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# Existence Theory for Single and Multiple Solutions to Singular Positone Boundary Value Problems

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In this paper we present some old and new existence results for singular boundary value problems. Our nonlinearity may be singular in its dependent variable.


## 1. INTRODUCTION

The study of singular boundary value problems (singular in the dependent variable) is relatively new. Indeed it was only in the middle 1970's that researchers realised that large numbers of applications [7, 8] in the study of nonlinear phenomena gave rise to singular boundary value problems (singular in the dependent variable). However, in our opinion, it was the 1979 paper of Taliaferro [15] that generated the interest of many researchers in singular problems in the 1980's and 1990's. In [15] Taliaferro showed that the singular boundary value problem

$$(1.1)\qquad \begin{aligned} &y'' + q(t)\, y^{-\alpha} = 0, \qquad 0 < t < 1 \\ &y(0) = 0 = y(1), \end{aligned}$$

has a $C[0, 1] \cap C^1(0, 1)$ solution; here $\alpha > 0$, $q \in C(0, 1)$ with $q > 0$ on $(0, 1)$ and $\int_0^1 t(1-t)\, q(t)\, dt < \infty$. Problems of the form (1.1) arise frequently in the study of nonlinear phenomena, for example in non-Newtonian



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fluid theory, such as the transport of coal slurries down conveyor belts [8], and boundary layer theory [7]. It is worth remarking here that we could consider Sturm Liouville boundary data in (1.1); however since the arguments are essentially the same (in fact easier) we will restrict our discussion to Dirichlet boundary data.

In the 1980's and 1990's many papers were devoted to singular boundary value problems of the form

$$\begin{aligned} &y'' + q(t)\, f(t, y) = 0, \qquad 0 < t < 1 \\ &y(0) = 0 = y(1). \end{aligned} \tag{1.2}$$

Almost all singular problems in the literature up to 1994 discussed the existence of one solution to positone problems i.e. problems where $f: [0, 1] \times (0, \infty) \to (0, \infty)$. In Section 2 we present a very general result for the existence of a solution to the positone singular problem (1.2). Our result includes those in [2, 3, 5, 6, 10, 11, 16] (we refer the reader to [4, 14] for some results when the differential equation involves the derivative term $y'$). In 1999 the question of multiplicity for positone singular problems was discussed for the first time by Agarwal and O'Regan [1]. The second half of Section 2 discusses multiplicity and some new results will be presented here.

To conclude the introduction we present an existence principle for the nonsingular boundary value problem which will be needed in Section 2. We use Schauder's fixed point theorem and a nonlinear alternative of Leray–Schauder type to obtain a general existence principle for the Dirichlet boundary value problem

$$\begin{aligned} &y'' + f(t, y) = 0, \qquad 0 < t < 1 \\ &y(0) = a, \qquad y(1) = b. \end{aligned} \tag{1.3}$$

THEOREM 1.1. *Suppose the following two conditions are satisfied:*

$$\text{the map } y \mapsto f(t, y) \text{ is continuous for a.e. } t \in [0, 1]. \tag{1.4}$$

*and*

$$\text{the map } t \mapsto f(t, y) \text{ is measurable for all } y \in \mathbf{R}. \tag{1.5}$$

(I) *Assume*

$$\begin{aligned} &\text{for each } r > 0 \text{ there exists } h_r \in L^1_{loc}(0, 1) \text{ with} \\ &\int_0^1 t(1-t)\, h_r(t)\, dt < \infty \text{ such that } |y| \leqslant r \text{ implies} \\ &|f(t, y)| \leqslant h_r(t) \text{ for a.e. } t \in (0, 1) \end{aligned} \tag{1.6}$$

*holds. In addition suppose there is a constant* $M > |a|+|b|$, *independent of* $\lambda$, *with*

$$(1.7) \qquad |y|_0 = \sup_{t \in [0,1]} |y(t)| \neq M$$

*for any solution* $y \in AC[0,1]$ *(with* $y' \in AC_{loc}(0,1)$*) to*

$$(1.8)_\lambda \qquad \begin{aligned} & y'' + \lambda f(t,y) = 0, \qquad 0<t<1 \\ & y(0) = a, \qquad y(1) = b, \end{aligned}$$

*for each* $\lambda \in (0,1)$. *Then* (1.3) *has a solution* $y$ *with* $|y|_0 \leqslant M$.

(II) *Assume*

(1.9) *there exists* $h \in L^1_{loc}(0,1)$ *with* $\int_0^1 t(1-t)\,h(t)\,dt < \infty$

*such that* $|f(t,y)| \leqslant h(t)$ *for a.e.* $t \in (0,1)$ *and* $y \in \mathbf{R}$

*holds. Then* (1.3) *has a solution.*

*Proof.* (I) We begin by showing that solving $(1.8)_\lambda$ is equivalent to finding a solution $y \in C[0,1]$ to

$$(1.10)_\lambda \qquad y(t) = a\,(1-t) + bt + \lambda\,(1-t) \int_0^t s f(s, y(s))\,ds + \lambda t \int_t^1 (1-s)\, f(s,y(s))\,ds.$$

To see this notice if $y \in C[0,1]$ satisfies $(1.10)_\lambda$ then it is easy to see (since (1.7) holds; see [12, 14]) that $y' \in L^1[0,1]$. Thus $y \in AC[0,1]$, $y' \in AC_{loc}(0,1)$ and note

$$y'(t) = -a + b - \lambda \int_0^t s f(s,y(s))\,ds + \lambda \int_t^1 (1-s)\, f(s,y(s))\,ds.$$

Next integrate $y'(t)$ from 0 to $x$ ($x \in (0,1)$) and interchange the order of integration to get

$$
\begin{aligned}
y(x)-y(0) &= \int_0^x y'(t)\,dt \\
&= -ax+bx-\lambda\int_0^x\int_0^t sf(s, y(s))\,ds\,dt \\
&\quad +\lambda\int_0^x\int_t^1 (1-s)\,f(s, y(s))\,ds\,dt \\
&= -ax+bx+\lambda\,(1-x)\int_0^x sf(s, y(s))\,ds \\
&\quad +\lambda x\int_x^1 (1-s)\,f(s, y(s))\,ds \\
&= -a+y(x),
\end{aligned}
$$

so $y(0)=a$. Similarly integrate $y'(t)$ from $x$ ($x\in(0,1)$) to 1 and interchange the order of integration to get $y(1)=b$. Thus if $y\in C[0,1]$ satisfies $(1.10)_\lambda$ then $y$ is a solution of $(1.8)_\lambda$.

Define the operator $N\colon C[0,1]\to C[0,1]$ by

$$
\text{(1.11)}\qquad Ny(t)=a(1-t)+bt+(1-t)\int_0^t sf(s, y(s))\,ds + t\int_t^1 (1-s)\,f(s, y(s))\,ds.
$$

Then $(1.10)_\lambda$ is equivalent to the fixed point problem

$$
(1.12)_\lambda \qquad y=(1-\lambda)\,p+\lambda Ny \qquad \text{where} \qquad p=a(1-t)+bt.
$$

It is easy to see [12, 14] that $N\colon C[0,1]\to C[0,1]$ is continuous and completely continuous. Set

$$
U=\{u\in C[0,1]: |u|_0<M\}, \qquad K=E=C[0,1].
$$

Now the nonlinear alternative of Leray–Schauder type [14] guarantees that $N$ has a fixed point i.e., $(1.10)_1$ has a solution.

(II) Solving (1.3) is equivalent to the fixed point problem $y=Ny$ where $N$ is as in (1.11). It is easy to see that $N\colon C[0,1]\to C[0,1]$ is continuous and compact (since (1.9) holds). The result follows from Schauder's fixed point theorem [14]. ∎

## 2. SINGULAR BOUNDARY VALUE PROBLEMS

In Section 2 we discuss positone boundary value problems. Almost all singular papers in the 1980's and 1990's were devoted to such problems. In the late 1990's the question of multiplicity for singular positone problems was raised, and we discuss this question in the second half of Section 2.

Consider the Dirichlet boundary value problem

$$\begin{aligned} & y'' + q(t)\, f(t, y) = 0, \qquad 0 < t < 1 \\ & y(0) = 0 = y(1). \end{aligned} \tag{2.1}$$

Here the nonlinearity $f$ may be singular at $y = 0$ and $q$ may be singular at $t = 0$ and/or $t = 1$. We begin by showing that (2.1) has a $C[0, 1] \cap C^2(0, 1)$ solution. To do so we first establish, via Theorem 1.1, the existence of a $C[0, 1] \cap C^2(0, 1)$ solution, for each sufficiently large $m$, to the "modified" problem

$$\begin{aligned} & y'' + q(t)\, f(t, y) = 0, \qquad 0 < t < 1 \\ & y(0) = \frac{1}{m} = y(1). \end{aligned} \tag*{$(2.2)^m$}$$

To show that (2.1) has a solution we let $m \to \infty$; the key idea in this step is the Arzela–Ascoli theorem.

THEOREM 2.1. *Suppose the following conditions are satisfied:*

$$q \in C(0, 1),\ q > 0 \text{ on } (0, 1) \quad \text{and} \quad \int_0^1 t(1-t)\, q(t)\, dt < \infty \tag{2.3}$$

$$f\colon [0, 1] \times (0, \infty) \to (0, \infty) \quad \text{is continuous.} \tag{2.4}$$

$$\begin{aligned} & 0 \leqslant f(t, y) \leqslant g(y) + h(y) \ \text{on } [0, 1] \times (0, \infty) \text{ with} \\ & g > 0 \text{ continuous and nonincreasing on } (0, \infty), \\ & h \geqslant 0 \text{ continuous on } [0, \infty), \text{ and } \tfrac{h}{g} \\ & \text{nondecreasing on } (0, \infty) \end{aligned} \tag{2.5}$$

$$\begin{aligned} & \text{for each constant } H > 0 \text{ there exists a function } \psi_H \\ & \text{continuous on } [0, 1] \text{ and positive on } (0, 1) \text{ such that} \\ & f(t, u) \geqslant \psi_H(t) \text{ on } (0, 1) \times (0, H] \end{aligned} \tag{2.6}$$

*and*

$$\exists r > 0 \quad \text{with} \quad \frac{1}{\left\{1+\dfrac{h(r)}{g(r)}\right\}} \int_0^r \frac{du}{g(u)} > b_0 \tag{2.7}$$

*hold; here*

$$b_0 = \max\left\{2\int_0^{1/2} t(1-t)\, q(t)\, dt,\ 2\int_{1/2}^1 t(1-t)\, q(t)\, dt\right\}. \tag{2.8}$$

*Then* (2.1) *has a solution* $y \in C[0,1] \cap C^2(0,1)$ *with* $y > 0$ *on* $(0,1)$ *and* $|y|_0 < r$.

*Proof.* Choose $\varepsilon > 0$, $\varepsilon < r$, with

$$\frac{1}{\left\{1+\dfrac{h(r)}{g(r)}\right\}} \int_\varepsilon^r \frac{du}{g(u)} > b_0. \tag{2.9}$$

Let $n_0 \in \{1, 2, ...\}$ be chosen so that $\frac{1}{n_0} < \frac{\varepsilon}{2}$ and let $N_0 = \{n_0, n_0+1, ...\}$. To show $(2.2)^m$, $m \in N_0$, has a solution we examine

$$\begin{aligned} & y'' + q(t)\, F(t, y) = 0, \qquad 0 < t < 1 \\ & y(0) = y(1) = \frac{1}{m}, \; m \in N_0, \end{aligned} \tag{$2.10^m$}$$

where

$$F(t,u) = \begin{cases} f(t,u), \; u \geqslant \dfrac{1}{m} \\ f\left(t, \dfrac{1}{m}\right), \; u \leqslant \dfrac{1}{m}. \end{cases}$$

To show $(2.10)^m$ has a solution for each $m \in N_0$ we will apply Theorem 1.1. Consider the family of problems

$$\begin{aligned} & y'' + \lambda q(t)\, F(t, y) = 0, \qquad 0 < t < 1 \\ & y(0) = y(1) = \frac{1}{m}, \qquad m \in N_0, \end{aligned} \tag{$2.11_\lambda^m$}$$

where $0<\lambda<1$. Let $y$ be a solution of $(2.11)_\lambda^m$. Then $y''\leqslant 0$ on $(0,1)$ and $y\geqslant\frac{1}{m}$ on $[0,1]$. Also there exists $t_m\in(0,1)$ with $y'\geqslant 0$ on $(0,t_m)$ and $y'\leqslant 0$ on $(t_m,1)$. For $x\in(0,1)$ we have

$$-y''(x)\leqslant g(y(x))\left\{1+\frac{h(y(x))}{g(y(x))}\right\}q(x). \tag{2.12}$$

Integrate from $t(t\leqslant t_m)$ to $t_m$ to obtain

$$y'(t)\leqslant g(y(t))\left\{1+\frac{h(y(t_m))}{g(y(t_m))}\right\}\int_t^{t_m}q(x)\,dx$$

and then integrate from 0 to $t_m$ to obtain

$$\int_{1/m}^{y(t_m)}\frac{du}{g(u)}\leqslant\left\{1+\frac{h(y(t_m))}{g(y(t_m))}\right\}\int_0^{t_m}xq(x)\,dx.$$

Consequently

$$\int_\varepsilon^{y(t_m)}\frac{du}{g(u)}\leqslant\left\{1+\frac{h(y(t_m))}{g(y(t_m))}\right\}\int_0^{t_m}xq(x)\,dx$$

and so

$$\int_\varepsilon^{y(t_m)}\frac{du}{g(u)}\leqslant\left\{1+\frac{h(y(t_m))}{g(y(t_m))}\right\}\frac{1}{1-t_m}\int_0^{t_m}x(1-x)\,q(x)\,dx. \tag{2.13}$$

Similarly if we integrate (2.12) from $t_m$ to $t(t\geqslant t_m)$ and then from $t_m$ to 1 we obtain

$$\int_\varepsilon^{y(t_m)}\frac{du}{g(u)}\leqslant\left\{1+\frac{h(y(t_m))}{g(y(t_m))}\right\}\frac{1}{t_m}\int_{t_m}^1 x(1-x)\,q(x)\,dx. \tag{2.14}$$

Now (2.13) and (2.14) imply

$$\int_\varepsilon^{y(t_m)}\frac{du}{g(u)}\leqslant b_0\left\{1+\frac{h(y(t_m))}{g(y(t_m))}\right\}.$$

This together with (2.9) implies $|y|_0 \neq r$. Then Theorem 1.1 implies that $(2.10)^m$ has a solution $y_m$ with $|y_m|_0 \leqslant r$. In fact (as above),

$$\frac{1}{m} \leqslant y_m(t) < r \qquad \text{for} \quad t \in [0, 1].$$

Next we obtain a sharper lower bound on $y_m$, namely we will show that there exists a constant $k > 0$, independent of $m$, with

$$y_m(t) \geqslant kt(1-t) \qquad \text{for} \quad t \in [0, 1]. \tag{2.15}$$

To see this notice (2.6) guarantees the existence of a function $\psi_r(t)$ continuous on $[0, 1]$ and positive on $(0, 1)$ with $f(t, u) \geqslant \psi_r(t)$ for $(t, u) \in (0, 1) \times (0, r]$. Now, using the Green's function representation for the solution of $(2.10)^m$, we have

$$y_m(t) = \frac{1}{m} + t \int_t^1 (1-x)\, q(x)\, f(x, y_m(x))\, dx$$
$$+ (1-t) \int_0^t xq(x)\, f(x, y_m(x))\, dx$$

and so

$$y_m(t) \geqslant t \int_t^1 (1-x)\, q(x)\, \psi_r(x)\, dx \tag{2.16}$$
$$+ (1-t) \int_0^t xq(x)\, \psi_r(x)\, dx \equiv \Phi_r(t).$$

Now it is easy to check (as in Theorem 1.1) that

$$\Phi_r'(t) = \int_t^1 (1-x)\, q(x)\, \psi_r(x)\, dx - \int_0^t xq(x)\, \psi_r(x)\, dx \qquad \text{for} \quad t \in (0, 1)$$

with $\Phi_r(0) = \Phi_r(1) = 0$. If $k_0 \equiv \int_0^1 (1-x)\, q(x)\, \psi_r(x)\, dx$ exists then $\Phi_r'(0) = k_0$; otherwise $\Phi_r'(0) = \infty$. In either case there exists a constant $k_1$, independent of $m$, with $\Phi_r'(0) \geqslant k_1$. Thus there is an $\varepsilon > 0$ with $\Phi_r(t) \geqslant \frac{1}{2} k_1 t \geqslant \frac{1}{2} k_1 t(1-t)$ for $t \in [0, \varepsilon]$. Similarly there is a constant $k_2$, independent of $m$, with $-\Phi_r'(1) \geqslant k_2$. Thus there is a $\delta > 0$ with $\Phi_r(t) \geqslant \frac{1}{2} k_2 (1-t) \geqslant \frac{1}{2} k_2 t(1-t)$ for $t \in [1-\delta, 1]$. Finally since $\frac{\Phi_r(t)}{t(1-t)}$ is bounded away from 0 on $[\varepsilon, 1-\delta]$ there is a constant $k$, independent of $m$, with $\Phi_r(t) \geqslant kt(1-t)$ on $[0, 1]$ i.e. (2.15) is true.

Next we will show

$$\{y_m\}_{m \in N_0} \text{ is a bounded, equicontinuous family on } [0, 1]. \tag{2.17}$$

Returning to (2.12) (with $y$ replaced by $y_m$) we have

$$-y_m''(x) \leqslant g(y_m(x))\left\{1+\frac{h(r)}{g(r)}\right\}q(x) \qquad \text{for} \quad x \in (0, 1). \tag{2.18}$$

Now since $y_m'' \leqslant 0$ on $(0, 1)$ and $y_m \geqslant \frac{1}{m}$ on $[0, 1]$ there exists $t_m \in (0, 1)$ with $y_m' \geqslant 0$ on $(0, t_m)$ and $y_m' \leqslant 0$ on $(t_m, 1)$. Integrate (2.18) from $t$ $(t < t_m)$ to $t_m$ to obtain

$$\frac{y_m'(t)}{g(y_m(t))} \leqslant \left\{1+\frac{h(r)}{g(r)}\right\}\int_t^{t_m} q(x)\,dx. \tag{2.19}$$

On the other hand integrate (2.18) from $t_m$ to $t$ $(t > t_m)$ to obtain

$$\frac{-y_m'(t)}{g(y_m(t))} \leqslant \left\{1+\frac{h(r)}{g(r)}\right\}\int_{t_m}^{t} q(x)\,dx. \tag{2.20}$$

We now claim that there exists $a_0$ and $a_1$ with $a_0 > 0$, $a_1 < 1$, $a_0 < a_1$ with

$$a_0 < \inf\{t_m : m \in N_0\} \leqslant \sup\{t_m : m \in N_0\} < a_1. \tag{2.21}$$

*Remark* 2.1. Here $t_m$ (as before) is the unique point in $(0, 1)$ with $y_m'(t_m) = 0$.

We now show $\inf\{t_m : m \in N_0\} > 0$. If this is not true then there is a subsequence $S$ of $N_0$ with $t_m \to 0$ as $m \to \infty$ in $S$. Now integrate (2.19) from 0 to $t_m$ to obtain

$$\int_0^{y_m(t_m)} \frac{du}{g(u)} \leqslant \left\{1+\frac{h(r)}{g(r)}\right\}\int_0^{t_m} xq(x)\,dx + \int_0^{1/m}\frac{du}{g(u)} \tag{2.22}$$

for $m \in S$. Since $t_m \to 0$ as $m \to \infty$ in $S$, we have from (2.22) that $y_m(t_m) \to 0$ as $m \to \infty$ in $S$. However since the maximum of $y_m$ on $[0, 1]$ occurs at $t_m$ we have $y_m \to 0$ in $C[0, 1]$ as $m \to \infty$ in $S$. This contradicts (2.15). Consequently $\inf\{t_m : m \in N_0\} > 0$. A similar argument shows $\sup\{t_m : m \in N_0\} < 1$. Let $a_0$ and $a_1$ be chosen as in (2.21). Now (2.19), (2.20) and (2.21) imply

$$\frac{|y_m'(t)|}{g(y_m(t))} \leqslant \left\{1+\frac{h(r)}{g(r)}\right\}v(t) \qquad \text{for} \quad t \in (0, 1) \tag{2.23}$$

where

$$v(t) = \int_{\min\{t,\, a_0\}}^{\max\{t,\, a_1\}} q(x)\,dx.$$

It is easy to see that $v \in L^1[0, 1]$. Let $I$: $[0, \infty) \to [0, \infty)$ be defined by

$$I(z) = \int_0^z \frac{du}{g(u)}.$$

Note $I$ is an increasing map from $[0, \infty)$ onto $[0, \infty)$ (notice $I(\infty) = \infty$ since $g > 0$ is nonincreasing on $(0, \infty)$) with $I$ continuous on $[0, A]$ for any $A > 0$. Notice

(2.24) $\{I(y_m)\}_{m \in N_0}$ is a bounded, equicontinuous family on $[0, 1]$.

The equicontinuity follows from (here $t, s \in [0, 1]$)

$$|I(y_m(t)) - I(y_m(s))| = \left| \int_s^t \frac{y_m'(x)}{g(y_m(x))}\, dx \right| \leqslant \left\{ 1 + \frac{h(r)}{g(r)} \right\} \left| \int_s^t v(x)\, dx \right|.$$

This inequality, the uniform continuity of $I^{-1}$ on $[0, I(r)]$, and

$$|y_m(t) - y_m(s)| = |I^{-1}(I(y_m(t))) - I^{-1}(I(y_m(s)))|$$

now establishes (2.17).

The Arzela–Ascoli Theorem guarantees the existence of a subsequence $N$ of $N_0$ and a function $y \in C[0, 1]$ with $y_m$ converging uniformly on $[0, 1]$ to $y$ as $m \to \infty$ through $N$. Also $y(0) = y(1) = 0$, $|y|_0 \leqslant r$ and $y(t) \geqslant kt(1-t)$ for $t \in [0, 1]$. In particular $y > 0$ on $(0, 1)$. Fix $t \in (0, 1)$ (without loss of generality assume $t \neq \frac{1}{2}$). Now $y_m$, $m \in N$, satisfies the integral equation

$$y_m(x) = y_m(\tfrac{1}{2}) + y_m'(\tfrac{1}{2})(x - \tfrac{1}{2}) + \int_{1/2}^x (s - x)\, q(s)\, f(s, y_m(s))\, ds$$

for $x \in (0, 1)$. Notice (take $x = \frac{2}{3}$) that $\{y_m'(\frac{1}{2})\}$, $m \in N$, is a bounded sequence since $ks(1-s) \leqslant y_m(s) \leqslant r$ for $s \in [0, 1]$. Thus $\{y_m'(\frac{1}{2})\}_{m \in N}$ has a convergent subsequence; for convenience let $\{y_m'(\frac{1}{2})\}_{m \in N}$ denote this subsequence also and let $r_0 \in \mathbf{R}$ be its limit. Now for the above fixed $t$,

$$y_m(t) = y_m(\tfrac{1}{2}) + y_m'(\tfrac{1}{2})(t - \tfrac{1}{2}) + \int_{1/2}^x (s - t)\, q(s)\, f(s, y_m(s))\, ds$$

and let $m \to \infty$ through $N$ (we note here that $f$ is uniformly continuous on compact subsets of $[\min(\frac{1}{2}, t), \max(\frac{1}{2}, t)] \times (0, r]$) to obtain

$$y(t) = y(\tfrac{1}{2}) + r_0(t - \tfrac{1}{2}) + \int_{1/2}^t (s - t)\, q(s)\, f(s, y(s))\, ds.$$

We can do this argument for each $t \in (0, 1)$ and so $y''(t) + q(t)\, f(t, y(t)) = 0$ for $0 < t < 1$. Finally it is easy to see that $|y|_0 < r$ (note if $|y|_0 = r$ then following essentially the argument from (2.12)–(2.14) will yield a contradiction). ∎

Next we establish the existence of two nonnegative solutions to the singular second order Dirichlet problem

$$(2.25) \qquad \begin{aligned} y''(t) + q(t)[g(y(t)) + h(y(t))] = 0, \qquad 0 < t < 1 \\ y(0) = y(1) = 0; \end{aligned}$$

here our nonlinear term $g + h$ may be singular at $y = 0$. The results presented here improve those in Agarwal and O'Regan [1]. First we state the fixed point result we will use to establish multiplicity (see [9, 13] for a proof).

THEOREM 2.2. *Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in $E$. Also $r$, $R$ are constants with $0 < r < R$. Suppose $A : \overline{\Omega_R} \cap K \to K$ (here $\Omega_R = \{x \in E : \|x\| < R\}$) is a continuous, compact map and assume the following conditions hold:*

$$(2.26) \qquad x \neq \lambda A(x) \quad \text{for} \quad \lambda \in [0, 1) \quad \text{and} \quad x \in \partial_E \Omega_r \cap K$$

*and*

$$(2.27) \qquad \begin{aligned} &\text{there exists a } v \in K \setminus \{0\} \text{ with } x \neq A(x) + \delta v \\ &\text{for any } \delta > 0 \text{ and } x \in \partial_E \Omega_R \cap K. \end{aligned}$$

*Then $A$ has a fixed point in $K \cap \{x \in E : r \leqslant \|x\| \leqslant R\}$.*

*Remark* 2.2. In Theorem 2.2 if (2.26) and (2.27) are replaced by

$$(2.26)^\star \qquad x \neq \lambda A(x) \quad \text{for} \quad \lambda \in [0, 1) \quad \text{and} \quad x \in \partial_E \Omega_R \cap K$$

and

$$(2.27)^\star \qquad \begin{aligned} &\text{there exists a } v \in K \setminus \{0\} \text{ with } x \neq A(x) + \delta\, v \\ &\text{for any } \delta > 0 \text{ and } x \in \partial_E \Omega_r \cap K \end{aligned}$$

then $A$ has also a fixed point in $K \cap \{x \in E : r \leqslant \|x\| \leqslant R\}$.

THEOREM 2.3. *Let $E = (E, \|\cdot\|)$ be a Banach space, $K \subset E$ a cone and let $\|.\|$ be increasing with respect to $K$. Also $r$, $R$ are constants with $0 < r < R$. Suppose $A: \overline{\Omega_R} \cap K \to K$ (here $\Omega_R = \{x \in E : \|x\| < R\}$) is a continuous, compact map and assume the following conditions hold:*

$$x \neq \lambda A(x) \quad \textit{for} \quad \lambda \in [0, 1) \quad \textit{and} \quad x \in \partial_E \Omega_r \cap K \tag{2.28}$$

*and*

$$\|Ax\| > \|x\| \quad \textit{for} \quad x \in \partial_E \Omega_R \cap K. \tag{2.29}$$

*Then $A$ has a fixed point in $K \cap \{x \in E : r \leqslant \|x\| \leqslant R\}$.*

*Proof.* Notice (2.29) guarantees that (2.27) is true. This is a standard argument and for completeness we supply it here. Suppose there exists $v \in K \setminus \{0\}$ with $x = A(x) + \delta v$ for some $\delta > 0$ and $x \in \partial_E \Omega_R \cap K$. Then since $\|\cdot\|$ is increasing with respect to $K$ we have since $\delta v \in K$,

$$\|x\| = \|Ax + \delta v\| \geqslant \|Ax\| > \|x\|,$$

a contradiction. The result now follows from Theorem 2.2. ∎

*Remark* 2.3. In Theorem 2.3 if (2.28) and (2.29) are replaced by

$$x \neq \lambda A(x) \quad \text{for} \quad \lambda \in [0, 1) \quad \text{and} \quad x \in \partial_E \Omega_R \cap K \tag{2.28$^\star$}$$

and

$$\|Ax\| > \|x\| \quad \text{for} \quad x \in \partial_E \Omega_r \cap K. \tag{2.29$^\star$}$$

then $A$ has a fixed point in $K \cap \{x \in E : r \leqslant \|x\| \leqslant R\}$.

Now $E = (C[0, 1], |\cdot|_0)$ (here $|u|_0 = \sup_{t \in [0, 1]} |u(t)|$, $u \in C[0, 1]$) will be our Banach space and

$$K = \{y \in C[0, 1] : y(t) \geqslant 0,\ t \in [0, 1] \text{ and } y(t) \text{ concave on } [0, 1]\}. \tag{2.30}$$

Let $\theta: [0, 1] \times [0, 1] \to [0, \infty)$ be defined by

$$\theta(t, s) = \begin{cases} \dfrac{t}{s} & \text{if} \quad 0 \leqslant t \leqslant s \\[2ex] \dfrac{1-t}{1-s} & \text{if} \quad s \leqslant t \leqslant 1. \end{cases}$$

The following result is easy to prove and is well known.

THEOREM 2.4. *Let $y \in K$ (as in (2.30)). Then there exists $t_0 \in [0, 1]$ with $y(t_0) = |y|_0$ and*

$$y(t) \geqslant \theta(t, t_0)\, |y|_0 \geqslant t\,(1-t)\, |y|_0 \qquad \textit{for} \quad t \in [0, 1].$$

*Proof.* The existence of $t_0$ is immediate. Now if $0 \leqslant t \leqslant t_0$ then since $y(t)$ is concave on $[0, 1]$ we have

$$y(t) = y\left(\left(1-\frac{t}{t_0}\right)0+\frac{t}{t_0}\,t_0\right) \geqslant \left(1-\frac{t}{t_0}\right)y(0)+\frac{t}{t_0}\,y(t_0).$$

That is

$$y(t) \geqslant \frac{t}{t_0}\,y(t_0) = \theta(t, t_0)\,|y|_0 \geqslant t(1-t)\,|y|_0.$$

A similar argument establishes the result if $t_0 \leqslant t \leqslant 1$. ∎

From Theorem 2.1 we have immediately the following existence result for (2.25).

THEOREM 2.5. *Suppose the following conditions are satisfied:*

(2.31) $q \in C(0, 1), q > 0$ *on* $(0, 1)$ *and* $\int_0^1 t(1-t)\,q(t)\,dt < \infty$

(2.32) $g > 0$ *is continuous and nonincreasing on* $(0, \infty)$

(2.33) $h \geqslant 0$ *continuous on* $[0, \infty)$ *with* $\dfrac{h}{g}$ *nondecreasing on* $(0, \infty)$

*and*

(2.34) $\exists r > 0$ *with* $\dfrac{1}{\left\{1+\dfrac{h(r)}{g(r)}\right\}}\displaystyle\int_0^r \frac{du}{g(u)} > b_0;$

*here*

(2.35) $$b_0 = \max\left\{2\int_0^{1/2} t(1-t)\,q(t)\,dt,\; 2\int_{1/2}^1 t(1-t)\,q(t)\,dt\right\}.$$

*Then* (2.25) *has a solution* $y \in C[0, 1] \cap C^2(0, 1)$ *with* $y > 0$ *on* $(0, 1)$ *and* $|y|_0 < r$.

*Proof.* The result follows from Theorem 2.1 with $f(t,u)=g(u)+h(u)$. Notice (2.6) is clearly satisfied with $\psi_H(t)=g(H)$. ∎

THEOREM 2.6. *Assume* (2.31)–(2.34) *hold. Choose* $a\in(0,\frac{1}{2})$ *and fix it and suppose there exists* $R>r$ *with*

$$\frac{Rg(a(1-a)\,R)}{g(R)\,g(a(1-a)\,R)+g(R)\,h(a(1-a)\,R)}\leqslant\int_a^{1-a}G(\sigma,s)\,q(s)\,ds; \tag{2.36}$$

*here* $0\leqslant\sigma\leqslant 1$ *is such that*

$$\int_a^{1-a}G(\sigma,s)\,q(s)\,ds=\sup_{t\in[0,1]}\int_a^{1-a}G(t,s)\,q(s)\,ds \tag{2.37}$$

*and*

$$G(t,s)=\begin{cases}(1-t)\,s, & 0\leqslant s\leqslant t\\ (1-s)\,t, & t\leqslant s\leqslant 1.\end{cases}$$

*Then* (2.25) *has a solution* $y\in C[0,1]\cap C^2(0,1)$ *with* $y>0$ *on* $(0,1)$ *and* $r<|y|_0\leqslant R$.

*Proof.* To show the existence of the solution described in the statement of Theorem 2.6 we will apply Theorem 2.3. First however choose $\varepsilon>0$ and $\varepsilon<r$ with

$$\frac{1}{\left\{1+\dfrac{h(r)}{g(r)}\right\}}\int_\varepsilon^r\frac{du}{g(u)}>b_0. \tag{2.38}$$

Let $m_0\in\{1,2,...\}$ be chosen so that $\frac{1}{m_0}<\frac{\varepsilon}{2}$ and $\frac{1}{m_0}<a(1-a)\,R$ and let $N_0=\{m_0,m_0+1,...\}$. We first show that

$$\begin{aligned}&y''(t)+q(t)[g(y(t))+h(y(t))]=0, \qquad 0<t<1\\ &y(0)=y(1)=\frac{1}{m}\end{aligned} \tag*{$(2.39)^m$}$$

has a solution $y_m$ for each $m\in N_0$ with $y_m>\frac{1}{m}$ on $(0,1)$ and $r\leqslant|y_m|_0\leqslant R$. To show $(2.39)^m$ has such a solution for each $m\in N_0$, we will look at

$$\begin{aligned}&y''(t)+q(t)[g^\star(y(t))+h(y(t))]=0, \qquad 0<t<1\\ &y(0)=y(1)=\frac{1}{m}\end{aligned} \tag*{$(2.40)^m$}$$

with

$$g^{\star}(u) = \begin{cases} g(u), & u \geqslant \dfrac{1}{m} \\ g\left(\dfrac{1}{m}\right), & 0 \leqslant u \leqslant \dfrac{1}{m}. \end{cases}$$

*Remark* 2.4. Notice $g^{\star}(u) \leqslant g(u)$ for $u > 0$.
Fix $m \in N_0$. Let $E = (C[0, 1], |\cdot|_0)$ and

(2.41) $K = \{u \in C[0, 1] : u(t) \geqslant 0,\ t \in [0, 1]$ and $u(t)$ concave on $[0, 1]\}$.

Clearly $K$ is a cone of $E$. Let $A$: $K \to C[0, 1]$ be defined by

$$Ay(t) = \frac{1}{m} + \int_0^1 G(t, s)\, q(s)[g^{\star}(y(s)) + h(y(s))]\, ds. \tag{2.42}$$

A standard argument implies $A$: $K \to C[0, 1]$ is continuous and completely continuous. Next we show $A$: $K \to K$. If $u \in K$ then clearly $Au(t) \geqslant 0$ for $t \in [0, 1]$. Also notice that

$$(Au)''(t) \leqslant 0 \qquad \text{on} \quad (0, 1)$$
$$Au(0) = Au(1) = \frac{1}{m}$$

so $Au(t)$ is concave on $[0, 1]$. Consequently $Au \in K$ so $A$: $K \to K$. Let

$$\Omega_1 = \{u \in C[0, 1] : |u|_0 < r\} \qquad \text{and} \qquad \Omega_2 = \{u \in C[0, 1] : |u|_0 < R\}.$$

We first show

$$y \neq \lambda A y \qquad \text{for} \quad \lambda \in [0, 1] \quad \text{and} \quad y \in K \cap \partial\Omega_1. \tag{2.43}$$

Suppose this is false i.e., suppose there exists $y \in K \cap \partial\Omega_1$ and $\lambda \in [0, 1)$ with $y = \lambda A y$. We can assume $\lambda \neq 0$. Now since $y = \lambda A y$ we have

$$\begin{aligned} & y''(t) + \lambda q(t)[g^{\star}(y(t)) + h(y(t))] = 0, \qquad 0 < t < 1 \\ & y(0) = y(1) = \frac{1}{m}. \end{aligned} \tag{2.44}$$

Since $y'' \leqslant 0$ on $(0, 1)$ and $y \geqslant \frac{1}{m}$ on $[0, 1]$ there exists $t_0 \in (0, 1)$ with $y' \geqslant 0$ on $(0, t_0)$, $y' \leqslant 0$ on $(t_0, 1)$ and $y(t_0) = |y|_0 = r$ (note $y \in K \cap \partial\Omega_1$). Also notice

$$g^\star(y(t)) + h(y(t)) \leqslant g(y(t)) + h(y(t)) \qquad \text{for} \quad t \in (0, 1)$$

since $g$ is nonincreasing on $(0, \infty)$. For $x \in (0, 1)$ we have

$$-y''(x) \leqslant g(y(x)) \left\{ 1 + \frac{h(y(x))}{g(y(x))} \right\} q(x). \tag{2.45}$$

Integrate from $t(t \leqslant t_0)$ to $t_0$ to obtain

$$y'(t) \leqslant g(y(t)) \left\{ 1 + \frac{h(r)}{g(r)} \right\} \int_t^{t_0} q(x)\, dx$$

and then integrate from 0 to $t_0$ to obtain

$$\int_{1/m}^{r} \frac{du}{g(u)} \leqslant \left\{ 1 + \frac{h(r)}{g(r)} \right\} \int_0^{t_0} xq(x)\, dx.$$

Consequently

$$\int_{\varepsilon}^{r} \frac{du}{g(u)} \leqslant \left\{ 1 + \frac{h(r)}{g(r)} \right\} \int_0^{t_0} xq(x)\, dx$$

and so

$$\int_{\varepsilon}^{r} \frac{du}{g(u)} \leqslant \left\{ 1 + \frac{h(r)}{g(r)} \right\} \frac{1}{1-t_0} \int_0^{t_0} x(1-x)\, q(x)\, dx. \tag{2.46}$$

Similarly if we integrate (2.45) from $t_0$ to $t(t \geqslant t_0)$ and then from $t_0$ to 1 we obtain

$$\int_{\varepsilon}^{r} \frac{du}{g(u)} \leqslant \left\{ 1 + \frac{h(r)}{g(r)} \right\} \frac{1}{t_0} \int_{t_0}^{1} x(1-x)\, q(x)\, dx. \tag{2.47}$$

Now (2.46) and (2.47) imply

$$\int_{\varepsilon}^{r} \frac{du}{g(u)} \leqslant b_0 \left\{ 1 + \frac{h(r)}{g(r)} \right\}, \tag{2.48}$$

where $b_0$ is as defined in (2.35). This contradicts (2.38) and consequently (2.43) is true.

Next we show

$$|Ay|_0 > |y|_0 \quad \text{for} \quad y \in K \cap \partial\Omega_2. \tag{2.49}$$

To see this let $y \in K \cap \partial\Omega_2$ so $|y|_0 = R$. Also since $y(t)$ is concave on $[0, 1]$ (since $y \in K$) we have from Theorem 2.4 that $y(t) \geqslant t(1-t)\,|y|_0 \geqslant t(1-t)\,R$ for $t \in [0, 1]$. Also for $s \in [a, 1-a]$ we have

$$g^\star(y(s)) + h(y(s)) = g(y(s)) + h(y(s))$$

since $y(s) \geqslant a(1-a)\,R > 1/m_0$ for $s \in [a, 1-a]$. Note in particular that

$$y(s) \in [a(1-a)\,R, R] \quad \text{for} \quad s \in [a, 1-a]. \tag{2.50}$$

With $\sigma$ as defined in (2.37) we have using (2.50) and (2.36),

$$\begin{aligned} Ay(\sigma) &= \frac{1}{m} + \int_0^1 G(\sigma, s)\, q(s)[g^\star(y(s)) + h(y(s))]\, ds \\ &> \int_a^{1-a} G(\sigma, s)\, q(s)[g^\star(y(s)) + h(y(s))]\, ds \\ &= \int_a^{1-a} G(\sigma, s)\, q(s)\, g(y(s)) \left\{1 + \frac{h(y(s))}{g(y(s))}\right\} ds \\ &\geqslant g(R) \left\{1 + \frac{h(a(1-a)\,R)}{g(a(1-a)\,R)}\right\} \int_a^{1-a} G(\sigma, s)\, q(s)\, ds \\ &\geqslant R = |y|_0, \end{aligned}$$

and so $|Ay|_0 > |y|_0$. Hence (2.49) is true.

Now Theorem 2.3 implies $A$ has a fixed point $y_m \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$ i.e., $r \leqslant |y_m|_0 \leqslant R$. In fact $|y_m|_0 > r$ (note if $|y_m|_0 = r$ then following essentially the same argument from (2.45)–(2.48) will yield a contradiction). Consequently $(2.40)^m$ (and also $(2.39)^m$) has a solution $y_m \in C[0, 1] \cap C^2(0, 1)$, $y_m \in K$, with

$$\frac{1}{m} \leqslant y_m(t) \quad \text{for} \quad t \in [0, 1], \quad r < |y_m|_0 \leqslant R \tag{2.51}$$

and (from Theorem 2.4, note $y_m \in K$)

$$y_m(t) \geqslant t(1-t)\, r \qquad \text{for} \quad t \in [0, 1]. \tag{2.52}$$

Next we will show

(2.53) $\{y_m\}_{m \in N_0}$ is a bounded, equicontinuous family on $[0, 1]$.

Returning to (2.45) (with $y$ replaced by $y_m$) we have

$$-y_m''(x) \leqslant g(y_m(x)) \left\{1 + \frac{h(R)}{g(R)}\right\} q(x) \qquad \text{for} \quad x \in (0, 1). \tag{2.54}$$

Now since $y_m'' \leqslant 0$ on $(0, 1)$ and $y_m \geqslant \frac{1}{m}$ on $[0, 1]$ there exists $t_m \in (0, 1)$ with $y_m' \geqslant 0$ on $(0, t_m)$ and $y_m' \leqslant 0$ on $(t_m, 1)$. Integrate (2.54) from $t$ $(t < t_m)$ to $t_m$ to obtain

$$\frac{y_m'(t)}{g(y_m(t))} \leqslant \left\{1 + \frac{h(R)}{g(R)}\right\} \int_t^{t_m} q(x)\, dx. \tag{2.55}$$

On the other hand integrate (2.54) from $t_m$ to $t$ $(t > t_m)$ to obtain

$$\frac{-y_m'(t)}{g(y_m(t))} \leqslant \left\{1 + \frac{h(R)}{g(R)}\right\} \int_{t_m}^{t} q(x)\, dx. \tag{2.56}$$

We now claim that there exists $a_0$ and $a_1$ with $a_0 > 0$, $a_1 < 1$, $a_0 < a_1$ with

$$a_0 < \inf\{t_m : m \in N_0\} \leqslant \sup\{t_m : m \in N_0\} < a_1. \tag{2.57}$$

*Remark* 2.5. Here $t_m$ (as before) is the unique point in $(0, 1)$ with $y_m'(t_m) = 0$.

We now show $\inf\{t_m : m \in N_0\} > 0$. If this is not true then there is a subsequence $S$ of $N_0$ with $t_m \to 0$ as $m \to \infty$ in $S$. Now integrate (2.55) from 0 to $t_m$ to obtain

$$\int_0^{y_m(t_m)} \frac{du}{g(u)} \leqslant \left\{1 + \frac{h(R)}{g(R)}\right\} \int_0^{t_m} x q(x)\, dx + \int_0^{1/m} \frac{du}{g(u)} \tag{2.58}$$

for $m \in S$. Since $t_m \to 0$ as $m \to \infty$ in $S$, we have from (2.58) that $y_m(t_m) \to 0$ as $m \to \infty$ in $S$. However since the maximum of $y_m$ on $[0, 1]$ occurs at $t_m$

we have $y_m \to 0$ in $C[0, 1]$ as $m \to \infty$ in $S$. This contradicts (2.52). Consequently $\inf\{t_m : m \in N_0\} > 0$. A similar argument shows $\sup\{t_m : m \in N_0\} < 1$. Let $a_0$ and $a_1$ be chosen as in (2.57). Now (2.55), (2.56) and (2.57) imply

$$\frac{|y_m'(t)|}{g(y_m(t))} \leqslant \left\{1+\frac{h(R)}{g(R)}\right\} v(t) \quad \text{for} \quad t \in (0, 1) \tag{2.59}$$

where

$$v(t) = \int_{\min\{t,\, a_0\}}^{\max\{t,\, a_1\}} q(x)\, dx.$$

It is easy to see that $v \in L^1[0, 1]$. Let $I$: $[0, \infty) \to [0, \infty)$ be defined by

$$I(z) = \int_0^z \frac{du}{g(u)}.$$

Note $I$ is an increasing map from $[0, \infty)$ onto $[0, \infty)$ (notice $I(\infty) = \infty$ since $g > 0$ is nonincreasing on $(0, \infty)$) with $I$ continuous on $[0, A]$ for any $A > 0$. Notice

(2.60) $\{I(y_m)\}_{m \in N_0}$ is a bounded, equicontinuous family on $[0, 1]$.

The equicontinuity follows from (here $t, s \in [0, 1]$)

$$|I(y_m(t)) - I(y_m(s))| = \left|\int_s^t \frac{y_m'(x)}{g(y_m(x))}\, dx\right| \leqslant \left\{1+\frac{h(R)}{g(R)}\right\} \left|\int_s^t v(x)\, dx\right|.$$

This inequality, the uniform continuity of $I^{-1}$ on $[0, I(R)]$, and

$$|y_m(t) - y_m(s)| = |I^{-1}(I(y_m(t))) - I^{-1}(I(y_m(s)))|$$

now establishes (2.53).

The Arzela–Ascoli Theorem guarantees the existence of a subsequence $N$ of $N_0$ and a function $y \in C[0, 1]$ with $y_m$ converging uniformly on $[0, 1]$ to $y$ as $m \to \infty$ through $N$. Also $y(0) = y(1) = 0$, $r \leqslant |y|_0 \leqslant R$ and $y(t) \geqslant t(1-t)\, r$ for $t \in [0, 1]$. In particular $y > 0$ on $(0, 1)$. Fix $t \in (0, 1)$ (without loss of generality assume $t \neq \frac{1}{2}$). Now $y_m, m \in N$, satisfies the integral equation

$$y_m(x) = y_m(\tfrac{1}{2}) + y_m'(\tfrac{1}{2})(x-\tfrac{1}{2}) + \int_{1/2}^x (s-x)\, q(s)[g(y_m(s)) + h(y_m(s))]\, ds$$

for $x \in (0, 1)$. Notice (take $x = \frac{2}{3}$) that $\{y_m'(\frac{1}{2})\}$, $m \in N$, is a bounded sequence since $rs(1-s) \leqslant y_m(s) \leqslant R$ for $s \in [0, 1]$. Thus $\{y_m'(\frac{1}{2})\}_{m \in N}$ has a convergent subsequence; for convenience let $\{y'_m(\frac{1}{2})\}_{m\in N}$ denote this subsequence also and let $r_0 \in \mathbf{R}$ be its limit. Now for the above fixed $t$,

$$y_m(t) = y_m(\tfrac{1}{2}) + y'_m(\tfrac{1}{2})(t-\tfrac{1}{2}) + \int_{1/2}^{t} (s-t)\, q(s)[g(y_m(s)) + h(y_m(s))]\, ds,$$

and let $m \to \infty$ through $N$ (we note here that $g+h$ is uniformly continuous on compact subsets of $[\min(\frac{1}{2}, t), \max(\frac{1}{2}, t)] \times (0, R]$) to obtain

$$y(t) = y(\tfrac{1}{2}) + r_0(t-\tfrac{1}{2}) + \int_{1/2}^{t} (s-t)\, q(s)[g(y(s)) + h(y(s))]\, ds.$$

We can do this argument for each $t \in (0, 1)$ and so $y''(t) + q(t)[g(y(t)) + h(y(t))] = 0$ for $0 < t < 1$. Finally it is easy to see that $|y|_0 > r$ (note if $|y|_0 = r$ then following essentially the argument from (2.45)–(2.48) will yield a contradiction). ∎

*Remark* 2.6. If in (2.36) we have $R < r$ then (2.25) has a solution $y \in C[0, 1] \cap C^2(0, 1)$ with $y > 0$ on $(0, 1)$ and $R \leqslant |y|_0 < r$. The argument is similar to that in Theorem 2.6 except here we use Remark 2.3.

*Remark* 2.7. It is also possible to use the ideas in Theorem 2.6 to discuss other boundary conditions; for example $y'(0) = y(1) = 0$.

*Remark* 2.8. If we use Krasnoselski's fixed point theorem in a cone we need more that (2.31)–(2.34), (2.36) to establish the existence of a solution $y \in C[0, 1] \cap C^2(0, 1)$ with $y > 0$ on $(0, 1)$ and $r < |y|_0 \leqslant R$. This is because (2.43) is less restrictive than $|Ay|_0 \leqslant |y|_0$ for $y \in K \cap \partial\Omega_1$.

THEOREM 2.7. *Assume* (2.31)–(2.34) *and* (2.36) *hold. Then* (2.25) *has two solutions* $y_1, y_2 \in C[0, 1] \cap C^2(0, 1)$ *with* $y_1 > 0$, $y_2 > 0$ *on* $(0, 1)$ *and* $|y_1|_0 < r < |y_2|_0 \leqslant R$.

*Proof.* The existence of $y_1$ follows from Theorem 2.5 and the existence of $y_2$ follows from Theorem 2.6. ∎

EXAMPLE 2.1. The singular boundary value problem

$$\begin{aligned} &y'' + \frac{1}{\alpha+1}(y^{-\alpha} + y^{\beta} + 1) = 0 \qquad \text{on} \quad (0, 1) \\ &y(0) = y(1) = 0, \qquad \alpha > 0,\ \beta > 1 \end{aligned} \tag{2.61}$$

has two solutions $y_1, y_2 \in C[0, 1] \cap C^2(0, 1)$ with $y_1 > 0$, $y_2 > 0$ on $(0, 1)$ and $|y_1|_0 < 1 < |y_2|_0$.

To see this we will apply Theorem 2.7 with $q = \frac{1}{\alpha+1}$, $g(u) = u^{-\alpha}$ and $h(u) = u^{\beta} + 1$. Clearly (2.31)–(2.33) hold. Also note

$$b_0 = \max\left\{\frac{2}{\alpha+1}\int_0^{1/2} t(1-t)\,dt,\ \frac{2}{\alpha+1}\int_{1/2}^{1} t(1-t)\,dt\right\} = \frac{1}{6(\alpha+1)}.$$

Consequently (2.34) holds (with $r = 1$) since

$$\frac{1}{\left\{1+\dfrac{h(r)}{g(r)}\right\}}\int_0^r \frac{du}{g(u)} = \frac{1}{(1+r^{\alpha+\beta}+r^{\alpha})}\left(\frac{r^{\alpha+1}}{\alpha+1}\right)$$

$$= \frac{1}{3(\alpha+1)} > b_0 = \frac{1}{6(\alpha+1)}.$$

Finally note (since $\beta > 1$), take $a = \frac{1}{4}$, that

$$\lim_{R\to\infty} \frac{Rg\left(\dfrac{3R}{16}\right)}{g(R)\,g\left(\dfrac{3R}{16}\right) + g(R)\,h\left(\dfrac{3R}{16}\right)}$$

$$= \lim_{R\to\infty} \frac{R^{\alpha+1}\left(\dfrac{3}{16}\right)^{-\alpha}}{\left(\dfrac{3}{16}\right)^{-\alpha} + \left(\dfrac{3}{16}\right)^{\beta} R^{\alpha+\beta} + R^{\alpha}} = 0$$

so there exists $R > 1$ with (2.36) holding. The result now follows from Theorem 2.7.

## REFERENCES


1. R. P. Agarwal and D. O'Regan, Twin solutions to singular Dirichlet problems, *J. Math. Anal. Appl.* **240** (1999), 433–445.
2. R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, "Positive Solutions of Differential, Difference and Integral Equations," Kluwer Acad. Publ., Dordrecht, 1999.
3. R. Aris, "The Mathematical Theory of Diffusion and Reaction of Permeable Catalysts," Clarendon Press, Oxford, 1975.
4. J. V. Baxley, A singular nonlinear boundary value problem: membrane response of a spherical cap, *SIAM J. Appl. Math.* **48** (1988), 497–505.
5. L. E. Bobisud, Behaviour of solutions for a Robin problem, *J. Differential Equations.* **85** (1990), 91–104.
6. L. E. Bobisud, J. E. Calvert, and W. D. Royalty, Some existence results for singular boundary value problems, *Differential Integral Equations* **6** (1993), 553–571.

7. A. Callegari and A. Nachman, Some singular nonlinear differential equations arising in boundary layer theory, *J. Math. Anal. Appl.* **64** (1978), 96–105.
8. A. Callegari and A. Nachman, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, *SIAM J. Appl. Math.* **38** (1980), 275–282.
9. K. Deimling, "Nonlinear Functional Analysis," Springer-Verlag, New York, 1985.
10. J. A. Gatica, V. Oliker, and P. Waltman, Singular nonlinear boundary value problems for second order differential equations, *J. Differential Equations.* **79** (1989), 62–78.
11. C. D. Luning and W. L. Perry, Positive solutions of negative exponent generalized Emden–Fowler boundary value problems, *SIAM J. Math. Anal.* **12** (1981), 874–879.
12. D. O'Regan, Existence principles and theory for singular Dirichlet boundary value problems, *Differential Equations Dynam. Systems* **3** (1995), 289–305.
13. D. O'Regan, Existence of nonnegative solutions to superlinear non-positone problems via a fixed point theorem in cones of Banach spaces, *Dynam. Continuous Discrete Impulsive Systems* **3** (1997), 517–530.
14. D. O'Regan, "Existence Theory for Nonlinear Ordinary Differential Equations," Kluwer Academic, Dordrecht, 1997.
15. S. Taliaferro, A nonlinear singular boundary value problem, *Nonlinear Anal.* **3** (1979), 897–904.
16. A. Tineo, Existence theorems for a singular two point Dirichlet problem, *Nonlinear Anal.* **19** (1992), 323–333.