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On the computation of parameter derivatives of solutions of linear difference equations

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ABSTRACT

A method is given to compute the parameter derivatives of recessive solutions of second-order inhomogeneous linear difference equations. The case of difference equations in which all solutions have the same rate of growth is also discussed.

The method is illustrated by numerical computations of parameter derivatives of incomplete gamma functions and confluent hypergeometric functions.

1. Introduction and summary

Many special functions depend on arguments and parameters. Usually they satisfy differential equations with respect to the arguments and difference equations with respect to the parameters. Special functions are useful tools in many applications, and in these applications the exceptional cases often involve, what can be seen, parameter derivatives of the special functions. For example, one of the incomplete gamma functions is defined as

$$\Gamma(a,z)=\int_z^\infty \mathrm{e}^{-t}t^{a-1}\mathrm{d}t. \tag{1.1}$$

See Section 11.2 in [3]. The parameter derivative is

$$\frac{\partial}{\partial a}\Gamma(a,z)=\int_z^\infty \mathrm{e}^{-t}t^{a-1}\ln t\,\mathrm{d}t, \tag{1.2}$$

and these functions are needed in hyperasymptotics, see appendix A3 in [1]. From this example it is also obvious that known integrals with additional logarithmic factors can often be seen as parameter derivatives. Another example is

$$\frac{\partial}{\partial c}U(a,c,z)=\frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-zt}t^{a-1}\,(1+t)^{c-a-1}\ln(1+t)\mathrm{d}t, \tag{1.3}$$

where $U(a,c,z)$ is one of the confluent hypergeometric functions. See Section 7.2 in [3].

The numerical computation of recessive solutions of second-order linear difference equations is well understood. See [2], or Section 2 of this paper where we summarise the results of [2]. We want to compute the parameter derivatives of the

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recessive solutions of these second-order inhomogeneous linear difference equations. Our method is based on the simple observation, made in Section 3, that the parameter derivative of the difference equation itself is of the same form as the original difference equation, that is, only the inhomogeneous term has changed. Hence, the methods of [2] can also be used to compute the parameter derivatives.

The method is illustrated in Section 4, where we discuss the computation of parameter derivatives of incomplete gamma functions, and in Section 5, where we discuss the computation of the a and c derivatives of the confluent hypergeometric function $U(a, c, z)$.

The results in Sections 2 and 3 are for the computation of recessive solutions. In Section 6 we illustrate, via the confluent hypergeometric function, the case in which all solutions of the linear difference equation have the same rate of growth. In that case the so-called Clenshaw averaging process has to be used. Since the parameter derivative of the difference equation is of the same form, the Clenshaw averaging process can also be used to compute the parameter derivatives.

Sections 5 and 6 include numerical illustrations.

2. Second-order linear difference equations

In this section we summarise the results in [2]. We will use the same notation and skip the proofs.

The difference equations in this paper will be second-order linear equations of the form

$$a_r(\nu)y_{r-1}(\nu) - b_r(\nu)y_r(\nu) + c_r(\nu)y_{r+1}(\nu) = d_r(\nu), \tag{2.1}$$

where the coefficients $a_r(\nu)$, $b_r(\nu)$ and $c_r(\nu)$ are analytic functions of a parameter ν and the solution $y_r(\nu)$ will also depend on ν. From now onwards we will abbreviate $y_r(\nu)$ to y_r.

We assume that the general solution of (2.1) has the form

$$y_r = Af_r + Bg_r + h_r, \tag{2.2}$$

in which A and B are arbitrary constants and the complementary functions f_r, g_r, and the particular solution h_r have the properties $f_0 \neq 0$, $g_r \neq 0$ for all sufficiently large r, and

$$f_r/g_r \to 0, \qquad h_r/g_r \to 0, \quad \text{as } r \to \infty. \tag{2.3}$$

We will be looking for a solution of (2.1) that satisfies the normalising condition

$$\sum_{r=0}^{\infty} m_r y_r = k, \tag{2.4}$$

in which $m_r(\nu)$ and $k(\nu)$ are analytic functions of ν.

For the functions on the right-hand side of (2.2) we assume that

$$\left|\sum_{r=0}^{N} m_r g_r\right| \to \infty, \quad \text{as } N \to \infty, \tag{2.5}$$

and

$$\sum_{r=0}^{\infty} m_r f_r = F, \qquad \sum_{r=0}^{\infty} m_r h_r = H, \tag{2.6}$$

where F and H are finite, and $F \neq 0$. Then (2.1) has a unique solution fulfilling (2.4). It is given by

$$y_r = \frac{k-H}{F} f_r + h_r. \tag{2.7}$$

To approximate this solution numerically we solve the system of linear algebraic equations given by

$$a_r y_{r-1}^{(N)} - b_r y_r^{(N)} + c_r y_{r+1}^{(N)} = d_r, \quad r = 1, 2, \ldots, N-1, \tag{2.8}$$

$$\sum_{r=0}^{N} m_r y_r^{(N)} = k, \tag{2.9}$$

and

$$y_N^{(N)} = 0. \tag{2.10}$$

Theorem 1. *In addition to the other conditions of this section, assume that for all sufficiently large N the system of equations (2.8)–(2.10) has a solution, that $g_N \neq 0$, and that*

$$\frac{f_N}{g_N}\sum_{r=0}^{N} m_r g_r \to 0, \qquad \frac{h_N}{g_N}\sum_{r=0}^{N} m_r g_r \to 0, \quad \text{as } N \to \infty. \tag{2.11}$$

Then if r is fixed and $N \to \infty$, $y_r^{(N)} \to y_r$.

This results follows from the fact that

$$y_r^{(N)} = A_N f_r + B_N g_r + h_r, \tag{2.12}$$

where

$$A_N = \frac{h_N \sum_{r=0}^{N} m_r g_r - g_N \left(\sum_{r=0}^{N} m_r h_r - k\right)}{g_N \sum_{r=0}^{N} m_r f_r - f_N \sum_{r=0}^{N} m_r g_r},$$

$$B_N = \frac{f_N \left(\sum_{r=0}^{N} m_r h_r - k\right) - h_N \sum_{r=0}^{N} m_r f_r}{g_N \sum_{r=0}^{N} m_r f_r - f_N \sum_{r=0}^{N} m_r g_r}. \tag{2.13}$$

Thus $B_N \to 0$ and $A_N \to (k - H)/F$.

To compute the $y_r^{(N)}$ we introduce

$$q_0 = 1, \qquad q_r = \frac{a_1 a_2 \cdots a_r}{c_1 c_2 \cdots c_r}, \qquad p_0 = 0, \qquad p_1 = m_0, \qquad e_0 = k, \tag{2.14}$$

and for $r \geq 1$

$$p_{r+1} = \frac{b_r p_r - a_r p_{r-1}}{c_r} + q_r m_r, \qquad e_r = \frac{a_r e_{r-1} - d_r p_r}{c_r}. \tag{2.15}$$

Then $y_{N-1}^{(N)} = e_{N-1}/p_N$ and the remaining $y_r^{(N)}$ can be computed from

$$p_{r+1} y_r^{(N)} - p_r y_{r+1}^{(N)} + q_r \sum_{s=r+1}^{N-1} m_s y_s^{(N)} = e_r, \quad r = N-2, \ldots, 1, 0 \tag{2.16}$$

by back-substitution.

Note that in the homogeneous case we have $d_r = h_r = H = 0$ and $e_r = k q_r$.

3. Parameter derivatives

In this section we want to introduce a method to compute $y_r' = \frac{\mathrm{d}}{\mathrm{d}\nu} y_r(\nu)$ with the aid of the defining difference equation. It is based on the simple observation that the derivative of (2.1), which is

$$a_r y_{r-1}' - b_r y_r' + c_r y_{r+1}' = d_r' - a_r' y_{r-1} + b_r' y_r - c_r' y_{r+1}, \tag{3.1}$$

is again of the form (2.1) with d_r replaced by the right-hand side of (3.1). The accompanying normalising condition is

$$\sum_{r=0}^{\infty} m_r y_r' = k' - \sum_{r=0}^{\infty} m_r' y_r, \tag{3.2}$$

which is again of the form (2.4). Hence, at the moment that the y_r are known, the results of Section 2 can be used to compute the y_r'. In practice, we can only approximate the y_r via the $y_r^{(N)}$ and we will use the results of Section 2 to compute $y'^{(N)}_r = \mathrm{d}y_r^{(N)}/\mathrm{d}\nu$ as follows. We differentiate the system of equations (2.8)–(2.10) and obtain

$$a_r y'^{(N)}_{r-1} - b_r y'^{(N)}_r + c_r y'^{(N)}_{r+1} = \tilde{d}_r, \quad r = 1, 2, \ldots, N-1, \tag{3.3}$$

$$\sum_{r=0}^{N} m_r y'^{(N)}_r = \tilde{k}, \tag{3.4}$$

and

$$y'^{(N)}_N = 0, \tag{3.5}$$

where

$$\tilde{d}_r = d'_r - a'_r y^{(N)}_{r-1} + b'_r y^{(N)}_r - c'_r y^{(N)}_{r+1}, \quad \text{and} \quad \tilde{k} = k' - \sum_{r=0}^{N} m'_r y^{(N)}_r. \tag{3.6}$$

Hence, the derivatives of the $y^{(N)}_r$ are the solutions of the system of equations (3.3)–(3.5) and can be computed via the method described at the end of Section 2.

Note that when we compare the system of equations (3.3)–(3.5) with the system of equations (2.8)–(2.10) we see that only the k and d_r have changed. It follows that when we use the method given in the final paragraph of Section 2, the p_r and q_r will be the same for the two systems.

To show that for fixed r we have $y'^{(N)}_r \to y'_r$, as $N \to \infty$, we have to show that for the A_N and B_N in (2.13) we have $A'_N \to \frac{\mathrm{d}}{\mathrm{d}\nu}\left((k-H)/F\right)$ and $B'_N \to 0$ as $N \to \infty$. This can be guaranteed when the following hold

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\left(f_r/g_r\right) \to 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\nu}\left(h_r/g_r\right) \to 0, \quad \text{as } r \to \infty, \tag{3.7}$$

$$\sum_{r=0}^{N} \frac{\mathrm{d}}{\mathrm{d}\nu}\left(m_r f_r\right) \to F', \qquad \sum_{r=0}^{N} \frac{\mathrm{d}}{\mathrm{d}\nu}\left(m_r h_r\right) \to H', \quad \text{as } N \to \infty, \tag{3.8}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{f_N}{g_N}\sum_{r=0}^{N} m_r g_r\right) \to 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{h_N}{g_N}\sum_{r=0}^{N} m_r g_r\right) \to 0, \quad \text{as } N \to \infty. \tag{3.9}$$

The original and additional conditions can often be checked by studying the rate of growth of f_r, g_r, h_r, m_r and their ν-derivatives, as $r \to \infty$.

4. Incomplete gamma functions

The incomplete gamma functions satisfy first-order inhomogeneous linear difference equations. For that result and notation see Section 11.2 in [3]. Here we will use the homogeneous second-order difference equation

$$z(r+a-1)y_{r-1} - (r+a+z)y_r + y_{r+1} = 0. \tag{4.1}$$

It has solutions

$$f_r = \gamma(r+a, z), \qquad g_r = \Gamma(r+a). \tag{4.2}$$

The rate of growth of the first of these is

$$f_r \sim \mathrm{e}^{-z} z^{a+r} r^{-1}, \quad \text{as } r \to \infty,\ |\mathrm{ph}\, z| \le \pi, \tag{4.3}$$

and it has the normalising condition

$$\sum_{r=0}^{\infty} \frac{1}{r!} f_r = a^{-1} z^a. \tag{4.4}$$

For these two results use (11.2) and (11.9) in [3]. It follows that the conditions for Theorem 1 are satisfied and that the results of Section 2 can be used to compute the f_r numerically.

In the remainder of this section we will use the notation $f'_r = \frac{\partial}{\partial a} f_r$. The a-derivative of (4.1) is

$$z(r+a-1)f'_{r-1} - (r+a+z)f'_r + f'_{r+1} = f_r - z f_{r-1}, \tag{4.5}$$

with normalising condition

$$\sum_{r=0}^{\infty} \frac{1}{r!} f'_r = a^{-2} z^a \left(a \ln(z) - 1\right). \tag{4.6}$$

Thus the results of Section 3 can be used to compute the parameter derivative of the incomplete gamma function $\gamma(a, z)$. For the parameter derivative of in the incomplete gamma function $\Gamma(a, z)$ we combine the above results with the identity $\Gamma(a) = \Gamma(a, z) + \gamma(a, z)$.

5. Confluent hypergeometric functions: The recessive case

The difference equation

$$(r+a-1)y_{r-1}-(2r+2a-c+z)y_r+(r+a-c+1)y_{r+1}=0, \tag{5.1}$$

has solutions

$$f_r=(a)_r\,U(a+r,c,z), \qquad g_r=\frac{(a)_r}{(a-c+1)_r}M(a+r,c,z), \tag{5.2}$$

where the Pochhammer symbol is $(a)_r=\Gamma(a+r)/\Gamma(a)$ and $U(a,c,z)$ and $M(a,c,z)$ are the confluent hypergeometric functions. See chapter 7 in [3]. The notation in (5.2) already indicates that f_r is the recessive solution, and g_r is a dominant solution for $r\to\infty$. This follows from

$$\begin{aligned} f_r &\sim \frac{\sqrt{\pi}}{\Gamma(a)}\mathrm{e}^{z/2}z^{(1-2c)/4}r^{(2c-3)/4}\mathrm{e}^{-2\sqrt{zr}}, \\ g_r &\sim \frac{\Gamma(a-c+1)\Gamma(c)}{2\sqrt{\pi}\Gamma(a)}\mathrm{e}^{z/2}z^{(1-2c)/4}r^{(2c-3)/4}\mathrm{e}^{2\sqrt{zr}}, \end{aligned} \tag{5.3}$$

as $r\to\infty$, $|\mathrm{ph}\,z|<\pi$. See Section 5.2 in [4]. The f_r have the normalising condition

$$\sum_{r=0}^{\infty}\frac{(a-c+1)_r}{r!}f_r=z^{-a}. \tag{5.4}$$

The reader can check that the conditions for Theorem 1 are satisfied. It follows that the results of Section 2 can be used to compute the f_r numerically. This result is well-known. Now we will compute the derivatives. The f_r is a function of a, c and z. The z-derivative is easy since

$$\frac{\mathrm{d}}{\mathrm{d}z}(a)_r\,U(a+r,c,z)=-(a)_{r+1}\,U(a+r+1,c+1,z). \tag{5.5}$$

5.1. The a-derivative

In this subsection we use the notation $f_r'=\frac{\partial}{\partial a}f_r$ and obtain from (5.1) that

$$(r+a-1)f_{r-1}'-(2r+2a-c+z)f_r'+(r+a-c+1)f_{r+1}'=-f_{r-1}+2f_r-f_{r+1}, \tag{5.6}$$

with normalising condition

$$\sum_{r=0}^{\infty}\frac{(a-c+1)_r}{r!}f_r'=-\ln(z)z^{-a}+\sum_{r=1}^{\infty}\frac{(a-c+1)_r}{r!}\left(\Psi(a-c+1)-\Psi(a-c+1+r)\right)f_r, \tag{5.7}$$

where $\Psi(a)=\Gamma'(a)/\Gamma(a)$ is the logarithmic derivative of the gamma function. Hence, the results of Section 3 can be used.

5.2. The c-derivative

Now we use the notation $f_r'=\frac{\partial}{\partial c}f_r$ and obtain from (5.1) that

$$(r+a-1)f_{r-1}'-(2r+2a-c+z)f_r'+(r+a-c+1)f_{r+1}'=f_{r+1}-f_r, \tag{5.8}$$

with normalising condition

$$\sum_{r=0}^{\infty}\frac{(a-c+1)_r}{r!}f_r'=\sum_{r=1}^{\infty}\frac{(a-c+1)_r}{r!}\left(\Psi(a-c+1+r)-\Psi(a-c+1)\right)f_r. \tag{5.9}$$

5.3. Numerical results

We take $N=50$ and use the difference equations (5.1), (5.6), (5.8) and the methods discussed in Sections 2 and 3. The results are given in Table 1.

Table 1
Numerical calculations with $a = 0.2$, $c = 0.3$, $z = 1.4$ and $N = 50$.

Function		|error|
f_0	0.8596259193	
$f_0^{(N)}$	0.8596259476	2.8×10^{-8}
$\frac{\partial}{\partial a} f_0$	−0.7093488450	
$\frac{\partial}{\partial a} f_0^{(N)}$	−0.7093485813	2.6×10^{-7}
$\frac{\partial}{\partial c} f_0$	0.0688571930	
$\frac{\partial}{\partial c} f_0^{(N)}$	0.0688571149	7.8×10^{-8}

6. Confluent hypergeometric functions: The oscillatory case

From (5.3) it follows that the method of Section 5 can be used in the sector $|\mathrm{ph}\, z| < \pi$. On the line $\mathrm{ph}\, z = \pi$ all the solutions of (5.1) have the same rate of growth, and we have to use the so-called Clenshaw averaging process. See Section 4.7 in [4].

Since all the solutions of (5.1) have the same rate of growth we can use (5.1) in the forward direction. The main difference is that now two normalising conditions are needed. We will use

$$\sum_{r=0}^{\infty} \frac{(a-c+1)_r}{r!} f_r = z^{-a}, \qquad \sum_{r=0}^{\infty} \frac{(a-c)_r}{r!} f_r = \mathrm{e}^z \Gamma(1-a, z), \tag{6.1}$$

$\Re(c-2a) > \frac{1}{2}$, where $\Gamma(1-a, z)$ is one of the incomplete gamma functions, see Section 4. Note that in contrast with (5.4), we have a constraint in (6.1), indicating that the convergence in (6.1) is also much slower. The second normalising condition can be obtained from the first via the identity $\int_z^\infty \mathrm{e}^{-t} U(a, c, t)\mathrm{d}t = \mathrm{e}^{-z} U(a, c-1, z)$.

We introduce two sequences $\left\{y_r^{(0)}\right\}$ and $\left\{y_r^{(1)}\right\}$ via

$$y_0^{(0)} = 1, \qquad y_1^{(0)} = 0, \qquad y_0^{(1)} = 0, \qquad y_1^{(1)} = 1, \tag{6.2}$$

and forward recursion in (5.1). For positive integer N let

$$K_N^{(j)} = \sum_{r=0}^{N-1} \frac{(a-c+1)_r}{r!} y_r^{(j)}, \qquad L_N^{(j)} = \sum_{r=0}^{N-1} \frac{(a-c)_r}{r!} y_r^{(j)}, \quad j = 0, 1, a, c, \tag{6.3}$$

and let the sequence $\left\{f_r^{(N)}\right\}$ be a solution of (5.1) such that

$$\sum_{r=0}^{N-1} \frac{(a-c+1)_r}{r!} f_r^{(N)} = z^{-a}, \qquad \sum_{r=0}^{N-1} \frac{(a-c)_r}{r!} f_r^{(N)} = \mathrm{e}^z \Gamma(1-a, z). \tag{6.4}$$

Then there are constants A_N and B_N such that

$$f_r^{(N)} = A_N y_r^{(0)} + B_N y_r^{(1)}. \tag{6.5}$$

Combining (6.5) with (6.3) and (6.4) it follows that

$$A_N K_N^{(0)} + B_N K_N^{(1)} = z^{-a}, \qquad A_N L_N^{(0)} + B_N L_N^{(1)} = \mathrm{e}^z \Gamma(1-a, z), \tag{6.6}$$

and these two linear equations can be used to compute A_N and B_N.

It is not difficult to show that for fixed r we have $f_r^{(N)} \to f_r$ as $N \to \infty$.

6.1. The a-derivative

We take the $f_r^{(N)}$ from above and introduce the sequence $\left\{y_r^{(a)}\right\}$ via $y_0^{(a)} = 0$, $y_1^{(a)} = 1$ and forward recursion in

$$(r+a-1)y_{r-1}^{(a)} - (2r+2a-c+z)y_r^{(a)} + (r+a-c+1)y_{r+1}^{(a)} = -f_{r-1}^{(N)} + 2f_r^{(N)} - f_{r+1}^{(N)}, \tag{6.7}$$

$r = 1, 2, \ldots, N-2$; compare with (5.6). Then there exist constants A_N and B_N such that

$$\frac{\partial}{\partial a} f_r^{(N)} = y_r^{(a)} + A_N y_r^{(0)} + B_N y_r^{(1)}. \tag{6.8}$$

Table 2
Numerical calculations with $a=-1.2$, $c=5.3$, $z=0.4\mathrm{e}^{\pi\mathrm{i}}$ and $N=400$.

Function		|relative error|
f_0	22.47933265 − 44.97489233i	
$f_0^{(N)}$	22.47933096 − 44.97489274i	3.5×10^{-8}
$\frac{\partial}{\partial a}f_0$	−141.6664316 + 221.1635858i	
$\frac{\partial}{\partial a}f_0^{(N)}$	−141.6664276 + 221.1635870i	1.6×10^{-8}
$\frac{\partial}{\partial c}f_0$	−59.94909351 − 185.4813249i	
$\frac{\partial}{\partial c}f_0^{(N)}$	−59.94910020 − 185.4813235i	3.5×10^{-8}

The a-derivative of the normalising conditions (6.1) are (5.7) and

$$\sum_{r=0}^{\infty}\frac{(a-c)_r}{r!}f_r'=\mathrm{e}^z\frac{\partial}{\partial a}\Gamma(1-a,z)+\sum_{r=1}^{\infty}\frac{(a-c)_r}{r!}\left(\Psi(a-c)-\Psi(a-c+r)\right)f_r, \tag{6.9}$$

where, again, $f_r'=\frac{\partial}{\partial a}f_r$. The a-derivative of the incomplete gamma function can be computed via the methods discussed in Section 4. Combining the normalising conditions (5.7) and (6.9) with (6.8) we obtain the equations

$$A_NK_N^{(0)}+B_NK_N^{(1)}+K_N^{(a)}=-\ln(z)z^{-a}+\sum_{r=1}^{N-1}\frac{(a-c+1)_r}{r!}\left(\Psi(a-c+1)-\Psi(a-c+1+r)\right)f_r^{(N)},$$

$$A_NL_N^{(0)}+B_NL_N^{(1)}+L_N^{(a)}=\mathrm{e}^z\frac{\partial}{\partial a}\Gamma(1-a,z)+\sum_{r=1}^{N-1}\frac{(a-c)_r}{r!}\left(\Psi(a-c)-\Psi(a-c+r)\right)f_r^{(N)}, \tag{6.10}$$

which can be used to compute the constants A_N and B_N.

6.2. The c-derivative

Now we define the sequence $\left\{y_r^{(c)}\right\}$ via $y_0^{(c)}=0$, $y_1^{(c)}=1$ and forward recursion in

$$(r+a-1)y_{r-1}^{(c)}-(2r+2a-c+z)y_r^{(c)}+(r+a-c+1)y_{r+1}^{(c)}=f_{r+1}^{(N)}-f_r^{(N)}, \tag{6.11}$$

$r=1,2,\ldots,N-2$; compare with (5.8). Then there exist constants A_N and B_N such that

$$\frac{\partial}{\partial c}f_r^{(N)}=y_r^{(c)}+A_Ny_r^{(0)}+B_Ny_r^{(1)}. \tag{6.12}$$

The c-derivative of the normalising conditions (6.1) are (5.9) and

$$\sum_{r=0}^{\infty}\frac{(a-c)_r}{r!}f_r'=\sum_{r=1}^{\infty}\frac{(a-c+r)_r}{r!}\left(\Psi(a-c+r)-\Psi(a-c)\right)f_r, \tag{6.13}$$

where $f_r'=\frac{\partial}{\partial c}f_r$. Combining these normalising conditions with (6.12) we obtain the equations

$$A_NK_N^{(0)}+B_NK_N^{(1)}+K_N^{(c)}=\sum_{r=1}^{N-1}\frac{(a-c+1)_r}{r!}\left(\Psi(a-c+1+r)-\Psi(a-c+1)\right)f_r^{(N)},$$

$$A_NL_N^{(0)}+B_NL_N^{(1)}+L_N^{(c)}=\sum_{r=1}^{N-1}\frac{(a-c)_r}{r!}\left(\Psi(a-c+r)-\Psi(a-c)\right)f_r^{(N)}, \tag{6.14}$$

which can be used to compute the constants A_N and B_N.

6.3. Numerical results

We take $N=400$ and use the methods discussed in this section. The results are given in Table 2. Note that due to the slow convergence of the normalising conditions we are forced to take a much larger N and make $\Re(c-2a)$ also relatively large; compare (6.1).

References

[1] G. Álvarez, C.J. Howls, H.J. Silverstone, Dispersive hyperasymptotics and the anharmonic oscillator, J. Phys. A 35 (2002) 4017–4042.
[2] F.W.J. Olver, Numerical solution of second-order linear difference equations, J. Res. Natl. Bur. Stand. B 71B (1967) 111–129.
[3] N.M. Temme, Special Functions: An Introduction to the Classical Functions of Mathematical Physics, John Wiley & Sons Inc., New York, 1996.
[4] J. Wimp, Computation with Recurrence Relations, Pitman (Advanced Publishing Program), Boston, MA, 1984.