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The medial graph and voltage-current duality

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Abstract

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The theories of current graphs and voltage graphs give powerful methods for constructing graph embeddings and branched coverings of surfaces. Gross and Alpert first showed that these two theories were dual, that is, that a current assignment on an embedded graph was equivalent to a voltage assignment on the embedded dual. In this paper we examine current and voltage graphs in the context of the medial graph, a 4-regular graph formed from an embedded graph which encodes both the primal and dual graphs. As a consequence we obtain new insights into voltage-current duality, including wrapped coverings. We also develop a method for simultaneously giving a voltage and a current assignment on an embedded graph in the case that the voltage-current group is abelian. We apply this technique to construct self-dual embeddings for a variety of graphs. We also construct orientable and non-orientable embeddings of $K_{p,q}$ with dual $K_{r,s}$ for all possible p, q, r, s even with $pq = rs$.

1. Introduction

History

A seminal question in topological graph theory was the map coloring problem, introduced by Heawood [20]. In this paper he showed that each surface S has a finite chromatic number, an integer $k(S)$ such that any map on that surface could be properly colored in $k(S)$ colors. Establishing a lower bound for the chromatic number of a surface was then reduced to calculating the genus of the complete graph. Ringel and Youngs [28] did just this, yielding a complete solution to the map color problem. The total effort was tremendous, occupying roughly 300 journal pages. The proof was later condensed and systematically presented in [26].

Heffter [19] gave the first construction for the genus of (some) complete graphs. His method of map schemes was further developed by Ringel. Edmonds

[13] introduced map schemes in their dual form as vertex rotations. In whichever form, the importance of these methods is the reduction of the embedding question to a combinatorial problem. Gustin [17] introduced current graphs as a method of generating embedding schemes. Originally current graphs were not considered independently interesting; rather they were mere 'nomograms' used in generating the rows of embedding schemes. To cover the different cases of the map color problem many modifications were made to the basic construction. Youngs [34] introduced excess currents at vertices, and introduced cascades [35] for the non-orientable case. Finally, in 1974 Gross and Alpert [12] unified all of these modifications and gave the general theory of current graphs.

Gross invented the voltage graph, the dual of a current graph [10, 12]. He also interpreted the graphical projection maps as coverings [11] and found the related branched coverings of surfaces. This was extended to nonregular coverings via permutation-voltage assignments by Gross and Tucker [16]. Finally, the full symmetry of voltage-current duality was given by Jackson et al. [21], who studied the dual of the derived graph and wrapped coverings of graphs.

Our goal

The purpose of this paper is to give a unified treatment of the dual current-voltage graph construction in the context of the medial graph. We consider both orientable and non-orientable surfaces. Similarly our current-voltage groups may be either abelian or non-abelian. We do however, consider only ordinary (as opposed to permutation) voltage assignments. Hence we get regular (as opposed to arbitrary) branched coverings. Our treatment will give properties of the derived graph and the dual derived graph, together with the aforementioned covering maps and wrapped coverings maps. We also extend current and voltage constructions to allow (in the abelian case) a simultaneous voltage and current assignment. This leads to several nice applications, including the construction of self-dual embeddings.

The paper is organized as follows. In Section 2 we give the requisite background information on graph embeddings and define the medial graph of an embedding. In Section 3 we give a brief review of the theories of current and voltage graphs. In Section 4 we then show how to transfer a voltage or current assignment to a voltage assignment on the medial graph. We do this so that the graph derived from the medial graph is the medial of the graph derived from the original assignment. We also introduce simultaneous voltage and current assignments on the same graph. In Section 5 we give some properties of the derived graph and its dual, including a special case which is especially easy to analyze. In Section 6 we introduce wrapped coverings, and interpret wrapped coverings in our context of the medial graph. We also define a special kind of wrapped covering called a composition. In Section 7 we examine how to assign excess currents to the graph so as to derive embeddings of compositions. In Section 8 we give applications of our theory, including coverings with both the primal and dual graph compositions. These coverings give a variety of self-dual embeddings. We also note several known genus formulas as corollaries of our theorems, including genus embeddings of $K_{n,m}$ and of $K_{n,n,n}$. Finally, in Section 9 we give some concluding remarks, primarily on our restriction to ordinary current-voltages.

2. Embeddings and medial graphs

In this section we show how to use signed graphs and rotation schemes to describe embeddings of graphs into surfaces. We refer the reader to [15, 16] for more details. We also describe the medial graph of an embedding—a 4-regular graph which simultaneously encodes the primal and dual graphs. The medial graph has also been called the *web* graph [25].

Embeddings as signed rotations

Following the usual convention of topological graph theorists, our graphs are connected and may have loops and multiple edges. What is occasionally referred to in the literature as a simple graph, where loops and multiple edges are not allowed, is called here *simplicial.* We begin by replacing each edge with a pair of oppositely directed edges, e^+ and e^-, where e^+ runs in some *preferred direction.* Two graphs are considered *equivalent* if one can be obtained from the other by reversing preferred directions (switching e^+ and e^-) and/or the usual graph isomorphism.

We begin with a description of cellular embeddings into orientable surfaces. Let G be a graph embedded on an orientable surface S. Fix an orientation on S, say counterclockwise. This orientation determines a cyclic permutation ρ_v of the outwardly directed incident edges at each vertex v, called a *local rotation.* The collection of local rotations, one per vertex, is called a *rotation* on G, denoted ρ. Note that ρ is just a permutation of the directed edges whose orbits cyclically permute the outwardly directed edges at each vertex.

Conversely, if we are given a rotation on a graph, then there exists a unique cellular embedding in an oriented surface such that at each vertex the local rotation on the outwardly directed edges coincides with the cyclic permutation induced by the fixed orientation [13]. Let δ be the fixed-point-free involution which switches e^+ and e^- for each edge. Then a rotation ρ induces a rotation ρ^* defined by the composition $\rho^* = \rho\delta$. The faces of the embedding can be recovered by tracing out the action on the directed edges of the graph of ρ^*. If we replace each local rotation ρ by ρ^{-1}, then we obtain the embedding of the graph in the same surface, but with the opposite orientation.

It is a bit more difficult to describe embeddings of a graph into non-orientable surfaces, as we cannot consistently distinguish between local rotations π and π^{-1}. In addition to the rotation scheme, we need a *signature* on G, defined as a mapping σ from the edge set of G to $\{+, -\}$. In the literature a positively signed edge is called *type* 0, while a negatively signed edge is called *type* 1. We prefer the more descriptive terminology (coming from the band decomposition of an embedding) where a positively signed edge is *straight* and a negatively signed edge is *twisted*, although we still refer to the sign as the *type* of the edge.

Suppose that the graph G is embedded on a non-orientable surface S. Then we can arbitrarily fix an orientation at each vertex of the graph. This collection of local orientations determines a rotation scheme on G where the local rotation at a vertex is induced by the choice of local orientation. It also determines a signature on G, as we define an edge to be straight if and only if the local orientations on the ends agree.

Conversely, suppose that we are given a graph G with a signature and a rotation scheme. Then there exists a unique embedding of G into a surface together with a local orientation at each vertex so that the procedure above recovers the rotation scheme and signature [30]. To recover the faces of the embedded graph we proceed as in the orientable case, except that traversing a twisted edge 'toggles' between states which use the rotations ρ and ρ^{-1}.

If we modify the signature and rotation scheme on a graph by switching all signs on the edges incident with a vertex v and replacing the local rotation ρ_v at v with ρ_v^{-1}, then we get the same embedded graph. This corresponds to choosing a different local orientation at v in the embedded graph. Hence two embeddings will be considered *equivalent* if one can be obtained from another by a sequence of these *local sign switches*.

The graph G with signature σ and rotation scheme ρ describe an embedding into an orientable surface if and only if (σ, ρ) is equivalent (under local switchings) to (σ^+, ρ') where σ^+ is the signature with every edge positive. Thus the *signed rotation* (σ, ρ) can be used to describe both orientable and non-orientable embeddings of G. In practice, it is easier to dispense with the signature if the surface is orientable, since we can find an equivalent embedding with each edge straight.

Medial graphs

Let G be a graph embedded in a surface S. A *corner* is a pair of consecutive edges $\{e_1, e_2\}$ in some face boundary (so that $\rho^*(e_1) = e_2$ or vice versa). The *medial graph*, $M(G)$, is the graph whose vertex set is the edge set of G, and having an edge joining each pair vertices which correspond to the edges in a corner. For example, Fig. 1 shows the octahedron (dashed lines) as the medial graph of the tetrahedron (solid lines).

The medial graph is 4-regular, as each face creates two adjacencies for each edge in its boundary. It inherits an embedding in S from the embedding of G. The faces of this embedding fall into two classes, those corresponding to the vertices of G and those corresponding to the faces of G.

Define the *dual* G^* of an embedded graph G as the graph with vertex set the faces of G, and with an edge e^* for each edge e of G joining the two faces on

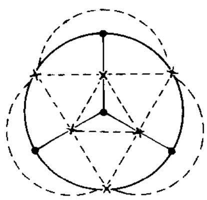

Fig. 1.

either side of e. In this context we call G the *primal* graph. The dual has a natural embedding into the same surface as the primal. With this embedding, the taking of duals is involutory, that is, $(G^*)^* = G$. Note that the medial graph of the dual is the medial graph of the primal.

The following theorem characterizes medial graphs.

Theorem 2.1. *Any embedded 4-regular graph whose faces can be 2-colored is the medial graph of a unique dual pair of embedding graphs.*

The dual of the medial graph is called the *radial graph*. Radial graphs have vertices corresponding to the vertices and faces of the embedded G, with edges joining incident elements (use multiple edges for multiple incidences). The radial graph is bipartite, and embeds in S such that each face is a quadrilateral. In fact, as in the preceding theorem, any bipartite quadrangulation of a surface is a radial graph of a dual pair of embedded graphs. As before, the radial graph of the dual is the radial graph of the primal.

3. Current graphs and voltage graphs

In this section we briefly review the rich theories of voltage graphs and current graphs. We consider embeddings in an orientable or a non-orientable surface, using voltages and currents from an abelian or a non-abelian group. However, we consider only 'ordinary', not 'permutation', voltages. We refer the reader to [15, 16] for a more detailed development of the material sketched herein.

Voltage graphs

Let $G = (V, E)$ be a graph. A *voltage assignment* on G is a mapping v from the directed edges of G to elements in some group Γ such that oppositely directed edges receive inverse group elements. In practice we only define the voltage in one direction, which forces the voltage in the opposite direction. The elements of Γ are called *voltages* and Γ is called the *voltage group*.

We next describe the *derived graph* $\tilde{G}$ from a voltage assignment. Note that the derived graph depends only on the voltage assignment, not on an embedding.

The vertex set of $\tilde{G}$ is $\tilde{V} = V \times \Gamma$; each vertex of G *lifts* to $|\Gamma|$ vertices of $\tilde{G}$. Let $e = uv$ be a directed edge of G receiving voltage ν. Then e *lifts* to $|\Gamma|$ edges in $\tilde{G}$, joining vertices (u, α) to $(v, \alpha\nu)$. So oppositely directed edges in the base graph lift to oppositely directed edges in the derived graph.

Suppose that the base graph G is embedded in a surface S. Then there is a natural embedding of the derived $\tilde{G}$ in a surface $\tilde{S}$. The local rotation $\tilde{\rho}_v$ for this embedding is the lift of the local rotation ρ_v on the base graph. Fix a vertex $(v, \alpha) \in \tilde{V}$. Let $(e_1, e_2, \ldots, e_k)$ be the local rotation at v in the embedded G. Each e_i lifts to a unique edge $\tilde{e}_i$ incident with (v, α). The local rotation of (v, α) is then $(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k)$. Similarly the signature on $\tilde{G}$ is the lift of the signature on G. That is, we define a signature $\tilde{\sigma}$ so that an edge $\tilde{e}$ is twisted if and only if it lies above a twisted edge e.

There is a natural way to describe the faces of the derived embedding. In particular, it is easy to determine the number and size of the faces. Define the *excess voltage* on a face f as the product of the voltages on the edges in the order they appear in a boundary walk. If the voltage group is non-abelian, then the excess voltage will depend on the starting edge of the product, but any two such values are conjugate and hence have the same order. Similarly if the boundary is traversed in the opposite direction, the excess voltage is the inverse of its previous value, and hence has the same order. In the derived $\tilde{G}$ there are $\mathrm{index}_\Gamma(\nu_f)$ faces lying above f, each of size $|f| \cdot \mathrm{order}_\Gamma(\nu_f)$. If ν_f is zero for each v then we say that *Kirchhoff's voltage law* holds. Note that a triangulation whose voltage assignment satisfies Kirchhoff's voltage law lifts to a triangular embedding of the derived graph.

We will need the following sequence of lemmas which show how certain modifications of the base graph G affect the derived voltage graph $\tilde{G}$. The first shows that the derived embedding is well defined.

Lemma 3.1. *Let G_1 and G_2 be equivalent and embedded voltage graphs. Then the derived embedded graphs $\tilde{G}_1$ and $\tilde{G}_2$ are equivalent.*

Proof. A local switch of the signature in the base graph lifts to a local switch of the signatures in the derived graph. Simultaneously reversing the local orientation in the base graph reverses the local orientation in the derived graph. Hence the two derived embeddings are equivalent. □

The next lemma allows us to perform a *local voltage modification.*

Lemma 3.2. *Let G be an embedded voltage graph. Form G' from G by replacing each voltage ν with $\alpha\nu$ on the edges directed out from a vertex v. Then the embedded derived graph $\tilde{G}'$ is isomorphic to the embedded derived $\tilde{G}$.*

Proof. We merely use an isomorphism from $\tilde{G}$ to $\tilde{G}'$ which maps each vertex (v, γ) in the fiber above v to $(v, \gamma\alpha^{-1})$, while leaving all other vertices fixed. □

We say two voltage graphs are *equivalent* if we can form one from the other by a sequence of local voltage modifications.

For a graph G define the *subdivided graph* G' as the graph formed by replacing every edge e with a path e_1, e_2 of length two. The *subdivided voltage assignment* has voltage $v(e_1) = v(e)$ and $v(e_2) = \mathrm{id}$ (the identity element). Note that by the preceding lemma, the order of the path does not matter, as the two subdivided voltage assignments are equivalent under a local voltage modification at the new degree two vertex. The following is immediate.

Lemma 3.3. *Let G' the subdivided voltage graph of G. Then the derived $\tilde{G}'$ is the subdivision of the derived G.*

Of course, we could subdivide some subset of the edges, even a single edge, instead of every edge. Finally, the following lemma allows us to add edges across a face of the embedding.

Lemma 3.4. *Let G be an embedded voltage graph, and let u and v be two vertices on a face f. Let P be a portion of the boundary walk of f from u to v and $v(P)$ be the product of the voltages assigned to the edges of P. Form G' from G by adding in f the edge $e = uv$ with voltage $v(P)$. Then the derived graph $\tilde{G}'$ is the derived graph $\tilde{G}$ with edges $\tilde{e}$ added in the faces $\tilde{f}$ lying above f.*

Proof. By our choice of the voltage on the new edge we can always add in the corresponding covering edges in the derived graph. □

Current graphs

We begin with a graph G^* embedded in a surface S with signed rotation (ρ, σ). Let F^* denote the faces of the embedding. A *current assignment* on the embedded G^* is a mapping κ from the directed edges of G^* to elements in some Γ such that $\kappa(e^-) = \kappa(e^+)^{-1}$ for straight edges and $\kappa(e^-) = \kappa(e^+)$ for twisted edges. The elements of Γ are called the *currents*, and Γ is called the *current group*. Note that when assigning voltages and currents to twisted edges the two directions receive inverse voltages but identical currents.

We can form a *derived graph* $\tilde{G}$ which 'lies above the dual G' by using the current assignment and the signed rotation on G. The vertex set of $\tilde{G}$ is $\tilde{V} = F^* \times \Gamma$; we say that each face of G *lifts* to $|\Gamma|$ vertices of $\tilde{G}$. Let e^+ be an edge of G, and suppose that the rotation at the initial vertex of e^+ carries face f to face g across e^+. Then e^+ lifts to $|\Gamma|$ edges in $\tilde{G}$, joining vertices (f, α) to $(g, \alpha\kappa(e^+))$ for each $\alpha \in \Gamma$.

We must show that the derived graph is well defined, as we may choose the direction on e in two ways. But if the edge e is straight, then the rotation at the initial vertex of e^- carries g to f, and lifts to edges (g, α) to $(f, \alpha\kappa(e^-)) = (f, \alpha\kappa(e^+)^{-1})$. If e is twisted, then the rotation at the initial vertex of e^- still carries f to g, and lifts to edges joining (f, α) to $(f, \alpha\kappa(e^-)) = (f, \alpha\kappa(e^+))$. It is to make the derived graph well defined that we required reversing the direction on an edge to invert the current if the edge is straight but to leave the current unchanged if the edge is twisted.

It appears that the current assignment, as well as the derived graph, is dependent on the particular (ρ, σ) used to define the graph. Suppose that we form (ρ', σ') by switching signs at a vertex v. Then we define the *local current switch* κ' by $\kappa'(e^+) = \kappa(e^+)^{-1}$ if the initial vertex of e^+ is v, and leaving the current unchanged otherwise. Define two current graphs to be *equivalent* if we can form one from the other by a sequence of local sign switches with corresponding local current switches.

Lemma 3.5. *Two equivalent current graphs generate identical derived graphs.*

Proof. Switching at v toggles the type of each incident edge, but inverting the current on outgoing edges implies that κ' still has inverse currents on straight edges and identical currents on twisted edges. Hence κ' is a current assignment. We have also reversed the local orientation. Now the rotation at the initial vertex of e^+ carries g to f. It follows that (g, α) is adjacent to $(f, \alpha\kappa'(e^+))$ for each α. And so $(g, \alpha\kappa(e^+)) = (g, \alpha\kappa'(e^+)^{-1})$ is adjacent to (f, α) as desired. $\square$

By Lemma 3.5 the derived graph depends only on the embedding and the currents, not on the particular signed rotation used to describe the embedding.

The derived graph has a natural *derived embedding*. The rotation scheme for this embedding is given by lifting the face walks of the embedded G^*. We begin by fixing an orientation on each face $f \in F^*$, and let $(e_1, e_2, \ldots, e_k)$ be the boundary walk of f in the induced direction. Each vertex $(f, \alpha) \in \tilde{V}$ is incident with an edge $\tilde{e}_i$ lying above e_i. The local rotation of (f, α) is then $(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k)$. We next need to define the signature. Let f and g be the faces incident with an edge e. If the orientations on f and g induce opposite directions on e, then each edge $\tilde{e}$ above e is straight. If the induced directions agree each $\tilde{e}$ is twisted.

We must show that the derived embedding is well defined, as we may choose the orientation on f in two ways. But reversing the orientation on f reverses the local rotations at each (f, α) and changes the signature of each incident edge, so that it gives an equivalent derived embedding.

There is a natural way to describe the faces of the derived embedding. In particular, it is easy to determine the number and size of the faces. Define the *excess current* κ_n at a vertex v as the product of the currents directed out of v of G, in the order determined by the local rotation. The excess current will depend on the starting edge of the product (if the current group is non-abelian) as well as the direction induced by the local rotation, but any two such values are conjugate or inverses and hence have the same order. There are $\text{index}_\Gamma(\kappa_v)$ faces in the derived $\tilde{G}$ lying above v, each of size $\deg_G(v) \cdot \text{order}_\Gamma(\kappa_v)$. This has been most widely used in the case κ_v is the identing for each v (*Kirchhoff's current law* holds) and G is cubic. In this case, the embedding of the derived graph is triangular, and hence a genus embedding.

The following sequence of lemmas are similar to those for voltage graphs; in fact, we shall see that they are dual. Their proofs are omitted.

Define the *edge duplicated current graph* G' by replacing every edge with two edges in parallel. The edges lie alongside each other in the embedding; that is, they bound a face of size two. We define a *modified current assignment* on the derived graph by giving one copy of the edge the current inherited from the current on G, and the other copy the identity.

Lemma 3.6. *The derived graph $\tilde{G}'$ of the edge duplicated graph is the subdivision of the derived graph of G^*.*

Finally, define a splitting of a vertex, $\mathrm{sp}_v(G^*)$ in an embedded current graph. Divide the edges incident with v into sets E_1, E_2 compatible with the rotation at v; that is, so that there are only two edges such that $\rho(e)$ is not in the same part as e. The vertex set of the split is formed by replacing v with two adjacent vertices. Each new vertex v_i is incident with those edges in E_i (in place of their incidence with v). We modify the embedding locally as shown in Fig. 2. Contracting the new edge v_1v_2 in the surface recovers the original graph with the original embedding. Note that we do not define an arbitrary splitting, only one in which the bipartition of the edges respects the rotation. We give both new vertices the same local orientation as the original vertex, make the new edge straight, and assign it an identity current.

Lemma 3.7. *Let $\mathrm{sp}_v(G^*)$ be a split of an embedded voltage graph G. Then the derived graph is formed from the derived graph of G by adding an edge across each face corresponding to v such that E_1 and E_2 together with this new edge are the two new face boundaries.*

Current-voltage duality

Let G be a current graph embedded in a surface S and let G^* be its dual. Choose a local orientation at each vertex of G and G^*; these local orientations determine rotation schemes and signatures for the embeddings. We will show

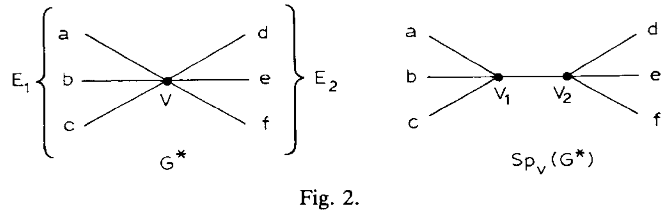

Fig. 2.

how to *transfer* a current assignment on G^* to a voltage assignment on G which generates the same derived graph. Likewise, we will transfer a voltage assignment to a dual current assignment.

We begin with a current assignment on G^*. Let e be an edge of G^* and fix a direction e^+. Let f and g be the faces incident with e, and suppose that the local rotation at the initial vertex of e^+ carries f to g across e. Let e^{*+} be the dual edge directed from f to g. Then we assign the voltage $v(e^{*+}) = \kappa(e^+)$.

We need to verify that the transferred voltage assignment is well defined, as there is a choice in the direction e^+. But if e is a straight edge, then the local rotation at the initial vertex of e^- carries g to f, and the directed dual edge e^{*-} is assigned voltage $v(e^{*-}) = \kappa(e^-)$. Since the opposite directed arcs e^+ and e^- receive inverse elements, so do the opposite directed arcs e^{*+} and e^{*-}. Likewise, if e is a twisted edge then the local rotation at the initial vertex of e^- carries f to g, and the directed dual edge again gets voltage $\kappa(e^-)$. In this case the two directed dual edges are identical, and receive the same voltage.

In Fig. 3 we illustrate how to transfer the current α on the solid edge to a voltage α on the dashed dual edge. The four cases depend on the type and direction of $e \in E(G^*)$.

It is a straightforward application of the definitions to verify that the derived graphs of the dual current and voltage assignments are the same.

We next investigate how to reverse the above process, that is, how to transfer a voltage assignment to a current assignment. As before, let G be an embedded graph with dual G^* and with voltage assignment v. Moreover, let e^* be an edge of G^* with dual edge e.

Fix a direction e^{*+} and suppose that the local rotation at the initial vertex carries face f to face g across e. Give the primal edge a direction e^+ which runs from f to g in G (recall that the faces of G^* correspond to the vertices of G). Then we can define $\kappa(e^{*+}) = v(e^+)$.

In Fig. 3 we show a voltage α on the dashed edge transferred to a current α on the dual solid edge. The four cases arise from the direction and type of the dual edge. From this figure we deduce that the transfer is well defined.

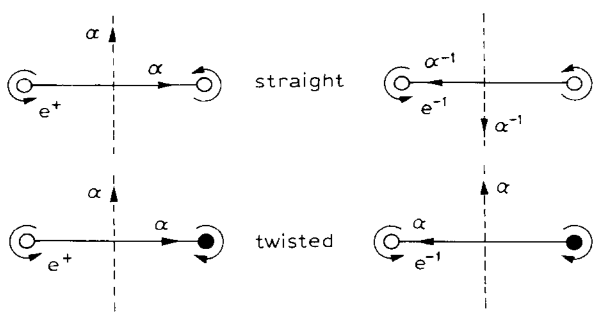

Fig. 3.

If we first transfer a current assignment to a voltage assignment and then transfer back again, we regain the original current assignment. Similarly, the two processes are inverses of each other in the other order. This shows that our transference of a voltage assignment to a current assignment preserves the derived graph. If an embedded voltage graph and dual embedded current graph are related in this way we call them *dual embedded voltage-current graphs.* We can then summarize the above as follows.

Theorem 3.8. *Let (G, v) and (G^*, κ) be dual embedded voltage-current graphs. Then the two derived embeddings are identical.*

Under this relationship two theories of current graphs and voltage graphs are essentially the same.

4. Transferring voltages and currents to the medial

Voltages on medial graphs

Let M be a medial graph in S of some dual pair of graphs G and G^*. Let v be a voltage assignment on M from the voltage group Γ. Form the derived graph $\tilde{M}$ with a derived embedding in a surface $\tilde{S}$. Note that $\tilde{M}$ is 4-regular, since voltage assignments preserve degrees in the derived graph. Also note that the faces of the embedding can be 2-colored; we merely lift the 2-coloring on the faces of M. Hence by Theorem 2.1, M is the medial graph of some dual pair of embedded graphs $\tilde{G}$ and $\tilde{G}^*$. Name these graphs so that $V(\tilde{G})$ covers $V(G)$ and $V(\tilde{G}^*)$ covers $V(G^*)$.

Transferring a voltage assignment to the medial graph

We now describe a method for transferring a voltage assignment v on an embedded graph G to a voltage assignment v_M on its medial graph M. We want the embedded derived medial graph $\tilde{M}$ to be the medial graph of the derived embedding $\tilde{G}$. For notational convenience we first transfer the voltage assignment to the subdivided medial graph M'. This corresponds to a voltage assignment on M, as we can perform a local voltage modfication (Lemma 3.2) on each degree-two vertex so that one incident edge receives the identity voltage, then collapse that edge using Lemma 3.3.

Let e be an edge of G and fix a direction e^+ on e. Let v_e be the corresponding vertex in the subdivided medial graph. There are two corners of the original embedding centered at the initial vertex of e and containing e. These correspond to two edges in the subdivided medial graph incident with v_e. Direct these two edges e_1^+ and e_2^+ so that they have terminal vertex v_e. The voltage $v(e^+)$ is then assigned to both e_1^+ and e_2^+. Of course, the oppositely directed e_1^- and e_2^- receive the inverse voltage element.

We need to check that the transferred voltage assignment on M' is well defined. We first note that each edge of M' is incident with a unique vertex of

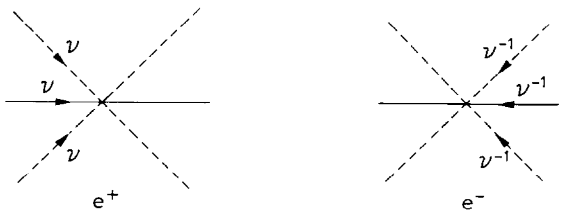

Fig. 4.

degree 4 and hence can be assigned a voltage transferred from at most one edge e. In other words, if we transfer all of the voltages simultaneously we do not get conflicting voltage assignments on an edge of M'. We also need to check that the voltage assigned to M' does not depend on the choice of a direction on e. But if we reverse the direction e^-, the other two edges incident with v_e receive transferred current v^{-1}. Hence the two subdivided medial voltage assignments generate isomorphic graphs and isomorphic embeddings by Lemma 3.2. Figure 4 illustrates the transference of a voltage v on G (solid lines) to voltages v on the medial (dotted lines).

We now show that the derived $\tilde{M}$ under the transferred voltage assignment is the medial graph of the derived $\tilde{G}$. Define the *total graph* $T(G)$ of an embedded graph G as the graph formed by subdividing each edge of G and adding in edges across each corner. Thus the total graph has both the medial graph and the subdivided primal graph as subgraphs.

Theorem 4.1. *Let G be an embedded voltage graph and let M be its medial graph with the transferred voltage. Then the derived medial $\tilde{M}$ is the medial of the derived $\tilde{G}$.*

Proof. We begin first with the embedded graph G. By Lemma 3.3 the subdivided graph G' has a voltage assignment which generates the subdivided derived graph $\tilde{G}'$. We now add, one by one, the edges across the corners using Lemma 3.4. After adding each such edge, we obtain an embedding of the total graph $T(G)$. Figure 5 shows a voltage on the subdivided G' extended to a voltage on T, where some edges of T have been subdivided for clarity.

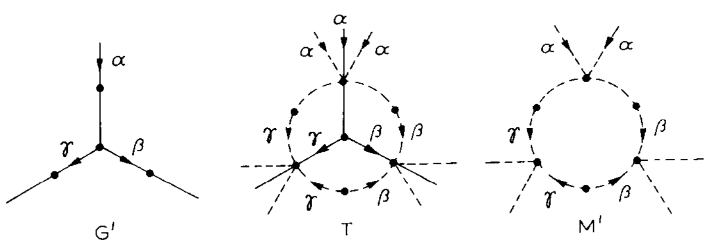

Fig. 5.

Similarly, consider the medial graph M with the transferred voltage assignment. Then across each face of M which corresponds to a vertex of g we add in edges (with the first one subdivided) using Lemma 3.4 until we again build an embedding of the total graph $T(G)$. It is easy to check that the two embeddings of $T(G)$ are equivalent under a sequence of local voltage changes, hence they generate isomorphic embeddings by Lemma 3.2. For example, Fig. 5 shows the voltage on G' transferred to voltages on a subdivided medial M', with the extension to T.

When we undo the operations in the derived graph which formed the total graph, we obtain in the first case the derived graph $\tilde{G}$ and in the second case the derived graph $\tilde{M}$. It follows that $\tilde{M}$ is the medial of $\tilde{G}$ as desired. □

Characterization of a transferred voltage assignment

When is a voltage assignment on the medial graph equivalent to one transferred from a voltage assignment on the primal graph? Before answering this question, define a *black*, or *primal-vertex face* of the medial graph as one which corresponds to a vertex of the primal, while a *white*, or *dual-vertex face* corresponds to a vertex of the dual.

Theorem 4.2. *A voltage assignment v_M is a transfer of a voltage assignment v on the primal graph if and only if the excess voltage on each primal-vertex face is the identity.*

Proof. Let v be a vertex of G, let e^+ be an edge directed from v, and let f_v be the primal-vertex face in M corresponding to v. Then in the subdivided medial graph the voltage $\nu(e^+)$ transfers to two consecutive voltages $\nu(e^+)$ and $\nu(e^+)^{-1}$. These voltages cancel when calculating the excess voltage on f_v, so that the excess voltage on f_v is the identity.

Conversely, suppose that we are given a voltage assignment on M with no excess current on any primal-vertex face. Subdivide each edge of M to form M'. Now, fix a primal-vertex face f_v of M and let $(e_1, \ldots, e_n)$ be the boundary cycle of f_v. Each e_i corresponds to two edges e_i' and e_i'' in M'. Direct these edges so that their initial vertices are of degree 2. Thus the boundary walk in f_v is $e_1'^-, e_1'', \ldots, e_n'^-, e_n''$. Define the voltage assignment on M' by $\nu(e_k'^-) = \prod_{i=1}^{i=j-1} \nu(e_i)$ (so that $\nu(e_1')$ is the identity), and $\nu(e_k'') = \prod_{i=1}^{i=k} \nu(e_i)$. (See Fig. 6 for

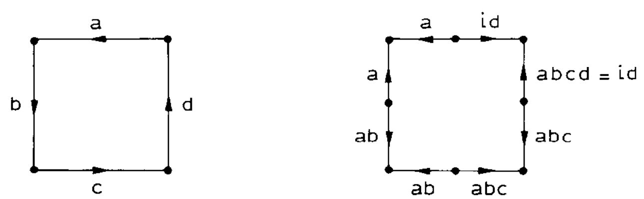

Fig. 6.

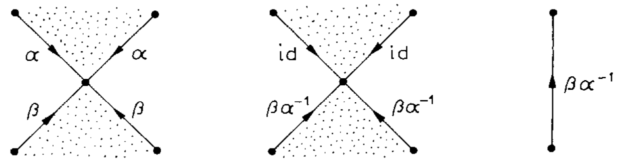

Fig. 7.

an example.) Note that $v(e_i')v(e_i'') = v(e_i)$, so that this voltage assignment yields the same derived graph by Lemmas 3.2 and 3.3. Also note that at each degree 4 vertex of the two incoming voltages are identical; in particular, this is true at the vertex between e_1 and e_n since the excess voltage of f_v is the identity.

We now show how this equivalent voltage assignment on M' arises from a voltage assignment on G. At each vertex v of G we have 4 edges directed in towards v. These edges receive voltages α, α, β, β where two edges with the same voltage lie on the corner of a black face. We now change the local voltage assignment at this degree 4 vertex, so that the edges now get voltages id, id, $\beta\alpha^{-1}$, $\beta\alpha^{-1}$ (see Fig. 7). This does not change the derived graph by Lemma 3.2. If we let u be the vertex corresponding to the black face with the corner receiving voltage $\beta\alpha^{-1}$ and v be the other black corner, then this voltage assignment arises (by subdividing, as in Lemma 3.3) from a voltage $\alpha^{-1}\beta$ on the edge uv in G. □

Transferring current assignments to the medial graph

We next describe a method for transferring a voltage assignment κ on an embedded graph G^* to a voltage assignment v_M on its medial graph M. Again, we want the embedded derived medial graph $\tilde{M}$ to be the medial graph of the derived embedding $\tilde{G}$. As in the transfer of a voltage assignment to the medial, we will use the subdivided medial M'.

Let e be an edge of G^* and fix a direction e^+ on e. Suppose that the rotation at the initial vertex of e^+ carries face f to face g across e. There are two corners of g containing e. These correspond to two edges in the subdivided medial graph. Direct these two edges e_1^+ and e_2^+ so that they have terminal vertex v_e, the vertex of the medial corresponding to e. The current $v(e^+)$ is then assigned as a voltage to both e_1^+ and e_2^+. Of course, the oppositely directed e_1^- and e_2^- receive the inverse voltage element.

We need to check that this voltage assignment on M' is well defined. But the proof is similar to the proof in the voltage case, and is omitted. We do, however, refer the reader to Fig. 8, where we show a current α on G^* (solid lines) transferred to a voltage α on the medial (dashed lines). The four cases depend on the type and direction of e.

Theorem 4.3. *The derived graph $\tilde{M}$ from the transferred current assignment is the medial graph of the derived graph.*

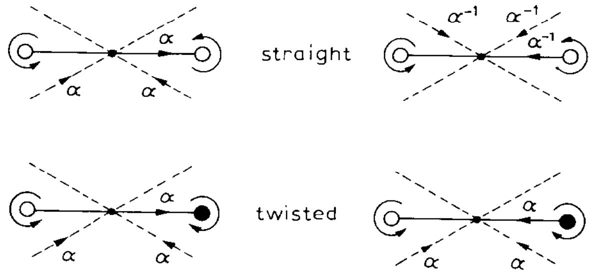

Fig. 8.

Proof. The proof is similar to that of Theorem 4.1. By the duplication of edges and a sequence of vertex splittings on G^* we could form a related current graph $G^{*\prime}$ whose derived graph is the total graph $T(\tilde{G})$. We can recover $\tilde{G}$ by deleting the edges and suppressing the degree two vertices created by the edge duplications and vertex splittings on G^*. But by starting with the medial graph with the voltage assignment transferred from the current graph, we can build the total graph of $T(G^*)$, which lifts to the same $T(\tilde{G})$. □

We could also prove the preceding theorem by citing the voltage-current duality shown in Section 3. The current assignment on G^* transfers to a voltage assignment on G with the same derived graph. This voltage assignment transfers to a voltage on the medial graph M which builds the medial graph of the derived graph. We would need only check that the composition of the two transfers gives the same voltages on M as does transferring the currents directly form G^* to M.

The proof of the following is identical to that of Theorem 4.2.

Theorem 4.4. *A voltage assignment v_M is a transfer of a current assignment v on the dual graph G^* if and only if the excess voltage on any black face is the identity.*

We could use the medial graph to give a new proof of voltage-current duality. We start with a current assignment on G^*. If we transfer it to a voltage assignment on M then no black face has excess current by Theorem 4.4. It follows from Theorem 4.2 that this medial assignment also arises from a voltage assignment on G. Likewise, a voltage assignment on G gives the same derived graph as some current assignment on G^*. These dual current-voltage assignments are, of course, the ones which we get by transferring directly.

Simultaneous voltage and current assignments

We now introduce one of the most important ideas of this paper; we show how to simultaneously make a voltage and a current assignment on a graph. Under duality, this is equivalent to making a current assignment to both the primal and dual graphs, or to making a voltage assignment to both the primal and dual. The trick is to transfer both assignments to the medial graph, take the lift of that medial graph to covering medial graph, and then to return back to the primal and dual of this derived medial. We consider only the case when the voltage-current group is abelian, as there is a much cleaner analysis of the derived graph in this case.

We begin with an embedded graph G and an abelian group Γ. Let κ be a current assignment using currents from Γ, and let ν be a voltage assignment using voltages from Γ. Thus Γ plays both the role of the current group and the voltage group. We would like to transfer both assignments directly to the subdivided medial graph, but there now is a problem. An edge of the subdivided medial graph may receive both a transferred current and a transferred voltage. For suppose that a directed edge e receives voltage ν and current κ. Let v_e be the vertex in the subdivided medial corresponding to the edge e. Then when we transfer the voltage, two edges incident with v_e will receive voltage ν, one in each face of G. But when we transfer the current two edges incident with v_e will receive voltage κ, both in the same face. So one edge should receive both voltage κ and voltage ν (see Fig. 9, where the edge indicated receives a voltage ν and the current κ has been transferred to a voltage κ on the dual vertical edge). In this case assign the voltage $\nu + \kappa = \kappa + \nu$ (the group is abelian). By Lemma 3.3 this voltage assignment on the subdivided medial corresponds to a voltage assignment on the medial graph.

Let v be a vertex of G, and let f_v be the black face of the subdivided medial corresponding to v. What is the excess voltage around f_v? Let $e_1, \ldots, e_n$ be the edges incident with v directed away from v. Then each $\nu(e_i)$ appears twice in the boundary walk of f_v, once in each direction. Since the group is abelian these transferred voltages cancel. On the other hand, each $\kappa(e_i)$ occurs once in the boundary walk. We have the following.

Lemma 4.5. *The excess voltage on a black face of the medial is the excess current on the corresponding vertex in the primal.*

Similarly we get the next lemma.

Lemma 4.6. *The excess voltage on a white face of the medial is the excess voltage on the corresponding face of the primal.*

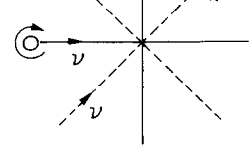

Fig. 9.

This independence will prove very useful in the applications. The number and degrees of the vertices depend only on current assignment. Likewise the number and sizes of the faces depend only on the voltage assignment. However, we emphasize that the adjacencies in the derived graph depend on both the current and voltage assignments. We will introduce a special case in the next section in which these adjacencies are especially easy to describe.

We note that the previous two lemmas are false if we allow a non-abelian current-voltage group. For when an edge of the subdivided medial is to receive both a voltage ν transferred from a voltage assignment and a voltage κ transferred from a current assignment we cannot assign a voltage which ensures that the ν's cancel on the black faces and the κ's cancel on the white faces. We could introduce an additional requirement that the voltage and current assigned to an edge always commute, but this does not seem to gain much advantage. If we arbitrarily establish that the transferred currents should cancel, it is difficult to determine the excess voltage around a white face. Similarly we cannot arbitrarily establish that the transferred voltages should cancel. If the reader feels that the use of a non-abelian group is essential in an application, then the author recommends working directly with a voltage assignment on the medial graph.

5. Properties of the derived graph

In Section 4 we saw how to make simultaneous current and voltage assignments on a graph G from an abelian group Γ and how to find the derived graph $\tilde{G}$. In this section we study the general relationship between G and $\tilde{G}$, where the derived graph is formed from a voltage assignment on the medial graph M of G. We do not require that the voltage group be abelian, although in practice most of our medial voltage assignments will come from simultaneous current and voltage assignments and hence the group will be abelian.

We first examine the lift of a vertex in G to vertices in $\tilde{G}$. Let v be a vertex of G and let f_v be its corresponding face in the medial M. Suppose that f_v gets an excess voltage of ν_v. Then in the derived graph $\tilde{M}$, f_v lifts to $\text{index}_\Gamma(\nu_v)$ faces, each of size $\text{order}_\Gamma(\nu_v) \cdot |f_v|$. It follows that $\tilde{G}$ has $\text{index}_\Gamma(\nu_v)$ vertices lying above v, each of degree $\text{order}_\Gamma(\nu_v) \cdot \deg_G(v)$.

We next examine the lift on an edge $e = uv$ of G. This corresponds in M to black faces f_u and f_v incident with a common vertex v_e. These in turn lift to sets of faces $f_{u,i}$ and $f_{v,j}$ incident with vertices of the form (v_e, α) for $\alpha \in \Gamma$. If (v_e, α) is on the boundary of $f_{u,i}$ and $f_{v,j}$ then it corresponds to an edge between vertices u_i and v_j in $\tilde{G}$.

We begin with the vertex (x_e, id) (where id is the identity element in the group) incident with say faces $f_{u,0}$ and $f_{v,0}$. If we walk once around the lift of the boundary of f_u we arrive at the vertex (v_e, ν_u) in $\tilde{M}$. Continuing the walk a second time gets us to (v_e, ν_u^2). It follows that the face $f_{u,0}$ contains exactly those vertices of the form (v_e, α) where α is in the subgroup generated by ν_u, denoted $\langle \nu_u \rangle$.

More strongly, other faces $f_{u,i}$ will contain vertices of the form (v_e, α) where α is in the ith coset of $\langle \nu_u \rangle$. Thus the faces above f_u are in a bijective correspondence with the cosets of $\langle \nu_u \rangle$. A similar statement holds for the faces $f_{v,j}$. Hence the lift of an edge e in G is determined by the intersection pattern among cosets of the groups generated by the excess voltages at the medial faces corresponding to the ends of e. We formalize this in the following theorem.

Theorem 5.1. *Let $e = uv$ be an edge in G where the faces corresponding to u and v in M get excess currents ν_u and ν_v respectively. Then e lifts to a subgraph having* $\text{index}_\Gamma(\langle \nu_u, \nu_v \rangle)$ *components, each a complete bipartite subgraph with vertex parts of sizes* $\text{index}_{\langle \nu_u, \nu_v \rangle}(\langle \nu_u \rangle)$ *above u and* $\text{index}_{\langle \nu_u, \nu_v \rangle}(\langle \nu_v \rangle)$ *above v, with each edge duplicated $|\langle \nu_u \rangle \cap \langle \nu_v \rangle|$ times.*

For arbitrary values ν_u and ν_v the derived graph is difficult to describe. In the following special cases the description is easier. We say that two subgroups of Γ are *disjoint* if they have only the identity element in common. The following two special cases of Theorem 5.1 are of special interest.

Corollary 5.2. (i) *There is at most one edge joining $\tilde{u}$, $\tilde{v}$ provided that ν_u and ν_v generate disjoint subgroups.*

(ii) *There is at least one edge joining $\tilde{u}$, $\tilde{v}$ provided that ν_u and ν_v together generate all of Γ.*

Even more special, we say that two elements ν_u and ν_v are *disjoint generators* for a group Γ if together they generate the group, but individually they generate disjoint subgroups. Then we have the following.

Corollary 5.3. *If $e = uv$ has u and v receiving disjoint generators in the transferred voltage assignment on M, then e lifts to a complete bipartite subgraph with vertex parts the vertices above u and above v.*

6. Wrapped coverings and compositions

The dual derived graph

There is a basic asymmetry in the theory of voltage-current duality; both the voltage on the primal graph and the current on the dual graph are used to construct the same derived graph. Moreover there is a graph covering from the derived $\tilde{G}$ to the primal G. What can we say about the dual $\tilde{G}^*$ of the derived graph? Is there a map from $\tilde{G}^*$ to G^*? This issue was first addressed by Jackson, Parsons and Pisanski [21, 22] (see also [1]), who developed wrapped covering maps. We now interpret their theory using the medial graph.

Wrapped coverings

Let G and $\tilde{G}$ be graphs. A *wrapped covering of order n* is a homomorphism ω from $\tilde{G}$ to G such that $E(\tilde{G})$ maps to $E(G)$ in an $n-1$ fashion and for each $\tilde{v} \in V(\tilde{G})$ there is a positive integer $\delta(\tilde{v})$ such that the edges incident to $\tilde{v}$ map to the edges incident with $v = \omega(\tilde{v})$ in a $\delta(\tilde{v})$-1 fashion. This δ is called the *wrapping index* of ω at $\tilde{v}$. The sum of the wrapping indices above a vertex $v \in G$ is n. If the wrapping index is 1 for each vertex of $\tilde{G}$, then the wrapped covering is a covering in the usual sense. For example, Fig. 10 shows a wrapped covering of the 3-wheel by the 6-wheel (vertices on the left are labeled by their image on the right). Every vertex of the 6-wheel has wrapping index 1, except the hub which has wrapping index 2.

Suppose that $\tilde{G}$ is a wrapped covering of G, and that these graphs are embedded in surfaces $\tilde{S}$ and S respectively using signed rotations $(\tilde{\rho}, \tilde{\sigma})$ and (ρ, σ) respectively. We say that $(\tilde{\rho}, \tilde{\sigma})$ is a *wrapped signed rotation* provided that σ preserves the type of each edge and $\omega\tilde{\rho} = \rho\omega$. Thus if the local rotation at v looks like $(e_1, \ldots, e_n)$, then the local rotation at the covering vertex $\tilde{v}$ looks like $(\tilde{e}_1^1, \ldots, \tilde{e}_n^1, \tilde{e}_1^2, \ldots, \tilde{e}_n^2, \ldots, \tilde{e}_1^k, \ldots, \tilde{e}_n^k)$, where k is the wrapping index at $\tilde{v}$ and each $\tilde{e}_i^j$ maps to e_i under ω. For example, Fig. 10 also illustrates a wrapped generalized rotation scheme, where the graphs are embedded in the plane as shown. Note that this wrapped covering extends to a branched covering of the sphere by itself with two branch points, one at the hub and one at ∞. Jackson, Parsons and Pisanski [21] have shown that in the orientable case a wrapped covering always extends to a branched covering of the surfaces. The corresponding result for non-orientable wrapped coverings was shown in [1]. The following holds for both orientable and non-orientable surfaces.

Theorem 6.1. *Let $\tilde{G}$ be an embedded wrapped covering of an embedded G using a wrapped signed rotation. Then the dual $\tilde{G}^*$ is an embedded wrapped cover of the dual G^*, with wrapped signed rotations. Moreover, these wrapped coverings extend to a branched covering of the surfaces, where the prebranch points occur precisely at the vertices of $\tilde{G}$ and $\tilde{G}^*$ with wrapping index exceeding* 1.

Proof. The wrapped covering and wrapped signed rotation can be used to show that the medial graph $\tilde{M}$ is a covering graph of M. Each face boundary in $\tilde{M}$ maps

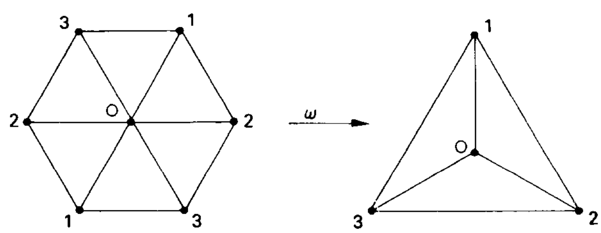

Fig. 10.

to the corresponding face boundary in M with wrapping since the signed rotations are wrapped. The result now follows by extending this wrapping to a disk with a single branch point as in [10]. □

The extension to embedded wrapped coverings restores a full symmetry to current-voltage duality by the following theorem, whose proof is omitted.

Theorem 6.2. *Let G^* be a current graph. Then the dual $\tilde{G}^*$ of the derived graph $\tilde{G}$ is an embedded wrapped covering of G^*. Moreover the branch points are at precisely those vertices of G^* with nontrivial excess current, and the order of the branching is the order of the excess current.*

Compositions

Among all wrapped coverings the easiest to describe—and perhaps the most interesting—are the following.

Let G be a graph and let n be an integer. The *composition* $G^{(n)}$ is the graph formed from G be replacing each vertex with n independent vertices, and replacing each edge of G with a complete bipartite graph $K_{n,n}$ whose vertex parts are the new vertices replacing the ends of e. For example, the composition $K_n^{(m)}$ is the regular multipartite graph with n sets of m vertices, $K_{n(m)}$.

There is a natural projection map from $G^{(n)}$ to G, and it is easy to check that this is a wrapped covering. In particular this projection maps the edges of $G^{(n)}$ to the edges of G in an n^2-to-1 fashion. Observe that $(G^{(n)})^{(m)} = G^{(nm)}$. This will prove useful, as the upcoming constructions are easier for n prime.

If $G_{A,B}$ is bipartite with vertex parts A and B, then we define a *bipartite composition* $G_{A,B}^{(n,m)}$ as the graph formed by replacing each vertex in A by n independent vertices, each vertex in B by m independent vertices, and each edge by the complete bipartite graph $K_{n,m}$ on these new vertices. For example, the composition $K_{a,b}^{(c,d)}$ is $K_{ac,bd}$, where we let vertex part A be the a vertices of degree b.

As before, the natural projection map is a wrapped covering. It maps the edges of the composition to the edges of the base graph in an nm-to-1 fashion. We will use $G_A^{(n)}$ to denote the bipartite composition $G_{A,B}^{(n,1)}$. In the upcoming theorems it will prove useful to consider only bipartite compositions $G_A^{(p)}$ with p prime. But we can form any bipartite composition by iterating such unilateral prime bipartite compositions.

In general we would like to replace vertices of G with sets of independent vertices, and edges with complete bipartite graphs. However, if we want the projection map from the composition to the original graph to be a wrapped covering, then we must have nm constant for each edge, where n and m are the number of vertices above the ends of that edge. It follows that the compositions described in the preceding two paragraphs are the only ones where the projection map is a wrapped covering.

We close by noting that Harary [8] defines a general composition of two graphs. Comparing notations, our $G^{(m)}$ in his $G[mK_1]$.

7. Assigning excess currents

In this section we examine how to assign currents from an abelian group Γ to an embedded graph G^* so that the dual derived graph $\tilde{G}^*$ is a composition. Let ν_u and ν_v be the excess voltages on the faces of the medial corresponding to adjacent vertices u and v. Then by Lemma 5.3 we need that ν_u and ν_v are disjoint generators for the group Γ. Moreover, note that by Lemma 4.6 this excess voltage on the faces is determined solely by the current assignment on G^*, not by a simultaneous voltage assignment. This independence will allow us to construct (in Section 8) simultaneous voltage-current assignments for which both the primal and dual derived graphs are compositions respectively of the base primal and dual graphs.

What group Γ should be use? To use simultaneous current-voltage assignments we need Γ to be abelian. We also need Γ to have two disjoint generators. The only such groups are either cyclic or the product of two cyclic groups. It is more convenient to work with the latter. In fact, it is most convenient to work with $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is the prime. For in this group, any element has order p, and any two non-identity elements which do not generate the same subgroup must be disjoint generators. Moreover, as discussed in Section 6, we can iterate to form nonprime compositions. Except for the following Lemma 7.1, we will assume that the current group is $\mathbb{Z}_p \times \mathbb{Z}_p$.

Feasible current assignments

Define an *excess current assignment* on G as a function ϵ from $V(G)$ to the current group. We say that ϵ is feasible provided that there is a current assignment κ which has excess current $\epsilon(v)$ for each vertex v. The following lemma characterizes feasible excesss current assignments.

Lemma 7.1. *An excess current assignment ϵ is feasible if and only if $\sum_v \epsilon(v) = 0$. Moreover if $\mathcal{D}$ is a set of edges whose removal does not disconnect the graph, then ϵ is realizable by a current assignment which is identically 0 on $\mathcal{D}$.*

Proof. The proof proceeds by induction on n, the number of vertices with nonzero excess voltage. If $n = 0$ then we can realize ϵ with the current assignment which is identically 0 on every edge. Suppose that $n > 0$, and select a vertex u with $\epsilon(u) \neq 0$. Then since $\sum_v \epsilon(v) \neq 0$ there exists a second vertex v with $\epsilon(v) \neq 0$.

Define an excess current assignment ϵ' by $\epsilon'(u) = 0$, $\epsilon'(v) = \epsilon(v) - \epsilon(u)$, and $\epsilon'(w) = \epsilon(w)$ for all other vertices. Then ϵ' has at most $n - 1$ vertices with a nonzero excess current. By induction there is a current assignment κ' which realizes ϵ'. Moreover we may assumed that κ' is identically 0 on $\mathcal{D}$.

Since $\mathcal{D}$ does not disconnect the graph there is a directed path P from u to v missing $\mathcal{D}$. By a sequence of local switchings we may assume that each edge in P is straight. Define a current assignment κ by $\kappa(e^+) = \kappa'(e^+) + \epsilon(u)$ and $\kappa(e^-) = -\kappa(e^+)$ if e is a directed edge in P, and $\kappa(e) = \kappa'(e)$ otherwise. It is routine to verify that κ realizes ϵ and that κ is identically 0 on $\mathcal{D}$. $\square$

An excess current assignment ϵ is *disjoint* if $\epsilon(u)$ and $\epsilon(v)$ generate disjoint subgroups for each edge uv. Similarly ϵ is *generating* if $\epsilon(u)$ and $\epsilon(v)$ together generate Γ for each edge uv. For the derived graph to be a composition, we need to find a feasible disjoint generating excess current assignment. This phrase is rather unwieldy; call such an assignment a *composition assignment*. Easy ways to find composition assignments are given in the following three propositions.

Proposition 7.2. *Let G be a graph with at least three vertices and with chromatic number k, and let p be a prime at least 5. Then G has a composition assignment in $\mathbb{Z}_p \times \mathbb{Z}_p$ if and only if $p \geq k - 1$.*

Proof. We begin by noting that each element of $\mathbb{Z}_p \times \mathbb{Z}_p$ is of order p. It follows that there are exactly $p + 1$ subgroups of order p, each containing exactly $p - 1$ nonzero elements. Moreover, any two such subgroups are disjoint and generating.

In any composition assignment each vertex has an *associated* order p subgroup, the one generated by its excess current. Since adjacent vertices are associated with disjoint subgroups, the composition assignment gives a proper coloring using the order p subgroups as colors. It follows that the chromatic number is at most $p + 1$.

Conversely, suppose that we have a k coloring of G using the order p subgroups as colors. Then we need to pick one element from each subgroup at a vertex so that their sum is 0. If so, then the disjoint generating assignment is feasible by Lemma 7.1.

Let the vertices of G be $v_1, \ldots, v_n$. Assume without loss of generality that v_{n-1} is associated with $\langle(1, 0)\rangle$, v_n with $\langle(0, 1)\rangle$, and v_1 with $\langle(1, 1)\rangle$. We will define ϵ vertex by vertex. We will require that for each $1 \leq j < n - 1$, $\sum_{i \leq j} \epsilon(v_i)$ is nonzero in each component. Begin by assigning v_1 the excess current $(1, 1)$. Let $j < n - 1$ and suppose that we have defined $\epsilon(v_i)$ for each $i < j$. Let $(a, b) = \sum_{i<j} \epsilon(v_i)$. Since $p - 1 > 2$, there is a non-identity element (c, d) in the subgroup associated with v_j with $c + a \neq 0$ and $d + b \neq 0$ (at most one element makes the first sum 0, and at most one makes the second sum 0). We define $\epsilon(e_j) = (c, d)$. We have left only to define $\epsilon(v_{n-1})$ and $\epsilon(v_n)$. Again let $(a, b) = \sum_{i<n-1} \epsilon(v_i)$. Define $\epsilon(v_{n-1}) = (-a, 0)$ and $\epsilon(v_n) = (0, -b)$. It is clear that the sum of the excess current assignments is 0, and hence it is feasible. $\square$

The cases $p = 2$ and $p = 3$ are somewhat different, and are covered in the following two propositions.

Proposition 7.3. *A graph G has a composition assignment in $\mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if it has a proper vertex 3-coloring with all three color classes having the same parity.*

Proof. There are three pairwise disjoint generating subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$, each with a single nonzero element. Suppose we have a proper 3-coloring of G with color classes of size a, b, and c all of the same parity. Consider the excess current assignment of $(1, 0)$, $(0, 1)$, $(1, 1)$ to each vertex in the first, second and third color classes respectively. Then the first coordinate in the sum of the excess currents is 0 since a and c have the same parity. Similarly the second coordinate is 0 since b and c have the same parity. By Lemma 7.4 the excess current assignment is feasible; it is also disjoint and separating.

Conversely, a composition assignment gives a proper 3-coloring with all three color classes of the same parity. □

Proposition 7.4. *A graph G has a composition assignment in $\mathbb{Z}_3 \times \mathbb{Z}_3$ if and only if it has chromatic number at most 4 and is not K_4 or K_2.*

Proof. That the chromatic number is at most 4 follows as in the preceding propositions.

We proceed as in Proposition 7.2, defining the excess voltage assignment vertex by vertex. Again we assume that v_{n-1} is associated with $\langle(1, 0)\rangle$, and v_n with $\langle(0, 1)\rangle$. We begin by defining $\epsilon(v_1) = (0, 1)$ and $\epsilon(v_2) = (1, 1)$, so that the sum of the excess voltages is nonzero in both coordinates. To define $\epsilon(v_j)$ we pick an element in the associated subgroup so that the sum of the excess currents defined to date is nonzero in the first coordinate. If perchance the second coordinate is zero, we redefine $\epsilon(v_1)$ as the other nonzero element in $\langle(0, 1)\rangle$. We define $\epsilon(v_{n-1})$ and $\epsilon(v_n)$ as before so that the assignment is feasible.

The proof breaks down if we cannot find at least two vertices in at least one color class (to serve as v_1 and v_n), so that the graph is complete. But excess currents $(0, 1)$, $(1, 0)$, and $(2, 2)$ are feasible for K_3. It is straightforward to check that K_2 and K_4 do not have the desired assignment. □

In Section 6 we noted that we lose no generality by considering only compositions $G^{(p)}$ for p prime. But could it be that by using $\mathbb{Z}_n \times \mathbb{Z}_n$ we could find appropriate excess current assignments, even if there was no such assignment for a prime factor p of n? No, as the following argument shows. Let G be a graph with chromatic number k.

To form a composition $G^{(n)}$ using a voltage group $\mathbb{Z}_n \times \mathbb{Z}_n$, where $n = p_1 p_2$, the excess voltage assigned to a vertex must form a subgroup of $\mathbb{Z}_n \times \mathbb{Z}_n$ of order

n. How many such groups are there? Each such group must contain $p_1 - 1$ elements of order p_2. There are $p_1^2 - 1$ such elements. Hence the number of such subgroups is at most $p_1 + 1$. It follows that the chromatic number of G can be at most $p_1 + 1$. A similar argument works for the product of an arbitrary number of primes.

The following theorem summarizes the preceding three propositions.

Theorem 7.5. *Let G be a graph with chromatic number k. Then there exists a sequence of composition assignments yielding $G^{(n)}$ provided that each prime factor of n is at least $k - 1$, except when*

(i) *n is even and every 3-coloring of G has color classes of different partity, and*
(ii) *$G = K_4$ and $n = 3$.*

Proof. Suppose that $G \neq K_4$. By Propositions 7.2–7.4 we can form a composition of order p for each prime at least $k - 1$ provided that (i) does not hold. Note that the graph $G^{(p)}$ also has chromatic number k. Moreover, if p is odd then $G^{(p)}$ has a 3-coloring with all classes of equal parity if and only if G does. We iteratively compose by each prime factor to obtain $G^{(n)}$, noting that $(G^{(p_1)})^{(p_2)} = G^{(p_1 p_2)}$.

If $G = K_4$ we proceed as before, except that we first compose by a prime factor not equal to 3. The resulting composition is not K_4, so that we can compose by 3. If $G = K_4$ and n is a power of three, we begin with the composition assignment $(1,2)$, $(7,8)$, $(1,0)$, and $(0,8)$ in $\mathbb{Z}_9 \times \mathbb{Z}_9$, and then compose by any remaining powers of three. □

The bipartite case

We begin with a bipartite graph G having vertex parts A and B. We restrict our attention to unilateral prime bipartite compositions $G_A^{(p)}$, in which each vertex in one part A is replaced with p vertices having the same adjacencies. We use bipartite composition assignments from the current group $\mathbb{Z}_p$ for p prime. For the derived graph to be $G_A^{(p)}$ we need that each vertex in A receives the identity excess current and each vertex in B receives a non-identity excess current.

Theorem 7.6. *Let G be a bipartite graph with vertex parts A and B. Then there is a sequence of bipartite composition assignments giving the derived graph $G_A^{(n)}$ unless $|B|$ is odd and n is even.*

Proof. That we can make a bipartite composition in $\mathbb{Z}_p$, p prime, follows as in Propositions 7.2–7.4. Note that doing a composition with an odd prime does not change the parity of the vertex part corresponding to B. The proof of Theorem 7.6 now follows as in the proof of Theorem 7.5. □

8. Applications

In this section we apply the theory of composition assignments to construct a wide variety of graph embeddings. We start with quadrilateral embeddings of $K_{m,n}$.

Example 8.1. Consider the 3-cycle embedded in the plane. Then each face is a triangle. Moreover, in a 3-coloring of the vertices all three parts have exactly one vertex. Hence by Theorem 7.5 we can find a sequence of composition assignments giving an orientable triangular embedding of $K_{n,n,n}$ for all n. Thus the genus of this graph is $(n-1)(n-2)/2$, as first shown in (29).

Example 8.2. In [21] Jackson et al. showed that a triangular embedding of G lifts to one of $G^{(m)}$ for all m relatively prime to some M which depends on the chromatic number of G. This result now follows as a corollary to our Theorem 7.5. In fact, we have a smaller M which improves the result.

Example 8.3. Abu-Sbeih and Parsons [2] have results on lifting quadrilateral embeddings of bipartite graphs to quadrilateral embeddings of bipartite compositions. Their results follow as a special case of our Theorem 7.6. They did not find the analogue of our Theorem 7.1, which complicated the statement and proof of their results.

Example 8.4. Consider a quadrilateral embedding of $K_{2,n}$ in the plane. Let A be the vertex part with two vertices of degree n, and let B be the n vertices of degree 2.

If $n = |B|$ is even, then by Theorem 7.6 we can find a sequence of bipartite composition assignments giving the derived graph $G_A^{(m)} = K_{n,2m}$ for all m. Moreover, as the dual base graph is 4-regular, the dual derived graph is 4-regular. So the embedding constructed has every face a 4-cycle. In other words, for all bipartite graphs with both parts even we can find quadrilateral, and hence genus, embeddings.

If n is odd, then again by Theorem 7.6 we can find a sequence of bipartite composition assignments giving $G_A^{(m)} = K_{n,2m}$ for all odd m. As before, we have constructed quadrilateral, and hence genus, embeddings for all complete bipartite graphs with one part odd and the other part 2 modulo 4. (See [27] for the full proof of the genus of $K_{n,m}$.)

It is an easy application of Euler's formula to show that these are the only possible congruence classes of m and n for which quadrilateral embeddings of complete bipartite graphs are possible.

Example 8.5. Figure 11 shows a quadrilateral embedding of $G = K_{4,4}$ in the torus (solid lines). The medial graph is $C_4 \times C_4$ (dotted lines). Note that there is a map

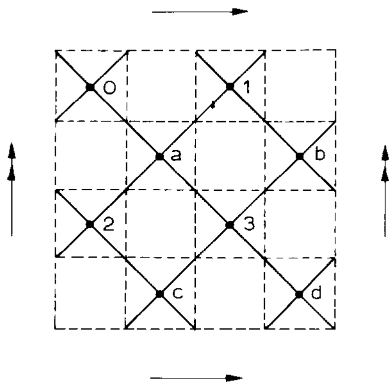

Fig. 11.

isomorphism of the embedded $C_4 \times C_4$ which switches the primal-vertex faces and the dual-vertex faces. It follows that the dual of G is also $K_{4,4}$.

By Theorem 7.6 there exists a current assignment κ on G in $\mathbb{Z}_p$ such that the derived graph is $K_{4p,4}$. Similarly there exists a dual current assignment κ^* on G^* so that the dual derived graph is $K_{4p,4}$. But a current assignment on both G and G^* corresponds to a simultaneous voltage-current assignment on G. By Lemma 4.5 the excess voltage on a primal-vertex face depends only on the excess current on the primal vertex. In particular, it is independent of the dual current assignment κ^*. Similarly, by Lemma 4.6 the excess voltage on a dual-vertex face is independent of κ. It follows that when making both current assignments simultaneously both the derived and dual derived graphs are still $K_{4p,4}$. Hence we have constructed a self-dual embedding of this graph. By iterating this process, switching the vertex parts if needed, we conclude the following.

Theorem 8.6. *$K_{4n,4m}$ has an orientable self-dual embedding for all n and m.*

Such embeddings were also given in [3]. But we can do much more here, as the following shows.

Theorem 8.7. *Let p,q,r,s be even integers exceeding 2 with $pq = rs$. Then there exist both orientable and non-orientable embeddings of $K_{p,q}$ with dual $K_{r,s}$, except that there is no orientable self-dual embedding of $K_{6,6}$.*

Proof. The proof proceeds inductively on the number of prime factors in $n = pq/4$. Since p and q are even integers exceeding 2, n has at least 2 prime factors.

Suppose that n has exactly two prime factors. Then there is a unique way to factor $4n = pq$ into even numbers exceeding 2. Hence we may assume that $r = p$ and $s = q$, so that we are looking for a self-dual embedding of $K_{p,q}$. Both orientable and non-orientable self-dual embeddings of $K_{p,q}$ are given in [3], except that there is no orientable self-dual embedding of $K_{6,6}$. These embeddings give the start of our induction.

Next suppose that n has at least three prime factors. Then at most two of p, q, r, s are twice a prime. Hence we can find a prime factor a of n which divides (without loss of generality) both p and r such that $p' = p/a$ and $r' = r/a$ are even numbers exceeding 2.

In the orientable case we can find an embedding of $K_{p',q}$ with dual $K_{r',s}$, unless $p' = q = r' = s = 6$. But in this case the desired embedding is $K_{6a,6}$ with dual $K_{6a,6}$ which exists by [3]. So we can assume that there is an embedding of $K_{p',q}$ with dual $K_{r',s}$. By Theorem 7.6 we can find a bipartite composition assignment from $\mathbb{Z}_a$ on $K_{p',q}$ which gives the derived graph $K_{p,q}$. Likewise, we can find a bipartite composition assignment on $K_{r',s}$ which gives the derived $K_{r,s}$. By making these current assignments on the embedded graph and its dual simultaneously, we get a derived orientable embedding of $K_{p,q}$ with dual $K_{r,s}$ as desired.

In the non-orientable case we can always find the embedding of $K_{p',q}$ with dual $K_{r',s}$. Similarly we can always find bipartite composition assignments on $K_{p',q}$ and $K_{r',s}$ to get a derived embedding of $K_{p,q}$ with dual $K_{r,s}$. But we need to ensure that the derived embedding is still non-orientable. To do this we first find an orientation-reversing 4-cycle in the base embedding. This is possible, since if every 4-cycle were orientation-preserving the embedding is orientable (the 4-cycles generate the cycle space, so that every cycle would then be orientation-preserving). The four edges in this 4-cycle are not all incident with a common vertex in either G or in G^*. Hence they do not disconnect either graph. By Lemma 7.1 we can realize the bipartite composition assignments so that these 4 edges all get 0 voltage. This implies that the orientation-reversing 4-cycle lifts to an orientation-reversing 4-cycle. Hence the derived embedding is non-orientable, and the theorem is demonstrated. □

The following theorem lifts self-dual embeddings in the nonbipartite case.

Theorem 8.8. *Let G be an embedded self-dual graph with chromatic number k. Suppose that n has no prime factor less than $k-1$, and if n is divisible by three that there is a vertex 3-coloring with all three color classes of equal parity. Then $G^{(n)}$ has a self-dual embedding.*

Proof. The proof follows as in the proof of Theorem 8.6, except that we need to use Theorem 7.5 in place of Theorem 7.6. The special case $G = K_4$ and $n = 3$ is covered in Example 8.10.

The following is a special case of Theorem 8.8.

Example 8.9. It follows from Euler's formula that K_n can have an orientable self-dual embedding only when n is 0 or 1 modulo 4. White [33] constructed such embeddings for $n \equiv 1$ modulo 4; Pengelley [24] did the remaining case. Note that the composition $K_n^{(m)}$ is the regular complete multipartite graph $K_{n(m)}$. By Theorem 8.8 we can lift these embeddings to construct self-dual embeddings of $K_{n(m)}$ for all $n \equiv 0$ or 1 modulo 4 and all m with no prime factor less than $n-1$. In general, this covers only a small fraction of the possible complete multipartite graphs. Stahl [31] has also constructed self-dual embeddings of regular complete multipartite graphs. There is some overlap between his work and ours, but each covers some cases the other cannot.

The special case $n = 4$ is worthy of note, as we get self-dual embeddings of $K_{m,m,m,m}$ for all m odd except $m = 3$.

Example 8.10. We examine the omitted case in the previous example. Specifically, is there a self-dual embedding of $K_{3,3,3,3}$? We begin with the tetrahedron, a self-dual embedding of K_4 into the sphere. We know (Theorem 7.5) that we cannot give a feasible excess current assignment from $\mathbb{Z}_3 \times \mathbb{Z}_3$ to the vertices so that the derived graph is $K_{3,3,3,3} = K_{4(3)}$. In particular, we cannot use Theorem 7.5 to construct a triangular embedding of $K_{4(3)}$. (Indeed, Jungerman [23] reports the results of a computer search which reveal that there is no orientable triangular embedding of $K_{4(3)}$.)

It would seem similarly hopeless to try to construct a self-dual embedding of $K_{4(3)}$ using excess current assignments, since we cannot assign either the primal or the dual requisite currents. So we cannot construct a self-dual embedding of $K_{4(3)}$ using a simultaneous current-voltage assignment on the tetrahedron. But surprisingly, we can construct such a self-dual embedding if we work directly from a current assignment on the medial graph. The medial graph of the tetrahedron is the octahedron. In Fig. 12 we show a voltage assignment on the octahedron. For this assignment it is easy to check that faces corresponding to the vertices of the primal K_4 receive excess voltages $(1,0)$, $(0,2)$, $(1,1)$ and $(2,1)$, while the remaining faces receive excess voltages $(2,0)$, $(0,1)$, $(2,2)$ and $(1,2)$. In both the primal and the dual, adjacent vertices correspond to faces in the medial with disjoint generating excess voltages. Hence the derived graph and its dual are both $K_{4(3)}$.

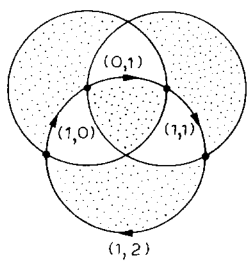

Fig. 12.

Note that the sum of the excess voltages around the medial faces corresponding to the vertices of the primal is not 0. Hence this does not contradict Lemma 7.1. But the total sum of the excess voltages is 0. This must hold in an abelian voltage group with an orientable surface, since we may arrange the sum so that each edge is covered once in each direction. With this arrangement the voltages cancel in pairs.

9. Conclusion

Any covering constructed from a voltage assignment is *regular,* that is, there is a group of automorphisms with the fibers as the vertex orbits. This is not necessarily the case for an arbitrary covering graph embedded with a lifted rotation scheme. In fact, there may not be any nontrivial fiber-preserving automorphisms. Gross and Tucker [5, 16] defined *permutation voltage assignments* which yield arbitrary coverings, not just regular coverings.

In this paper we have been concerned solely with regular voltage assignments. To what extent do our results extend to permutation voltage assignments? How can they be improved in the more general setting?

There is hope for substantial progress using nonregular voltage assignments. For example, Bouchet [5] constructs triangular embeddings of compositions from triangular embeddings of the base graph using nowhere-zero flows. These embeddings are nonregular. They have order p^2 (where p is an odd prime), with an order p fiber-preserving free automorphism, but also with a nonregular order p component. By iterating these compositions—as we have also done in Sections 7 and 8—he can lift triangular embeddings of G to ones of $G^{(m)}$ for all m relatively prime to 30. Moreover, if every 2-edge-connected graph has a 5-flow, as conjectured by Tutte [32], then the result is improved to all m relatively prime to 6.

Jackson has used a similar technique to construct triangular embeddings of $K_{i(m)}$. Here he starts with the base graph K_m triangularly embedded and lifts these to an embedding of the composition for i relatively prime to 6.

Both the results of Bouchet and Jackson rely on lifting triangulations only. Can the results in this paper be used to lift other embeddings? Can such compositions be done simultaneously on primal and dual graphs?

Voltage graphs construct coverings of surfaces in which the branch points lie in the interior of faces. Dually, current graphs construct coverings with branch points at the vertices. The techniques of this paper cover the general case with branch points in both vertices and faces. Note that if a branch point on an edge were desired, we would apply the present theory by first subdividing that edge, then placing the branch point on the new degree two vertex.

Bouchet [6] has constructed interesting graph embeddings using coverings with folds. Is there an easy way to describe coverings with folds in our context?

We have chosen to describe our results in terms of voltage assignments on the medial graph. By duality, we could have used current assignments on the radial graph. The two theories are equivalent, and in practice each method has its own advantages. For example, in current graphs the technique of excess currents gives a great flexibility in constructing covering triangulations. Likewise, current graphs are quite useful under the special conditions needed in the proof of the map color theorem [26]. On the other hand, when using voltage graphs the derived graph is independent of the embedding. This allows one to first search for suitable quotient graphs and then study their various embeddings.

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The author thanks Bruce Richter for enlightening discussions on this material. In particular, the idea of using voltages on the medial graph first arose in our joint work [4], where we needed a covering of the sphere by itself with one branch point at a vertex and the other at a face of an embedded graph. The author also thanks Nora Hartsfield for her help in understanding the full power of current graphs, and for helpful discussions on embeddings of complete bipartite graphs.

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