc} + \partial_j u^i_{\rm osc}\big)^2.$$

Here we recall that $(\mathbf{v}, \theta)$ is the solution to the system (1.11).

By virtue of (1.21), (1.23), (1.24) and (2.5), it is easy to prove the following existence results of solutions to the problems (2.2) and (2.6).

Proposition 2.1. *Let $T > 0$, $T < T^*$ be given. Let $\mathbf{v}, \theta \in C^i([0,T], H^{s+3-i})$, $i = 0, 1$, $s > 1 + N/2$, be the solution to the initial value problem* (1.11) *and* (1.22)*. Then the problem* (2.2) *admits a unique classical solution $(\rho_{\rm osc}, \mathbf{u}_{\rm osc}, \nabla\phi_{\rm osc})^{\rm T}$ for $t \in [0,T]$ satisfying*

$$\big\|\rho_{\rm osc}(t)\big\|_{H^{s+2}} + \big\|(\mathbf{u}_{\rm osc}, \nabla\phi_{\rm osc})(t)\big\|_{H^{s+3}(\mathbb{T}^N)} \leqslant C_T, \quad (2.7)$$

and the problem (2.6) *admits a unique classical solution $(\rho_{\rm cor}, \mathbf{u}_{\rm cor}, \theta_{\rm cor}, \nabla\phi_{\rm cor})^{\rm T}$ for $t \in [0,T]$ satisfying*

$$\big\|\rho_{\rm cor}(\tau)\big\|_{H^{s+1}} + \big\|(\mathbf{u}_{\rm cor}, \theta_{\rm cor}, \nabla\phi_{\rm cor})(\tau)\big\|_{H^{s+2}(\mathbb{T}^N)} \leqslant C_T, \quad (2.8)$$

where $C_T > 0$ depends only on T and the initial data $(\mathbf{v}_0, \theta_0, \mathcal{Q}\mathbf{u}_0, \nabla\phi_0)$, but is independent of λ.

According to Propositions 1.1 and 2.1, we can make the following asymptotic expansions of the solution $(\rho_\lambda, \mathbf{u}_\lambda, \theta_\lambda, \phi_\lambda)$

$$\begin{cases} \rho_\lambda(x,t) = 1 + \lambda\rho_{\mathrm{osc}}(x,t) + \lambda^2\big(\Delta\Pi(x,t) + \rho_{\mathrm{cor}}(x,t/\lambda)\big) + \lambda^2\rho_{\mathrm{rem}}(x,t), \\ \mathbf{u}_\lambda(x,t) = \mathbf{v} + \mathbf{u}_{\mathrm{osc}}(x,t) + \lambda\mathbf{u}_{\mathrm{cor}}(x,t/\lambda) + \lambda\mathbf{u}_{\mathrm{rem}}(x,t), \\ \theta_\lambda(x,t) = \theta(x,t) + \lambda\theta_{\mathrm{cor}}(x,t/\lambda) + \lambda\theta_{\mathrm{rem}}(x,t), \\ \phi_\lambda(x,t) = \phi_{\mathrm{osc}}(x,t) + \lambda\big(\Pi(x,t) + \phi_{\mathrm{cor}}(x,t/\lambda)\big) + \lambda\phi_{\mathrm{rem}}(x,t). \end{cases} \tag{2.9}$$

Substituting (2.9) into the Navier–Stokes–Poisson system (1.6)–(1.9), using (1.11), (2.2) and (2.6), and by tedious but direct computations, we can show that $(\rho_{\mathrm{rem}}, \mathbf{u}_{\mathrm{rem}}, \theta_{\mathrm{rem}}, \phi_{\mathrm{rem}})$ solves the following initial value problem

$$\begin{cases} \partial_t\rho_{\mathrm{rem}} + \mathbf{u}_\lambda\cdot\nabla\rho_{\mathrm{rem}} + \dfrac{1}{\lambda}\rho_\lambda \operatorname{div}\mathbf{u}_{\mathrm{rem}} = h_0, \\ \partial_t\mathbf{u}_{\mathrm{rem}} + (\mathbf{u}_\lambda\cdot\nabla)\mathbf{u}_{\mathrm{rem}} + \lambda\dfrac{\theta_\lambda}{\rho_\lambda}\nabla\rho_{\mathrm{rem}} + \nabla\theta_{\mathrm{rem}} \\ \quad -\mu\Delta\mathbf{u}_{\mathrm{rem}} - (\mu+\nu)\nabla\operatorname{div}\mathbf{u}_{\mathrm{rem}} = -\dfrac{1}{\lambda}\nabla\phi_{\mathrm{rem}} + \mathbf{f}_0, \\ \partial_t\theta_{\mathrm{rem}} + \mathbf{u}_\lambda\cdot\nabla\theta_{\mathrm{rem}} + \theta_\lambda\operatorname{div}\mathbf{u}_{\mathrm{rem}} - \kappa\Delta\theta_{\mathrm{rem}} = \lambda\nu(\operatorname{div}\mathbf{u}_{\mathrm{rem}})^2 \\ \quad + \dfrac{\mu\lambda}{2}\displaystyle\sum_{i,j=1}^{N}\big(\partial_i u^j_{\mathrm{rem}} + \partial_j u^i_{\mathrm{rem}}\big)^2 + g_0, \\ -\Delta\phi_{\mathrm{rem}} = \rho_{\mathrm{rem}} \end{cases} \tag{2.10}$$

with initial data

$$\begin{cases} \rho_{\mathrm{rem}}(x,0) = \dfrac{1}{\lambda^2}\big[\rho_{0\lambda}(x) - 1 + \lambda\Delta\phi_0(x)\big] - \Delta\Pi(x,0), \\ \mathbf{u}_{\mathrm{rem}}(x,0) = \dfrac{1}{\lambda}\big[\mathbf{u}_{0\lambda}(x) - \mathbf{v}_0(x) - \mathcal{Q}\mathbf{u}_0(x)\big], \\ \theta_{\mathrm{rem}}(x,0) = \dfrac{1}{\lambda}\big[\theta_{0\lambda}(x) - \theta_0(x)\big]. \end{cases} \tag{2.11}$$

In (2.10), we denote

$$\begin{aligned} h_0 = {} & -\mathbf{u}_{\mathrm{rem}}\cdot\nabla\rho_{\mathrm{osc}} - \rho_{\mathrm{rem}}\nabla\cdot(\mathbf{u}_{\mathrm{osc}} + \lambda\mathbf{u}_{\mathrm{cor}}) - \nabla\cdot(\rho_{\mathrm{osc}}\mathbf{u}_{\mathrm{cor}}) \\ & - (\mathbf{v} + \mathbf{u}_{\mathrm{osc}} + \lambda\mathbf{u}_{\mathrm{cor}} + \lambda\mathbf{u}_{\mathrm{rem}})\cdot\nabla\rho_{\mathrm{cor}} - \rho_{\mathrm{cor}}\nabla\cdot(\mathbf{u}_{\mathrm{osc}} + \lambda\mathbf{u}_{\mathrm{cor}}) \\ & - \Delta\Pi_t - \big(\nabla(\Delta\Pi)\big)(\mathbf{v} + \mathbf{u}_{\mathrm{osc}} + \lambda\mathbf{u}_{\mathrm{cor}} + \lambda\mathbf{u}_{\mathrm{rem}}) \\ & - \Delta\Pi\operatorname{div}(\mathbf{u}_{\mathrm{osc}} + \lambda\mathbf{u}_{\mathrm{cor}}), \end{aligned} \tag{2.12}$$

$$\mathbf{f}_0 = \mathbf{f}_{01} + \mathbf{f}_{02}, \tag{2.13}$$

$$g_0 = g_{01} + g_{02} \tag{2.14}$$

with

$$\begin{aligned} \mathbf{f}_{01} = {} & -\big((\mathbf{u}_{\mathrm{cor}} + \mathbf{u}_{\mathrm{rem}})\cdot\nabla\big)(\mathbf{v} + \mathbf{u}_{\mathrm{osc}}) - \big((\mathbf{v} + \mathbf{u}_{\mathrm{osc}} + \lambda\mathbf{u}_{\mathrm{cor}} + \lambda\mathbf{u}_{\mathrm{rem}})\cdot\nabla\big)\mathbf{u}_{\mathrm{cor}} \\ & - \frac{\theta_\lambda}{\rho_\lambda}\nabla\big(\rho_{\mathrm{osc}} + \lambda(\Delta\Pi + \rho_{\mathrm{cor}})\big) - \nabla\theta_{\mathrm{cor}}, \end{aligned}$$

$$\begin{aligned}
\mathbf{f}_{02} &= \mu\Delta\mathbf{u}_{\rm cor} + (\mu+\nu)\nabla\ \operatorname{div}\mathbf{u}_{\rm cor}\\
&\quad - \frac{\mu}{\rho_\lambda}\big(\rho_{\rm osc} + \lambda(\Delta\Pi + \rho_{\rm cor}) + \lambda\rho_{\rm rem}\big)\Delta(\mathbf{v} + \mathbf{u}_{\rm osc} + \lambda\mathbf{u}_{\rm cor} + \lambda\mathbf{u}_{\rm rem})\\
&\quad - \frac{\mu+\nu}{\rho_\lambda}\big(\rho_{\rm osc} + \lambda(\Delta\Pi + \rho_{\rm cor}) + \lambda\rho_{\rm rem}\big)\nabla\operatorname{div}(\mathbf{v} + \mathbf{u}_{\rm osc} + \lambda\mathbf{u}_{\rm cor} + \lambda\mathbf{u}_{\rm rem}),
\end{aligned}$$

$$g_{01} = -(\mathbf{u}_{\rm cor} + \mathbf{u}_{\rm rem})\nabla\theta - (\mathbf{v} + \mathbf{u}_{\rm osc} + \lambda\mathbf{u}_{\rm cor} + \lambda\mathbf{u}_{\rm rem})\nabla\theta_{\rm cor} - (\theta_{\rm cor} + \theta_{\rm rem})\operatorname{div}\mathbf{u}_{\rm osc} + \theta_\lambda\operatorname{div}\mathbf{u}_{\rm cor},$$

$$\begin{aligned}
g_{02} &= \kappa\Delta\theta_{\rm cor} - \frac{\kappa}{\rho_\lambda}\big(\rho_{\rm osc} + \lambda(\Delta\Pi + \rho_{\rm cor}) + \lambda\rho_{\rm rem}\big)\Delta(\theta + \lambda\theta_{\rm cor} + \lambda\theta_{\rm rem})\\
&\quad + 2\nu\operatorname{div}\mathbf{u}_{\rm osc}(\operatorname{div}\mathbf{u}_{\rm cor} + \operatorname{div}\mathbf{u}_{\rm rem}) + \lambda\nu(\operatorname{div}\mathbf{u}_{\rm cor})^2 + 2\lambda\nu\operatorname{div}\mathbf{u}_{\rm cor}\operatorname{div}\mathbf{u}_{\rm rem}\\
&\quad + \mu\sum_{i,j=1}^N\big(\partial_i v_j + \partial_j v_i + \partial_i u^j_{\rm osc} + \partial_j u^i_{\rm osc}\big)\big(\partial_i u^j_{\rm cor} + \partial_j u^i_{\rm cor} + \partial_i u^j_{\rm rem} + \partial_j u^i_{\rm rem}\big)\\
&\quad + \frac{\mu\lambda}{2}\sum_{i,j=1}^N\big(\partial_i u^j_{\rm cor} + \partial_j u^i_{\rm cor}\big)^2 + \mu\lambda\sum_{i,j=1}^N\big(\partial_i u^j_{\rm cor} + \partial_j u^i_{\rm cor}\big)\big(\partial_i u^j_{\rm rem} + \partial_j u^i_{\rm rem}\big)\\
&\quad - \frac{1}{\rho_\lambda}(\rho_{\rm osc} + \lambda\Delta\Pi + \lambda\rho_{\rm cor} + \lambda\rho_{\rm rem})\left[\nu(\operatorname{div}\mathbf{u}_\lambda)^2 + \frac{\mu}{2}\sum_{i,j=1}^N\big(\partial_i u^j_\lambda + \partial_j u^i_\lambda\big)^2\right].
\end{aligned}$$

If we denote

$$U_{\rm rem} := (\rho_{\rm rem}, \mathbf{u}_{\rm rem}, \theta_{\rm rem})^{\rm T},$$

the problem (2.10)–(2.11) can be rewritten as follows

$$\begin{cases}
\partial_t U_{\rm rem} + \sum_{j=1}^N A_j(x,t,U_{\rm rem})\partial_{x_j}U_{\rm rem} - \mu\Delta\tilde{\mathbf{u}}_{\rm rem} - (\mu+\nu)\nabla\operatorname{div}\tilde{\mathbf{u}}_{\rm rem} - \kappa\Delta\tilde\theta_{\rm rem}\\
\quad = \lambda\nu J + \frac{\lambda\mu}{2}G + \frac{1}{\lambda}B + F(x,t,U_{\rm rem}),\\
-\Delta\phi_{\rm rem} = \rho_{\rm rem},\\
U_{\rm rem}(x,0) = \big(\rho_{\rm rem}(x,0), \mathbf{u}_{\rm rem}(x,0), \theta_{\rm rem}(x,0)\big)^{\rm T} := U_{\rm rem0}(x).
\end{cases} \tag{2.15}$$

Here the matrices A_j $(j = 1,\dots,N)$ are defined as

$$A_j(x,t,U_{\rm rem}) \equiv u^j_\lambda I_{(N+2)\times(N+2)} + \begin{pmatrix} 0 & \frac{1}{\lambda}\rho_\lambda e_j & 0\\ \frac{\lambda\theta_\lambda}{\rho_\lambda}e_j^{\rm T} & O & e_j^{\rm T}\\ 0 & \theta_\lambda e_j & 0\end{pmatrix}$$

and

$$\begin{gathered}
\tilde{\mathbf{u}}_{\rm rem} = (0, \mathbf{u}_{\rm rem}, 0)^{\rm T}, \qquad \tilde\theta_{\rm rem} = (0,\dots,0,\theta_{\rm rem})^{\rm T},\\
J = \big(0,\dots,0,(\operatorname{div}\mathbf{u}_{\rm rem})^2\big)^{\rm T}, \qquad F = (h_0, \mathbf{f}_0, g_0)^{\rm T},\\
G = \left(0,\dots,0,\sum_{i,j=1}^N\big(\partial_i u^j_{\rm rem} + \partial_j u^i_{\rm rem}\big)^2\right)^{\rm T}, \qquad B = (0, -\nabla\phi_{\rm rem}, 0)^{\rm T}.
\end{gathered}$$

3. Local existence of solution to the remainder system (2.15)

In this section we study the local existence of smooth solution to the remainder system (2.15), our result reads

Theorem 3.1. *Let $T > 0$, $T < T^*$ be given and $\mathbf{v}, \theta \in C^i([0,T], H^{s+3-i})$, $i = 0, 1$, $s > 2 + N/2$, be the solution to the problem* (1.11) *and* (1.22)*. Then there exists a constant $\delta_T > 0$ such that for any $\lambda \in (0, \delta_T]$, the initial value problem* (2.15) *admits a unique classical solution $(U_{\mathrm{rem}}, \phi_{\mathrm{rem}})$ in $[0,T]$ satisfying*

$$\sup_{0 \leqslant t \leqslant T} \left(\left\| (\lambda \rho_{\mathrm{rem}}, \mathbf{u}_{\mathrm{rem}}, \theta_{\mathrm{rem}})(t) \right\|_{H^s} + \left\| \nabla \phi_{\mathrm{rem}}(t) \right\|_{H^{s+1}} \right) \leqslant C(T), \tag{3.1}$$

where $C(T)$ is a positive constant independent of λ.

The proof of Theorem 3.1 proceeds via a priori energy estimates and the classical iteration scheme. The crucial step is to show the following energy estimates which can be obtained by performing the refined energy estimates for the quasilinear symmetric hyperbolic–parabolic system and the Poisson equation.

Lemma 3.2. *Let $T > 0$ be given and $s \geqslant N/2 + 2$. There exist positive constants δ_T, M, $\tilde{M}$ such that the classical solution $(U_{\mathrm{rem}}, \phi_{\mathrm{rem}})$ to the initial value problem* (2.15) *satisfies*

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant T} \left(\left\| (\lambda \rho_{\mathrm{rem}}, \mathbf{u}_{\mathrm{rem}}, \theta_{\mathrm{rem}})(t) \right\|_{H^s}^2 + \left\| \nabla \phi_{\mathrm{rem}}(t) \right\|_{H^{s+1}}^2 \right) \\ &\quad + \int_0^T \left\| \mathbf{u}_{\mathrm{rem}}(s) \right\|_{H^{s+1}}^2 dt + \int_0^T \left\| \theta_{\mathrm{rem}}(s) \right\|_{H^{s+1}}^2 dt \leqslant M^2, \end{aligned} \tag{3.2}$$

and

$$\sup_{0 \leqslant t \leqslant T} \left(\left\| \lambda \partial_t \rho_{\mathrm{rem}}(t) \right\|_{H^{s-1}} + \left\| \lambda \partial_t \mathbf{u}_{\mathrm{rem}}(t) \right\|_{H^{s-2}} + \left\| \partial_t \theta_{\mathrm{rem}}(t) \right\|_{H^{s-2}} + \left\| \lambda \partial_t \nabla \phi_{\mathrm{rem}}(t) \right\|_{H^s} \right) \leqslant \tilde{M} \tag{3.3}$$

uniformly with respect to $\lambda \in (0, \delta_T]$.

Proof. We assume a priori that the classical solution to initial value problem (2.15) satisfies (3.2) and (3.3). Then our task is to determine these unknown constants by energy estimates.

Noticing the matrices $A_j(x, t, U_{\mathrm{rem}})$, $j = 1, \ldots, N$, can be symmetrized by

$$A_0(x, t, U_{\mathrm{rem}}) = \begin{pmatrix} \lambda^2 \frac{\theta_\lambda}{\rho_\lambda} & O & 0 \\ O & \rho_\lambda I_{N \times N} & O \\ 0 & O & \frac{\rho_\lambda}{\theta_\lambda} \end{pmatrix},$$

we rewrite the system (2.15) in the following form

$$\begin{cases} A_0(U_{\mathrm{rem}}) \partial_t U_{\mathrm{rem}} + \displaystyle\sum_{j=1}^N \mathcal{A}_j(x, t, U_{\mathrm{rem}}) \partial_{x_j} U_{\mathrm{rem}} - \mu \rho_\lambda \Delta \tilde{\mathbf{u}}_{\mathrm{rem}} \\ \qquad - (\mu + \nu) \rho_\lambda \nabla \operatorname{div} \tilde{\mathbf{u}}_{\mathrm{rem}} - \dfrac{\kappa \rho_\lambda}{\theta_\lambda} \Delta \tilde{\theta}_{\mathrm{rem}} \\ \quad = \lambda \nu \tilde{J} + \dfrac{\lambda \mu}{2} \tilde{G} + \dfrac{1}{\lambda} \tilde{B} + \tilde{F}(x, t, U_{\mathrm{rem}}), \\ -\Delta \phi_{\mathrm{rem}} = \rho_{\mathrm{rem}}, \\ U_{\mathrm{rem}}(x, 0) = U_{\mathrm{rem}0}(x), \end{cases} \tag{3.4}$$

where $\mathcal{A}_j = A_0 A_j,\ j = 1, \dots, N$, are symmetric matrices given by

$$\mathcal{A}_j(U_{\mathrm{rem}}) = u_\lambda^j A_0(U_{\mathrm{rem}}) + \begin{pmatrix} 0 & \lambda\theta_\lambda e_j & 0 \\ \lambda\theta_\lambda e_j^{\mathrm{T}} & O & \rho_\lambda e_j^{\mathrm{T}} \\ 0 & \rho_\lambda e_j & 0 \end{pmatrix}$$

and

$$\tilde{J} := A_0 J = \left(0, \dots, 0, \frac{\rho_\lambda}{\theta_\lambda}(\operatorname{div}\mathbf{u}_{\mathrm{rem}})^2\right)^{\mathrm{T}},$$

$$\tilde{G} := A_0 G = \left(0, \dots, 0, \frac{\rho_\lambda}{\theta_\lambda}\sum_{i,j=1}^{N}\left(\partial_i u_{\mathrm{rem}}^j + \partial_j u_{\mathrm{rem}}^i\right)^2\right)^{\mathrm{T}},$$

$$\tilde{B} := A_0 B = (0, -\rho_\lambda \nabla\phi_{\mathrm{rem}}, 0)^{\mathrm{T}},$$

$$\tilde{F} := A_0 F = \left(\frac{\lambda^2\theta_\lambda h_0}{\rho_\lambda}, \rho_\lambda \mathbf{f}_0, \frac{\rho_\lambda g_0}{\theta_\lambda}\right)^{\mathrm{T}}.$$

Next we perform energy estimates for the classical solution to the system (2.15) with initial data (2.11). Define the canonical energy by

$$\|U_{\mathrm{rem}}\|_E^2 := \int \left\langle A_0(U_{\mathrm{rem}})U_{\mathrm{rem}}, U_{\mathrm{rem}}\right\rangle dx.$$

Multiplying $(3.4)_1$ by U_{rem} and integrating the result by parts, we get the basic energy equality of Friedrich's

$$\begin{aligned}
&\frac{d}{dt}\|U_{\mathrm{rem}}\|_E^2 + 2\mu\int |\nabla\mathbf{u}_{\mathrm{rem}}|^2\,dx + 2(\mu+\nu)\int |\operatorname{div}\mathbf{u}_{\mathrm{rem}}|^2\,dx + 2\kappa\int \frac{\rho_\lambda}{\theta_\lambda}|\nabla\theta_{\mathrm{rem}}|^2\,dx \\
&\quad= \int \langle \Gamma U_{\mathrm{rem}}, U_{\mathrm{rem}}\rangle\,dx + 2\lambda\nu\int \frac{1}{\theta_\lambda}(\operatorname{div}\mathbf{u}_{\mathrm{rem}})^2\theta_{\mathrm{rem}}\,dx \\
&\qquad+ \lambda\mu\sum_{i,j=1}^{N}\int \frac{1}{\theta_\lambda}\left(\partial_i u_{\mathrm{rem}}^j + \partial_j u_{\mathrm{rem}}^i\right)^2\theta_{\mathrm{rem}}\,dx - \frac{2}{\lambda}\int \rho_\lambda\nabla\phi_{\mathrm{rem}}\mathbf{u}_{\mathrm{rem}}\,dx \\
&\qquad+ 2\int \langle A_0 F, U_{\mathrm{rem}}\rangle\,dx + R_1, \qquad (3.5)
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= 2(\mu+\nu)\int (\rho_\lambda - 1)\nabla\operatorname{div}\mathbf{u}_{\mathrm{rem}}\mathbf{u}_{\mathrm{rem}}\,dx + 2\mu\int (\rho_\lambda - 1)\Delta\mathbf{u}_{\mathrm{rem}}\mathbf{u}_{\mathrm{rem}}\,dx \\
&\quad- 2\kappa\int \nabla\left(\frac{\rho_\lambda}{\theta_\lambda}\right)\nabla\theta_{\mathrm{rem}}\theta_{\mathrm{rem}}\,dx \qquad (3.6)
\end{aligned}$$

and

$$\Gamma = (\partial_t, \nabla)\cdot(A_0, \mathcal{A}_1, \dots, \mathcal{A}_3).$$

Since $\mu > 0$, $2\mu + N\nu > 0$, there exists a positive constant ξ_1 such that

$$\mu \int |\nabla \mathbf{u}_{\mathrm{rem}}|^2\, dx + (\mu + \nu) \int |\operatorname{div} \mathbf{u}_{\mathrm{rem}}|^2\, dx \geqslant \xi_1 \int |\nabla \mathbf{u}_{\mathrm{rem}}|^2\, dx \tag{3.7}$$

in view of $\int (\operatorname{div} \mathbf{u}_{\mathrm{rem}})^2\, dx \leqslant \int |\nabla \mathbf{u}_{\mathrm{rem}}|^2\, dx$. Notice the fact that there is a $\delta_T > 0$ such that for $\lambda \in (0, \lambda_T]$ it holds that

$$0 < \rho_- \leqslant 1 + \lambda \rho_{\mathrm{osc}} + \lambda^2 \Delta \Pi + \lambda^2 \rho_{\mathrm{cor}} + \lambda^2 \rho_{\mathrm{rem}} \leqslant \rho_+, \tag{3.8}$$

$$0 < \theta_- \leqslant \theta + \lambda \theta_{\mathrm{cor}} + \lambda \theta_{\mathrm{rem}} \leqslant \theta_+, \tag{3.9}$$

where $\rho_\pm$ and $\theta_\pm$ are positive constants. Thus, the matrices A_0 and $\mathcal{A}_j$, $j = 1, \ldots, N$, together with their derivatives are continuous and bounded uniformly. Moreover, A_0 is uniformly positive definite, i.e. there exists a constant $c_0 > 0$ such that

$$\big\langle A_0(U_{\mathrm{rem}}) U_{\mathrm{rem}}, U_{\mathrm{rem}} \big\rangle \geqslant c_0 \big(\lambda^2 \rho_{\mathrm{rem}}^2 + \mathbf{u}_{\mathrm{rem}}^2 + \theta_{\mathrm{rem}}^2 \big) \tag{3.10}$$

for all U_{rem}.

Now we estimate the terms on the right-hand side of (3.5). Since Γ is bounded there exists a generic constant M_0, independent of $(\rho_{\mathrm{rem}}, \mathbf{u}_{\mathrm{rem}}, \theta_{\mathrm{rem}}, \phi_{\mathrm{rem}})$ and $\lambda > 0$, such that

$$\int \langle \Gamma U_{\mathrm{rem}}, U_{\mathrm{rem}} \rangle\, dx \leqslant M_0 \big(1 + \lambda (M + \tilde{M})\big) \|U_{\mathrm{rem}}\|_E^2. \tag{3.11}$$

By Sobolev's embedding inequality and the inequality (3.9) we obtain that

$$\begin{aligned} &2\lambda \nu \int \frac{1}{\theta_\lambda} (\operatorname{div} \mathbf{u}_{\mathrm{rem}})^2 \theta_{\mathrm{rem}}\, dx + \lambda \mu \sum_{i,j=1}^{N} \int \frac{1}{\theta_\lambda} \big(\partial_i u_{\mathrm{rem}}^j + \partial_j u_{\mathrm{rem}}^i \big)^2 \theta_{\mathrm{rem}}\, dx \\ &\quad \leqslant \lambda M_0 M (2\mu + \nu) \int \big(|\nabla \mathbf{u}_{\mathrm{rem}}|^2 + |\theta_{\mathrm{rem}}|^2 \big)\, dx. \end{aligned} \tag{3.12}$$

By integrating by parts, Cauchy's inequality and the equation for ρ_{rem} in (2.15), the forth term on the right-hand side of (3.5) is estimated as follows

$$\begin{aligned} &-\frac{2}{\lambda} \int \rho_\lambda \nabla \phi_{\mathrm{rem}} \mathbf{u}_{\mathrm{rem}}\, dx \\ &\quad = \frac{2}{\lambda} \int \rho_\lambda \operatorname{div} \mathbf{u}_{\mathrm{rem}} \phi_{\mathrm{rem}}\, dx + \frac{2}{\lambda} \int \nabla \rho_\lambda \mathbf{u}_{\mathrm{rem}} \phi_{\mathrm{rem}}\, dx \\ &\quad = -2 \int \partial_t \rho_{\mathrm{rem}} \phi_{\mathrm{rem}}\, dx - 2 \int (\mathbf{v} + \mathbf{u}_{\mathrm{osc}} + \lambda \mathbf{u}_{\mathrm{cor}} + \lambda \mathbf{u}_{\mathrm{rem}}) \nabla \rho_{\mathrm{rem}} \phi_{\mathrm{rem}}\, dx \\ &\qquad + 2 \iint h_0 \phi_{\mathrm{rem}}\, dx + 2 \int \nabla \big(\rho_{\mathrm{osc}} + \lambda (\Delta \Pi + \rho_{\mathrm{cor}}) + \lambda \rho_{\mathrm{rem}} \big) \mathbf{u}_{\mathrm{rem}} \phi_{\mathrm{rem}}\, dx \\ &\quad \leqslant -\partial_t \|\nabla \phi_{\mathrm{rem}}\|_{L^2}^2 + M_0 (1 + \lambda M) \big(\|\nabla \phi_{\mathrm{rem}}\|_{L^2}^2 + \|U_{\mathrm{rem}}\|_E^2 \big) + \epsilon_1 \|\nabla \mathbf{u}_{\mathrm{rem}}\|_{L^2}^2 \end{aligned} \tag{3.13}$$

for some sufficiently small constant $\epsilon_1 > 0$.

Now we deal with the term R_1. By integrating by parts and using Sobolev's inequality, we get

$$\begin{aligned} &2(\mu+\nu)\int(\rho_\lambda-1)\nabla\operatorname{div}\mathbf{u}_{\mathrm{rem}}\mathbf{u}_{\mathrm{rem}}\,dx+2\mu\int(\rho_\lambda-1)\Delta\mathbf{u}_{\mathrm{rem}}\mathbf{u}_{\mathrm{rem}}\,dx \\ &\quad\leqslant \lambda M_0(M+1)(2\mu+\nu)\int\big(|\nabla\mathbf{u}_{\mathrm{rem}}|^2+|\mathbf{u}_{\mathrm{rem}}|^2\big)\,dx. \end{aligned} \tag{3.14}$$

In view of (3.8), (3.9) and Cauchy's inequality, we obtain that

$$\begin{aligned} &-2\kappa\int\nabla\frac{\rho_\lambda}{\theta_\lambda}\nabla\theta_{\mathrm{rem}}\theta_{\mathrm{rem}}\,dx \\ &\quad=-2\kappa\int\frac{\nabla\rho_\lambda}{\theta_\lambda}\nabla\theta_{\mathrm{rem}}\theta_{\mathrm{rem}}\,dx+2\kappa\int\frac{\rho_\lambda}{(\theta_\lambda)^2}\nabla\theta_\lambda\nabla\theta_{\mathrm{rem}}\theta_{\mathrm{rem}}\,dx \\ &\quad\leqslant \lambda M_0(M+1)\kappa\int\big(|\nabla\theta_{\mathrm{rem}}|^2+|\theta_{\mathrm{rem}}|^2\big)\,dx \\ &\qquad+M_0\kappa\int|\theta_{\mathrm{rem}}|^2\,dx+\epsilon_2\kappa\int|\nabla\theta_{\mathrm{rem}}|^2\,dx \end{aligned} \tag{3.15}$$

for some sufficiently small constant $\epsilon_2>0$.

The estimate of the fifth term on the right-hand side of (3.5) is tedious but straightforward. In view of the definitions of h_0, $\mathbf{f}_0$, and g_0 in (2.12)–(2.14), and Propositions 1.1 and 2.1, we get

$$\begin{aligned} &2\lambda^2\int\frac{\theta_\lambda}{\rho_\lambda}h_0\rho_{\mathrm{rem}}\,dx+2\int\rho_\lambda\mathbf{f}_{01}\mathbf{u}_{\mathrm{rem}}\,dx+2\int\frac{\rho_\lambda}{\theta_\lambda}g_{01}\theta_{\mathrm{rem}}\,dx \\ &\quad\leqslant M_0\|U_{\mathrm{rem}}\|_E^2+M_0 \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &2\int\rho_\lambda\mathbf{f}_{02}\cdot\mathbf{u}_{\mathrm{rem}}\,dx+2\int\frac{\rho_\lambda}{\theta_\lambda}g_{02}\theta_{\mathrm{rem}}\,dx \\ &\quad\leqslant \lambda(2\mu+\nu+\kappa)M_0(1+M)\int\big(|\mathbf{u}_{\mathrm{rem}}|^2+|\nabla\mathbf{u}_{\mathrm{rem}}|^2+|\nabla\theta_{\mathrm{rem}}|^2\big)\,dx \\ &\qquad+(2\mu+\nu+k+1)M_0. \end{aligned} \tag{3.17}$$

We choose δ_T sufficiently small such that, for $\lambda\in(0,\delta_T]$,

$$\lambda M_0(M+1)(2\mu+\nu+\kappa)\leqslant\min\left\{\frac{\xi_1}{2},\frac{\kappa\rho_-}{2\theta_+}\right\}:=\eta_1. \tag{3.18}$$

Choosing ϵ_1 and ϵ_2 sufficiently small and combining (3.7)–(3.18) with (3.5), we obtain that

$$\begin{aligned} &\frac{d}{dt}\big(\|U_{\mathrm{rem}}\|_E^2+\|\nabla\phi_{\mathrm{rem}}\|_{L^2}^2\big)+\frac{\xi_1}{2}\int|\nabla\mathbf{u}_{\mathrm{rem}}|^2\,dx+\frac{k\rho_-}{2\theta_+}\int|\nabla\theta_{\mathrm{rem}}|^2\,dx \\ &\quad\leqslant M_0\big(1+\lambda(M+\tilde M)\big)\big(\|U_{\mathrm{rem}}\|_E^2+\|\nabla\phi_{\mathrm{rem}}\|_{L^2}^2\big)+3\eta\int\big(|\mathbf{u}_{\mathrm{rem}}|^2+|\theta_{\mathrm{rem}}|^2\big)\,dx \\ &\qquad+\kappa M_0\int|\theta_{\mathrm{rem}}|^2\,dx+(2\mu+\nu+\kappa+1)M_0. \end{aligned} \tag{3.19}$$

Next we shall obtain the energy estimates of higher order derivatives for the classical solutions to the initial value problem (2.15). For the multi-index α with $1 \leqslant |\alpha| \leqslant s$, we take the operator D^α to (2.15) and multiply the resulting equations by A_0 to obtain

$$\begin{cases} A_0(U_{\mathrm{rem}})\partial_t D^\alpha U_{\mathrm{rem}} + \sum_{j=1}^{N} \mathcal{A}_j(x,t,U_{\mathrm{rem}})\partial_{x_j} D^\alpha U_{\mathrm{rem}} - \rho_\lambda \mu \Delta D^\alpha \tilde{\mathbf{u}}_{\mathrm{rem}} \\ \quad - (\mu+\nu)\rho_\lambda \nabla \operatorname{div} D^\alpha \tilde{\mathbf{u}}_{\mathrm{rem}} - \dfrac{\kappa \rho_\lambda}{\theta_\lambda} \Delta D^\alpha \tilde{\theta}_{\mathrm{rem}} \\ = \lambda \nu A_0(U_{\mathrm{rem}}) D^\alpha J + \dfrac{\lambda\mu}{2} A_0(U_{\mathrm{rem}}) D^\alpha G + \dfrac{1}{\lambda} A_0(U_{\mathrm{rem}}) D^\alpha B \\ \quad + A_0(U_{\mathrm{rem}}) D^\alpha F + H^\alpha, \\ -\Delta D^\alpha \phi_{\mathrm{rem}} = D^\alpha \rho_{\mathrm{rem}} \end{cases} \tag{3.20}$$

with initial data

$$D^\alpha U_{\mathrm{rem}}(x,0) = D^\alpha U_{\mathrm{rem}0}(x), \tag{3.21}$$

where H^α consists of the commutating terms as

$$H^\alpha = -\sum_{j=1}^{N} A_0(U_{\mathrm{rem}})\big(D^\alpha\big(A_j(U_{\mathrm{rem}})\partial_{x_j} U_{\mathrm{rem}}\big) - A_j(U_{\mathrm{rem}})\partial_{x_j} D^\alpha U_{\mathrm{rem}}\big).$$

Taking the inner product between $(3.20)_1$ and $D^\alpha U_{\mathrm{rem}}$, we have the following differential equality

$$\begin{aligned} &\frac{d}{dt}\big\|D^\alpha U_{\mathrm{rem}}(t)\big\|_E^2 + 2\mu \int \big|\nabla D^\alpha \mathbf{u}_{\mathrm{rem}}\big|^2 dx + 2(\mu+\nu)\int \big|\operatorname{div} D^\alpha \mathbf{u}_{\mathrm{rem}}\big|^2 dx + 2\kappa \int \frac{\rho_\lambda}{\theta_\lambda}\big|D^{\alpha+1}\theta_{\mathrm{rem}}\big|^2 dx \\ &\quad = \int \big\langle \Gamma D^\alpha U_{\mathrm{rem}}, D^\alpha U_{\mathrm{rem}}\big\rangle dx + 2\lambda\nu \int \big\langle A_0(U_{\mathrm{rem}}) D^\alpha J, D^\alpha U_{\mathrm{rem}}\big\rangle dx \\ &\qquad + \lambda\mu \int \big\langle A_0(U_{\mathrm{rem}}) D^\alpha G, D^\alpha U_{\mathrm{rem}}\big\rangle dx + \frac{2}{\lambda}\int \big\langle A_0(U_{\mathrm{rem}}) D^\alpha B, D^\alpha U_{\mathrm{rem}}\big\rangle dx \\ &\qquad + 2\int \big\langle A_0(U_{\mathrm{rem}}) D^\alpha F(t), D^\alpha U_{\mathrm{rem}}\big\rangle dx + 2\int \big\langle H^\alpha(t), D^\alpha U_{\mathrm{rem}}\big\rangle dx + R_2, \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} R_2 = {}& 2\mu \int (\rho_\lambda - 1)\Delta D^\alpha \mathbf{u}_{\mathrm{rem}} D^\alpha \mathbf{u}_{\mathrm{rem}}\, dx - 2\kappa \int \nabla\left(\frac{\rho_\lambda}{\theta_\lambda}\right) \nabla D^\alpha \theta_{\mathrm{rem}} \theta_{\mathrm{rem}}\, dx \\ & + 2(\mu+\nu)\int (\rho_\lambda - 1)\nabla \operatorname{div} D^\alpha \mathbf{u}_{\mathrm{rem}} D^\alpha \mathbf{u}_{\mathrm{rem}}\, dx. \end{aligned}$$

It is easy to see that we also have the following estimate

$$\mu \int \big|\nabla D^\alpha \mathbf{u}_{\mathrm{rem}}\big|^2 dx + (\mu+\nu)\int \big|\operatorname{div} D^\alpha \mathbf{u}_{\mathrm{rem}}\big|^2 dx \geqslant \xi_2 \int \big|\nabla D^\alpha \mathbf{u}_{\mathrm{rem}}\big|^2 dx \tag{3.23}$$

for some constant $\xi_2 > 0$.

Now we deal with the right-hand side of (3.22). In the following the generic constant M_0 may depend on T and s. By integrating by part, Sobolev's inequality and Cauchy's inequality it holds, similar to (3.11) and (3.14)–(3.15), that

$$\int \langle \Gamma D^\alpha U_{\mathrm{rem}}, D^\alpha U_{\mathrm{rem}} \rangle \,dx \leqslant M_0\big(1+\lambda(M+\tilde{M})\big) \| D^\alpha U_{\mathrm{rem}} \|_E^2 \tag{3.24}$$

and

$$\begin{aligned} R_2 \leqslant{}& \lambda M_0(M+1)(2\mu+\nu+\kappa) \int \big(|\nabla D^\alpha \mathbf{u}_{\mathrm{rem}}|^2 + |D^{\alpha+1}\theta_{\mathrm{rem}}|^2 + |D^\alpha \mathbf{u}_{\mathrm{rem}}|^2 + |D^\alpha \theta_{\mathrm{rem}}|^2 \big)\,dx \\ &+ M_0\kappa \int |D^\alpha \theta_{\mathrm{rem}}|^2\,dx + \delta\kappa \int |D^{\alpha+1}\theta_{\mathrm{rem}}|^2 \end{aligned} \tag{3.25}$$

for some sufficiently small constant $\delta > 0$.

By the definition of A_0, G and J, it follows from the Sobolev's inequality that

$$\begin{aligned} &\lambda\nu \int \langle A_0(U_{\mathrm{rem}}) D^\alpha J, D^\alpha U_{\mathrm{rem}} \rangle\,dx + 2\lambda\mu \int \langle A_0(U_{\mathrm{rem}}) D^\alpha G, D^\alpha U_{\mathrm{rem}} \rangle\,dx \\ &\quad\leqslant \lambda M_0(2\mu+\nu) \left(\big\| (\operatorname{div} \mathbf{u}_{\mathrm{rem}})^2 \big\|_{H^\alpha} + \left\| \sum_{i,j=1}^N \big(\partial_i u^j_{\mathrm{rem}} + \partial_j u^i_{\mathrm{rem}}\big)^2 \right\|_{H^\alpha} \right) \|\theta_{\mathrm{rem}}\|_{H^\alpha} \\ &\quad\leqslant \lambda M_0(2\mu+\nu) \|\mathbf{u}_{\mathrm{rem}}\|_{H^\alpha}^{\frac14} \|\mathbf{u}_{\mathrm{rem}}\|_{H^{\alpha+1}}^{\frac74} \|\theta_{\mathrm{rem}}\|_{H^\alpha} \\ &\quad\leqslant \lambda M_0(2\mu+\nu) \|\mathbf{u}_{\mathrm{rem}}\|_{H^\alpha}^{\frac12} \|\mathbf{u}_{\mathrm{rem}}\|_{H^{\alpha+1}}^{\frac32} \|\theta_{\mathrm{rem}}\|_{H^\alpha}^2 + \lambda M_0(2\mu+\nu) \|\mathbf{u}_{\mathrm{rem}}\|_{H^{\alpha+1}}^2 \\ &\quad\leqslant \lambda M M_0(2\mu+\nu) \|\mathbf{u}_{\mathrm{rem}}\|_{H^{\alpha+1}}^{\frac32} \|\theta_{\mathrm{rem}}\|_{H^\alpha}^2 + \lambda M_0(2\mu+\nu) \|\mathbf{u}_{\mathrm{rem}}\|_{H^{\alpha+1}}^2. \end{aligned} \tag{3.26}$$

We deal with the fourth term on the right-hand side of (3.22). From (2.15), we can easily get the equation for $D^\alpha \rho_{\mathrm{rem}}$,

$$\partial_t D^\alpha \rho_{\mathrm{rem}} + \mathbf{u}_\lambda \cdot \nabla D^\alpha \rho_{\mathrm{rem}} + \frac{1}{\lambda} \rho_\lambda \operatorname{div} D^\alpha \mathbf{u}_{\mathrm{rem}} = D^\alpha h_0 + h^\alpha \tag{3.27}$$

with

$$h^\alpha = -D^\alpha(\mathbf{u}_\lambda \cdot \nabla \rho_{\mathrm{rem}}) + \mathbf{u}_\lambda \cdot \nabla D^\alpha \rho_{\mathrm{rem}} - \frac{1}{\lambda} D^\alpha(\rho_\lambda \operatorname{div} \mathbf{u}_{\mathrm{rem}}) + \frac{1}{\lambda} \rho_\lambda \operatorname{div} D^\alpha \mathbf{u}_{\mathrm{rem}}.$$

In view of (3.27) and the Poisson equation $(3.20)_2$, we get

$$\begin{aligned} &\frac{2}{\lambda} \int \langle A_0(U_{\mathrm{rem}}) D^\alpha B, D^\alpha U_{\mathrm{rem}} \rangle\,dx \\ &\quad= -\frac{2}{\lambda} \int \rho_\lambda \nabla D^\alpha \phi_{\mathrm{rem}} D^\alpha \mathbf{u}_{\mathrm{rem}}\,dx \\ &\quad= \frac{2}{\lambda} \int \rho_\lambda \operatorname{div} D^\alpha \mathbf{u}_{\mathrm{rem}} D^\alpha \phi_{\mathrm{rem}}\,dx + \frac{2}{\lambda} \int \nabla \rho_\lambda D^\alpha \mathbf{u}_{\mathrm{rem}} D^\alpha \phi_{\mathrm{rem}}\,dx \\ &\quad= -2\int \partial_t D^\alpha \rho_{\mathrm{rem}} D^\alpha \phi_{\mathrm{rem}}\,dx - 2\int \mathbf{u}_\lambda \nabla D^\alpha \rho_{\mathrm{rem}} D^\alpha \phi_{\mathrm{rem}}\,dx + 2\int D^\alpha h_0 D^\alpha \phi_{\mathrm{rem}}\,dx \\ &\qquad + 2\int \nabla\big(\rho_{\mathrm{osc}} + \lambda(\Delta \Pi + \rho_{\mathrm{cor}}) + \lambda \rho_{\mathrm{rem}}\big) D^\alpha \mathbf{u}_{\mathrm{rem}} D^\alpha \phi_{\mathrm{rem}}\,dx + 2\int h^\alpha D^\alpha \phi_{\mathrm{rem}}\,dx \end{aligned}$$

$$\leqslant -\frac{d}{dt}\left\| D^\alpha \nabla \phi_{\mathrm{rem}} \right\|_{L^2}^2 + M_0(1+\lambda M)\left(\left\| D^\alpha \nabla \phi_{\mathrm{rem}} \right\|_{L^2}^2 + \sum_{0 \leqslant |\beta| \leqslant |\alpha|} \left\| D^\beta U_{\mathrm{rem}} \right\|_E^2 \right)$$
$$+ \epsilon_3 \int \left\| \nabla D^\alpha \mathbf{u}_{\mathrm{rem}} \right\|^2 dx \tag{3.28}$$

for some sufficiently small constant $\epsilon_3 > 0$.

The fifth term on the right-hand side of (3.22) is very tedious. The main techniques involved are Leibniz's formula, Moser-type calculus inequalities (1.33)–(1.34), and Sobolev's embedding inequalities. Actually, after the tedious computations, we finally obtain the following estimate

$$2\int \left\langle A_0(U_{\mathrm{rem}}) D^\alpha F(x,t,U_{\mathrm{rem}}), D^\alpha U_{\mathrm{rem}} \right\rangle dx$$
$$\leqslant \lambda(2\mu+\nu+\kappa)M_0(1+M)\left[\sum_{0 \leqslant |\beta| \leqslant |\alpha|} \left(\left\| \nabla D^\beta \mathbf{u}_{\mathrm{rem}} \right\|_{L^2}^2 + \left\| \nabla D^\beta \theta_{\mathrm{rem}} \right\|_{L^2}^2 \right) + \sum_{0 \leqslant |\beta| \leqslant |\alpha|} \left\| D^\beta U_{\mathrm{rem}} \right\|_E^2 \right]$$
$$+ (2\mu+\nu+\kappa+1)M_0. \tag{3.29}$$

The commutating term H^α can be bounded by

$$\int \left\langle H^\alpha(t), D^\alpha U_{\mathrm{rem}} \right\rangle dx \leqslant \sum_{1 \leqslant |\beta| \leqslant |\alpha|} M_0(1+\lambda M)\left\| D^\beta U_{\mathrm{rem}} \right\|_E^2 + \left\| D^\alpha U_{\mathrm{rem}} \right\|_E^2 + M_0. \tag{3.30}$$

We now re-choose δ_T sufficiently small such that, for $\lambda \in (0, \delta_T]$,

$$\lambda s M_0(M+1)(2\mu+\nu+\kappa) \leqslant \min\left\{ \frac{\xi_2}{2}, \frac{\kappa\rho_-}{2\theta_+} \right\} := \eta_2. \tag{3.31}$$

Let

$$\Phi(t) = \lambda^2 \|\rho_{\mathrm{rem}}\|_{H^s}^2 + \|\mathbf{u}_{\mathrm{rem}}\|_{H^s}^2 + \|\theta_{\mathrm{rem}}\|_{H^s}^2. \tag{3.32}$$

Taking δ and ϵ_3 small enough and combining the estimates (3.24)–(3.30) with (3.22) and (3.19), we obtain that

$$c_0\Phi(t) + \|\nabla\phi_{\mathrm{rem}}\|_{H^s}^2 + \frac{\xi}{2}\int_0^t \|\mathbf{u}_{\mathrm{rem}}\|_{H^{s+1}}^2 dr + \frac{\kappa\rho_-}{2\theta_+}\int_0^t \|\theta\|_{H^{s+1}}^2 dr$$
$$\leqslant \int_0^t \left\{ M_0\left(M_0\left(1+\lambda(M+\tilde{M})\right) + 3\eta + M_0\kappa + \lambda(2\mu+\nu)MM_0\|\mathbf{u}_{\mathrm{rem}}\|_{H^{s+1}}^{\frac{3}{2}}\right) \right.$$
$$\left. \times \left(c_0\Phi(r) + \|\nabla\phi_{\mathrm{rem}}\|_{H^s}^2(r)\right) \right\} dr$$
$$+ c_0\Phi(0) + \left\|\nabla\phi_{\mathrm{rem}}(0)\right\|_{H^s}^2 + M_0(2\mu+\nu+\kappa)T, \tag{3.33}$$

where $\xi = \min\{\xi_1, \xi_2\}$ and $\eta = \max\{\eta_1, \eta_2\}$. By virtue of Gronwall's inequality, we obtain that

$$
\begin{aligned}
&c_0\Phi(t)+\|\nabla\phi_{\mathrm{rem}}\|_{H^s}^2\\
&\quad\leqslant\big(c_0\Phi(0)+\big\|\nabla\phi_{\mathrm{rem}}(0)\big\|_{H^s}^2+M_0(2\mu+\nu+\kappa)T\big)\\
&\qquad\times\exp\Bigg\{M_0\int_0^t\big[M_0\big(1+\lambda(M+\tilde M)\big)+3\eta+M_0\kappa+\lambda(2\mu+\nu)MM_0\|\mathbf{u}_{\mathrm{rem}}\|_{H^{s+1}}^{\frac32}\big]\,dr\Bigg\}. \qquad (3.34)
\end{aligned}
$$

From (3.2) and Hölder's inequality, we have

$$
\lambda(2\mu+\nu)MM_0\int_0^t\|\mathbf{u}_{\mathrm{rem}}\|_{H^{s+1}}^{\frac32}\,dr\leqslant\lambda M_0(2\mu+\nu)M^{\frac74}T^{\frac14}. \qquad (3.35)
$$

In view of (1.27) and (1.28), we obtain that

$$
\lambda^2\big\|\rho_{\mathrm{rem}}(0)\big\|_{H^s}^2\leqslant\tilde C\lambda^2,\qquad\big\|\mathbf{u}_{\mathrm{rem}}(0)\big\|_{H^s}^2+\big\|\theta_{\mathrm{rem}}(0)\big\|_{H^s}^2\leqslant\tilde C \qquad (3.36)
$$

and

$$
\|\nabla\phi_{\mathrm{rem}}\|_{H^s}^2\leqslant\tilde C. \qquad (3.37)
$$

We choose δ_T sufficiently small such that, for $\lambda\in(0,\delta_T]$, it holds that

$$
\lambda(M+\tilde M)+\lambda(2\mu+\nu)M^{\frac74}<1. \qquad (3.38)
$$

Set

$$
L_1=M_0\big(2M_0+3\eta+M_0\kappa+M_0T^{1/4}\big).
$$

Substituting (3.35)–(3.38) into (3.34), we obtain that

$$
\begin{aligned}
c_0\Phi(t)+\|\nabla\phi_{\mathrm{rem}}\|_{H^s}^2&\leqslant\big(c_0\Phi(0)+\big\|\nabla\phi_{\mathrm{rem}}(0)\big\|_{H^s}^2+M_0(2\mu+\nu+\kappa)T\big)e^{L_1T}\\
&\leqslant\big(M_0\tilde C+M_0(2\mu+\nu+\kappa)T\big)e^{L_1T}=:L_3. \qquad (3.39)
\end{aligned}
$$

In view of (3.33), we get that

$$
\frac{\xi}{2}\int_0^t\|\mathbf{u}_{\mathrm{rem}}\|_{H^{s+1}}^2\,dr+\frac{\kappa\rho_-}{2\theta_+}\int_0^t\|\theta\|_{H^{s+1}}^2\,dr\leqslant L_1L_3T+M_0\tilde C+M_0(2\mu+\nu+\kappa)T. \qquad (3.40)
$$

Therefore (3.2) is proved if we set

$$
M^2=:\big(L_3+L_1L_3T+M_0\tilde C+M_0(2\mu+\nu+\kappa)T\big)\cdot\max\left\{\frac{1}{c_0},1,\frac{2}{\xi},\frac{2\theta_+}{\kappa\rho_-}\right\}. \qquad (3.41)
$$

It follows from (3.20) that

$$
\sup_{0\leqslant t\leqslant T}\big(\lambda\big\|\partial_t\rho_{\mathrm{rem}}(t)\big\|_{H^{s-1}}+\lambda\big\|\partial_t\mathbf{u}_{\mathrm{rem}}(t)\big\|_{H^{s-2}}+\big\|\partial_t\theta_{\mathrm{rem}}(t)\big\|_{H^{s-1}}+\lambda\big\|\partial_t\nabla\phi_{\mathrm{rem}}(t)\big\|_{H^s}\big)\leqslant\tilde M \qquad (3.42)
$$

with

$$\tilde{M} := \big(M_0(1+2M)\big)^{1/2}. \tag{3.43}$$

The proof of Lemma 3.2 is completed. □

Proof of Theorem 3.1. With the a priori estimates (3.2) and (3.3), we now start the proof of Theorem 3.1. We first construct the approximate solutions. Define

$$\big(U_{\mathrm{rem}}^{n+1}, \phi_{\mathrm{rem}}^{n+1}\big) = \big(\rho_{\mathrm{rem}}^{n+1}, \mathbf{u}_{\mathrm{rem}}^{n+1}, \theta_{\mathrm{rem}}^{n+1}, \phi_{\mathrm{rem}}^{n+1}\big)^{\mathrm{T}} \quad (n \geqslant 0)$$

inductively as the solution of linear equations

$$\begin{cases} A_0\big(U_{\mathrm{rem}}^n\big)\partial_t U_{\mathrm{rem}}^{n+1} + \displaystyle\sum_{j=1}^N \mathcal{A}_j\big(x,t,U_{\mathrm{rem}}^n\big)\partial_{x_j}U_{\mathrm{rem}}^{n+1} - \mu\rho_\lambda^n \Delta\tilde{\mathbf{u}}_{\mathrm{rem}}^{n+1} \\ \qquad - (\mu+\nu)\rho_\lambda^n \nabla \operatorname{div}\tilde{\mathbf{u}}_{\mathrm{rem}}^{n+1} - \dfrac{\kappa\rho_\lambda^n}{\theta_\lambda^n}\Delta\tilde{\theta}_{\mathrm{rem}}^{n+1} \\ \quad = \lambda\nu\tilde{J}^n + \dfrac{\lambda\mu}{2}\tilde{G}^n + \dfrac{1}{\lambda}\tilde{B}^{n+1} + \tilde{F}^n, \\ -\Delta\phi_{\mathrm{rem}}^n = \rho_{\mathrm{rem}}^n \end{cases} \tag{3.44}$$

with initial data

$$U_{\mathrm{rem}}^n(x,0) = U_{\mathrm{rem}0}(x), \tag{3.45}$$

where

$$\rho_\lambda^n(x,t) = 1 + \lambda\rho_{\mathrm{osc}}(x,t) + \lambda^2\big(\Delta\Pi(x,t) + \rho_{\mathrm{cor}}(x,t/\lambda)\big) + \lambda^2\rho_{\mathrm{rem}}^n(x,t),$$
$$\mathbf{u}_\lambda^n(x,t) = \mathbf{v} + \mathbf{u}_{\mathrm{osc}}(x,t) + \lambda\mathbf{u}_{\mathrm{cor}}(x,t/\lambda) + \lambda\mathbf{u}_{\mathrm{rem}}^n(x,t),$$
$$\theta_\lambda^n(x,t) = \theta(x,t) + \lambda\theta_{\mathrm{cor}}(x,t/\lambda) + \lambda\theta_{\mathrm{rem}}^n(x,t),$$
$$\phi_\lambda^n(x,t) = \phi_{\mathrm{osc}}(x,t) + \lambda\big(\Pi(x,t) + \phi_{\mathrm{cor}}(x,t/\lambda)\big) + \lambda\phi_{\mathrm{rem}}^n(x,t),$$
$$\tilde{\mathbf{u}}_{\mathrm{rem}}^{n+1} = \big(0, \mathbf{u}_{\mathrm{rem}}^{n+1}, 0\big)^{\mathrm{T}}, \qquad \tilde{B}^{n+1} = A_0 B(x,t,U_{\mathrm{rem}}) = \big(0, -\rho_\lambda^n \nabla\phi_{\mathrm{rem}}^{n+1}, 0\big),$$
$$\tilde{J}^n := A_0 D\big(x,t,U_{\mathrm{rem}}^n\big) = \left(0,\ldots,0, \frac{\rho_\lambda^n}{\theta_\lambda^n}\big(\operatorname{div}\mathbf{u}_{\mathrm{rem}}^n\big)^2\right)^{\mathrm{T}},$$
$$\tilde{G}^n := A_0 G\big(x,t,U_{\mathrm{rem}}^n\big) = \left(0,\ldots,0, \frac{\rho_\lambda^n}{\theta_\lambda^n}\sum_{i,j=1}^N \Big(\big(\partial_i u_{\mathrm{rem}}^j\big)^n + \big(\partial_j u_{\mathrm{rem}}^i\big)^n\Big)^2\right)^{\mathrm{T}},$$
$$\tilde{F}^n = A_0 F\big(x,t,U_{\mathrm{rem}}^n\big).$$

It is standard to know that the approximate problem (3.44) admits a unique solution such that

$$\big(\rho_{\mathrm{rem}}^{n+1}, \mathbf{u}_{\mathrm{rem}}^{n+1}, \theta_{\mathrm{rem}}^{n+1}, \nabla\phi_{\mathrm{rem}}^{n+1}\big) \in C\big([0,T];H^s\big), \qquad \nabla\phi_{\mathrm{rem}}^{n+1} \in C\big([0,T];H^{s+1}\big),$$
$$\mathbf{u}_{\mathrm{rem}}^{n+1} \in L^2\big(0,T;H^{s+1}\big), \qquad \theta_{\mathrm{rem}}^{n+1} \in L^2\big(0,T;H^{s+1}\big),$$
$$\partial_t\rho_{\mathrm{rem}}^{n+1} \in C\big([0,T];H^{s-1}\big), \qquad \partial_t\mathbf{u}_{\mathrm{rem}}^{n+1} \in C\big([0,T];H^{s-2}\big),$$

$$\partial_t \theta_{\mathrm{rem}}^{n+1} \in C\big([0,T]; H^{s-2}\big), \qquad \partial_t \nabla\phi_{\mathrm{rem}}^{n+1} \in C\big([0,T]; H^{s}\big),$$

and satisfies the uniform estimates

$$\sup_{0\leqslant t\leqslant T}\big(\big\|\big(\lambda\rho_{\mathrm{rem}}^{n+1}, \mathbf{u}_{\mathrm{rem}}^{n+1}, \theta_{\mathrm{rem}}^{n+1}\big)(t)\big\|_{H^s}^2 + \big\|\nabla\phi_{\mathrm{rem}}^{n+1}\big\|_{H^{s+1}}^2\big) + \int_0^T \big\|\mathbf{u}_{\mathrm{rem}}^{n+1}\big\|_{H^{s+1}}^2\,dt + \int_0^T \big\|\theta_{\mathrm{rem}}^{n+1}\big\|_{H^{s+1}}^2\,dt \leqslant M^2, \tag{3.46}$$

$$\sup_{0\leqslant t\leqslant T}\big(\lambda^2\big\|\partial_t\rho_{\mathrm{rem}}^{n+1}(t)\big\|_{H^{s-1}}^2 + \lambda^2\big\|\partial_t\mathbf{u}_{\mathrm{rem}}^{n+1}(t)\big\|_{H^{s-2}}^2 + \big\|\partial_t\theta_{\mathrm{rem}}^{n+1}(t)\big\|_{H^{s-1}}^2 + \lambda^2\big\|\partial_t\nabla\phi_{\mathrm{rem}}^{n+1}(t)\big\|_{H^{s}}^2\big) \leqslant \tilde{M}^2. \tag{3.47}$$

It is standard to verify that the difference

$$\big(\bar\rho_{\mathrm{rem}}^{n+1}, \bar{\mathbf{u}}_{\mathrm{rem}}^{n+1}, \bar\theta_{\mathrm{rem}}^{n+1}, \bar\phi_{\mathrm{rem}}^{n+1}\big) = \big(\rho_{\mathrm{rem}}^{n+1} - \rho_{\mathrm{rem}}^{n}, \mathbf{u}_{\mathrm{rem}}^{n+1} - \mathbf{u}_{\mathrm{rem}}^{n}, \theta_{\mathrm{rem}}^{n+1} - \theta_{\mathrm{rem}}^{n}, \phi_{\mathrm{rem}}^{n+1} - \phi_{\mathrm{rem}}^{n}\big)$$

satisfies

$$\begin{cases} \partial_t\bar\rho_{\mathrm{rem}}^{n+1} + \mathbf{u}_\lambda^n\nabla\bar\rho_{\mathrm{rem}}^{n+1} + \dfrac{1}{\lambda}\rho_\lambda^n \operatorname{div}\bar{\mathbf{u}}_{\mathrm{rem}}^{n+1} \\ \quad = -\lambda\bar{\mathbf{u}}_{\mathrm{rem}}^{n}\nabla\rho_{\mathrm{rem}}^{n} - \lambda\bar\rho_{\mathrm{rem}}^{n}\operatorname{div}\mathbf{u}_{\mathrm{rem}}^{n} + h_0\big(x,t,\mathbf{u}_{\mathrm{rem}}^{n+1},\rho_{\mathrm{rem}}^{n+1}\big) - h_0\big(x,t,\mathbf{u}_{\mathrm{rem}}^{n},\rho_{\mathrm{rem}}^{n}\big), \\ \partial_t\bar{\mathbf{u}}_{\mathrm{rem}}^{n+1} + \big(\mathbf{u}_\lambda^n\cdot\nabla\big)\bar{\mathbf{u}}_{\mathrm{rem}}^{n+1} + \lambda\dfrac{\theta_\lambda^n}{\rho_\lambda^n}\nabla\bar\rho_{\mathrm{rem}}^{n+1} + \nabla\bar\theta_{\mathrm{rem}}^{n+1} - \mu\Delta\bar{\mathbf{u}}_{\mathrm{rem}}^{n+1} - (\mu+\nu)\nabla\operatorname{div}\bar{\mathbf{u}}_{\mathrm{rem}}^{n+1} \\ \quad = \dfrac{1}{\lambda}\nabla\phi_{\mathrm{rem}}^{n+1} - \lambda\big(\bar{\mathbf{u}}_{\mathrm{rem}}^{n}\cdot\nabla\big)\mathbf{u}_{\mathrm{rem}}^{n} - \lambda\bigg(\dfrac{\theta_\lambda^n}{\rho_\lambda^n} - \dfrac{\theta_\lambda^{n-1}}{\rho_\lambda^{n-1}}\bigg)\nabla\rho_{\mathrm{rem}}^{n} \\ \qquad + f_0\big(x,t,\mathbf{u}_{\mathrm{rem}}^{n+1},\rho_{\mathrm{rem}}^{n+1}\big) - f_0\big(x,t,\mathbf{u}_{\mathrm{rem}}^{n},\rho_{\mathrm{rem}}^{n}\big), \\ \partial_t\bar\theta_{\mathrm{rem}}^{n+1} + \mathbf{u}_\lambda^n\cdot\nabla\bar\theta_{\mathrm{rem}}^{n+1} + \theta_\lambda^n\operatorname{div}\bar{\mathbf{u}}_{\mathrm{rem}}^{n+1} \\ \quad = \lambda\nu\big(\tilde J^n - \tilde J^{n-1}\big) + 2\lambda\mu\big(\tilde G^n - \tilde G^{n-1}\big) - \big(\theta_\lambda^n - \theta_\lambda^{n-1}\big)\operatorname{div}\mathbf{u}_{\mathrm{rem}}^{n} \\ \qquad - \lambda\bar{\mathbf{u}}_{\mathrm{rem}}^{n}\nabla\theta_{\mathrm{rem}}^{n} + g_0\big(x,t,\mathbf{u}_{\mathrm{rem}}^{n+1},\rho_{\mathrm{rem}}^{n+1}\big) - g_0\big(x,t,\mathbf{u}_{\mathrm{rem}}^{n},\rho_{\mathrm{rem}}^{n}\big). \end{cases} \tag{3.48}$$

Observing that, for $|\alpha| \leqslant s$,

$$\begin{aligned} &\big|D^\alpha\big(\tilde J^n - \tilde J^{n-1}\big)\big| + \big|D^\alpha\big(\tilde G^n - \tilde G^{n-1}\big)\big| \\ &\quad\leqslant M_0 \sum_{|\alpha|-1=|\beta|+|\gamma|\leqslant s-1}\big[\big(\big|D^{\beta+1}\mathbf{u}_{\mathrm{rem}}^{n}\big| + \big|D^{\beta+1}\mathbf{u}_{\mathrm{rem}}^{n}\big|\big)\big|D^{\gamma+1}\bar{\mathbf{u}}_{\mathrm{rem}}^{n}\big|\big]. \end{aligned} \tag{3.49}$$

Then repeating the previous analysis used in the proof of Lemma 3.2 and using the interpolation inequalities, we can show that there is a $\delta_T > 0$ such that, for any $\lambda \in (0, \delta_T]$ and $s' < s$,

$$\sup_{0\leqslant t\leqslant T}\big(\big\|\big(\lambda\bar\rho_{\mathrm{rem}}^{n+1}, \bar{\mathbf{u}}_{\mathrm{rem}}^{n+1}, \bar\theta_{\mathrm{rem}}^{n+1}\big)(t)\big\|_{H^{s'}}^2 + \big\|\nabla\bar\phi_{\mathrm{rem}}^{n+1}(t)\big\|_{H^{s'+1}}^2\big) + \int_0^T \big\|\bar{\mathbf{u}}_{\mathrm{rem}}^{n+1}\big\|_{H^{s'+1}}^2\,dr + \int_0^T \big\|\bar\theta_{\mathrm{rem}}^{n+1}\big\|_{H^{s'+1}}^2 \leqslant C,$$

$$\sup_{0\leqslant t\leqslant T}\big(\lambda^2\big\|\partial_t\bar\rho_{\mathrm{rem}}^{n+1}(t)\big\|_{H^{s'-1}}^2 + \lambda^2\big\|\partial_t\bar{\mathbf{u}}_{\mathrm{rem}}^{n+1}(t)\big\|_{H^{s'-2}}^2 + \big\|\partial_t\bar\theta_{\mathrm{rem}}^{n+1}(t)\big\|_{H^{s'-2}}^2 + \lambda^2\big\|\partial_t\nabla\bar\phi_{\mathrm{rem}}^{n+1}(t)\big\|_{H^{s'}}^2\big) \leqslant C$$

for some constant $C > 0$. Then the Arzelà–Ascoli theorem implies that there exists a limit vector function

$$(\rho_{\mathrm{rem}}, \mathbf{u}_{\mathrm{rem}}, \theta_{\mathrm{rem}}, \nabla\phi_{\mathrm{rem}})^{\mathrm{T}} \in L^\infty\big(0, T; H^{s'}\big) \cap \mathrm{Lip}\big([0, T]; H^{s'-1}\big)$$

satisfying (3.2)–(3.3) such that

$$\sup_{0\leqslant t\leqslant T} \big\| \big(\rho^{n+1}_{\mathrm{rem}} - \rho_{\mathrm{rem}}, \mathbf{u}^{n+1}_{\mathrm{rem}} - \mathbf{u}_{\mathrm{rem}}, \theta^{n+1}_{\mathrm{rem}} - \theta_{\mathrm{rem}}, \nabla\phi^{n+1}_{\mathrm{rem}} - \nabla\phi_{\mathrm{rem}}\big)(t)\big\|_{H^{s'-2}} \to 0$$

as $n \to +\infty$ for any $\lambda \in (0, \delta_T]$. Furthermore, for $N/2 - [N/2] < \sigma < 1$, we have the convergence

$$\big(\rho^{n+1}_{\mathrm{rem}}, \mathbf{u}^{n+1}_{\mathrm{rem}}, \theta^{n+1}_{\mathrm{rem}}, \nabla\phi^{n+1}_{\mathrm{rem}}\big)^{\mathrm{T}} \to (\rho_{\mathrm{rem}}, \mathbf{u}_{\mathrm{rem}}, \theta_{\mathrm{rem}}, \nabla\phi_{\mathrm{rem}})^{\mathrm{T}}$$

in $C([0, T]; H^{s-\sigma})$ by the standard interpolation inequality. Moreover, by Sobolev's embedding theorem, we have

$$\begin{aligned}
&(\rho_{\mathrm{rem}}, \mathbf{u}_{\mathrm{rem}}, \theta_{\mathrm{rem}}, \phi_{\mathrm{rem}})^{\mathrm{T}} \in C\big([0, T]; H^{s'}\big) \cap C^1\big([0, T]; H^{s'-2}\big) \\
&\quad \hookrightarrow C^1\big([0, T] \times \mathbb{T}^N\big) \cap C\big([0, T]; C^2\big(\mathbb{T}^N\big)\big)
\end{aligned}$$

for any $\lambda \in (0, \delta_T]$, where we have used the fact $s' > N/2 + 2$. Then the existence of classical solutions to the initial value problem (2.15), (2.11) is proved. The uniqueness of the classical solutions can be proved easily by energy estimates for the difference of any two solutions. Thus the proof of Theorem 3.1 is finished. □

4. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. By the asymptotic expansion (2.9), Propositions 1.1 and 2.1, the existence and uniqueness of classical solutions to the initial value problem of Navier–Stokes–Poisson system (1.6)–(1.9) is proved and the solution satisfies

$$\sup_{0\leqslant t\leqslant T} \big\|(\rho_\lambda, \mathbf{u}_\lambda, \theta_\lambda)(t)\big\|_{H^s} + \sup_{0\leqslant t\leqslant T} \big\|\nabla\phi_\lambda(t)\big\|_{H^{s+1}} + \|\mathbf{u}_\lambda\|_{L^2(0,T;H^{s+1})} + \|\theta_\lambda\|_{L^2(0,T;H^{s+1})} \leqslant C(T),$$

$$\sup_{0\leqslant t\leqslant T} \big(\big\|\partial_t(\rho_\lambda, \mathbf{u}_\lambda, \theta_\lambda)(t)\big\|_{H^s} + \big\|\partial_t\nabla\phi_\lambda(t)\big\|_{H^{s+1}}\big) \leqslant C(T, \lambda),$$

where $C(T) > 0$ is a constant independent of λ and $C(T, \lambda) > 0$ is a constant dependent on λ. Moreover, it is easy to see that, for $\lambda \in (0, \delta_T]$,

$$\sup_{0\leqslant t\leqslant T} \big\|(\rho_\lambda - 1, \mathbf{u}_\lambda - \mathbf{v} - \mathbf{u}_{\mathrm{osc}}, \theta_\lambda - \theta)(t)\big\|_{H^s} + \sup_{0\leqslant t\leqslant T} \big\|(\nabla\phi_\lambda - \nabla\phi_{\mathrm{osc}})(t)\big\|_{H^{s+1}} \leqslant C(T)\lambda.$$

Thus the proof of Theorem 1.2 is finished. □

As far as the combined quasineutral, vanishing viscosity and vanishing heat conductivity limit is concerned, we can follow the same lines as the proof of Theorem 1.2. Recalling the uniformly bounded estimates obtained in Lemma 3.2, we are able to get the uniform bound with respect to λ, μ, ν and κ for the solutions. Thus Theorem 1.3 can be proved similarly with minor modifications of our previous arguments. We omit the details here for conciseness.

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