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Regular Line-Symmetric Graphs

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ABSTRACT

We say that a graph is *point-symmetric* if, given any two points of the graph, there is an automorphism of the graph that sends the first point to the second. Similarly, we say that a graph is *line-symmetric* if, given any two lines of the graph, there is an automorphism that sends the first line to the second.

In general a line-symmetric graph need not be point-symmetric. For example, any complete bipartite graph is line-symmetric, but if it is not regular then it is not point-symmetric. In this paper we investigate the extent to which line symmetry and regularity imply point symmetry.

We first give some conditions on the number of points and the degree of regularity under which line symmetry and regularity imply point symmetry. We then give some general methods for constructing graphs which are line-symmetric and regular but not point-symmetric. Finally we summarize what is known about the number of points that a regular line-symmetric graph which is not point-symmetric can have. We conclude with a list of unsolved problems in this area.

1. INTRODUCTION

Let G be a graph. An automorphism of G is a permutation of the points of G that preserves adjacency. We say points u and v of G are similar if there is an automorphism of G that sends u to v. Lines e and f in G are said to be similar if there is an automorphism that sends the end-points of e to the end-points of f. We say that G is point (line)-symmetric if all of its points (lines) are similar.

In [2], Dauber and Harary investigated the relationship between line symmetry and point symmetry. They give simple examples of graphs that are line-symmetric but not point-symmetric and vice versa. However, their line-symmetric graphs that are not point-symmetric fail to be point-symmetric because they are not regular. This raises the question of whether or not a regular line-symmetric graph must be point-symmetric.

Dauber and Harary give a partial answer to this question. They show that, if G is a line-symmetric graph with v points, which is regular of degree d, then G must be point-symmetric if v is odd or if $d \geqslant v/2$.

Here we investigate the case v even and $d < v/2$. We first show (*Theorem* 2) that, if $v = 2p$ or $2p^2$ where p is prime, then G must be point-symmetric. We then give some methods for constructing regular graphs that are line-symmetric but not point-symmetric (Theorems 3 and 4). Finally, we summarize what is known about the values of v for which there is a line-symmetric graph with v points that is not point-symmetric (Theorem 5). In the concluding section we give some problems that are still open.

2. Conditions for Line Symmetry to Imply Point Symmetry

To fix our notation we make the following formal definitions. *A graph* is an ordered pair (V, E), where V is a finite set (the *points* of the graph) and E is a collection of two element subsets of V (the *lines* of the graph). If $e = \{u, v\}$ is a line, then u and v are the *end-points of e*. The *degree* of a point u is the number of lines with u as an end-point. A graph is *regular of degree d* if every point of the graph has degree d.

An *automorphism* of the graph $G = (V, E)$ is a permutation σ of V with the property that if $\{u, v\} \in E$, then $\{\sigma(u), \sigma(v)\} \in E$. The automorphisms of G form a group $\mathscr{G}$. This group is a permutation group on V by definition. It acts as a permutation group on E if we set $\sigma(\{u, v\}) = \{\sigma(u), \sigma(v)\}$ for $\sigma \in \mathscr{G}, \{u, v\} \in E$.

Let $\mathscr{G}$ be a permutation group on a set S and let $x \in S$. We define $\mathscr{G}(x)$, the *orbit of x*, by

$$\mathscr{G}(x) = \{\sigma(x) \mid \sigma \in \mathscr{G}\}.$$

We say that $\mathscr{G}$ is *transitive on S* if $\mathscr{G}(x) = S$ for some (and hence all) $x \in S$. We define $\mathscr{G}_x$, the *stability subgroup of x*, by

$$\mathscr{G}_x = \{\sigma \in \mathscr{G} \mid \sigma(x) = x\}.$$

We have the relation $|\mathscr{G}(x)| = (\mathscr{G} : \mathscr{G}_x)$, where $|\mathscr{G}(x)|$ is the number of elements in $\mathscr{G}(x)$ and $(\mathscr{G} : \mathscr{G}_x)$ is the index of the subgroup $\mathscr{G}_x$ in the group $\mathscr{G}$. With this terminology it is clear that a graph $G = (V, E)$ is point (line)-symmetric if and only if its automorphism group is transitive on $V(E)$.

The following theorem and its corollary are due to Dauber and Harary [2] in a slightly different form. We include a proof here for completeness.

Theorem 1 (Dauber and Harary). *Let $G = (V, E)$ be a graph with no isolated points (i.e., no points of degree zero). Let $\mathscr{H}$ be a subgroup of the* *group of automorphisms of G, which is transitive on E but not on V. Then V is the disjoint union of subsets V_1 and V_2 with the following properties:*

(2.1) *$\mathscr{H}$ acts as a transitive permutation group on V_1 and V_2.*

(2.2) *Each line of G has one end-point in V_1 and the other in V_2.*

PROOF: Let $\{v_1, v_2\} \in E$. Set $V_1 = \mathscr{H}(v_1)$, $V_2 = \mathscr{H}(v_2)$. Let $u \in V$. Since u is not isolated, $\{u, u'\} \in E$ for some $u' \in V$. Now $\mathscr{H}$ is transitive on E, so $\{u, u'\} = \sigma\{v_1, v_2\}$ for some $\sigma \in \mathscr{H}$. Therefore either $u = \sigma(v_1)$ or $u = \sigma(v_2)$. In either case, $u \in V_1 \cup V_2$, so $V = V_1 \cup V_2$. If V_1 and V_2 were not disjoint we would have $\mathscr{H}(v_1) = \mathscr{H}(v_2) = V$, contradicting the assumption that $\mathscr{H}$ is not transitive on V.

Let $\sigma \in \mathscr{H}$ and let $u \in V_i$, $i = 1$ or 2. Then $u = \tau(v_i)$ for some $\tau \in \mathscr{H}$, so $\sigma(u) = \sigma\tau(v_i) \in \mathscr{H}(v_i) = V_i$. Hence, $\mathscr{H}$ acts as a permutation group on V_i. The action is transitive, since $V_i = \mathscr{H}(v_i)$.
Now let $e \in E$. Since $\mathscr{H}$ is transitive on E,

$$e = \sigma(\{v_1, v_2\}) = \{\sigma(v_1), \sigma(v_2)\}$$

for some $\sigma \in \mathscr{H}$. Now $\sigma(v_1) \in V_1$ and $\sigma(v_2) \in V_2$, so (2.2) holds. This completes the proof.

COROLLARY 1.1 (Dauber and Harary). *Let $G = (V, E)$ be a line-symmetric graph that is regular of degree $d > 0$. Let $v = |V|$. If v is odd or $d \geqslant v/2$, then G is point-symmetric.*

PROOF: Suppose G is not point-symmetric. Then we may apply Theorem 1 with $\mathscr{H}$ the entire group of automorphisms of G. By (2.2)

$$d\,|V_1| = |E| = d\,|V_2|,$$

so $|V_1| = |V_2|$ and $v = 2|V_1|$ is even. Hence, we must have $d \geqslant v/2$. Again by (2.2), a point in V_1 is an end-point of at most $|V_2| = v/2$ lines, so we must have $d = v/2$. But this implies that the lines of G are all pairs with one element in V_1 and the other in V_2. Since $|V_1| = |V_2|$, this graph is point-symmetric, so we have arrived at a contradiction.

The following result gives another condition which guarantees that a line-symmetric graph is point-symmetric.

THEOREM 2. *Let $G = (V, E)$ be a line-symmetric graph that is regular of degree d. If $|V| = 2p$ or $2p^2$, where p is prime, then G is point-symmetric.*

PROOF: Suppose G is not point-symmetric. A graph that is regular of degree 0 is clearly point-symmetric, so we must have $d > 0$. Hence, we may apply Theorem 1 with $\mathscr{H} = \mathscr{G}$, the entire group of automorphisms of G. By (2.2),

$$d \mid V_1 \mid = \mid E \mid = d \mid V_2 \mid ,$$

so $\mid V_1 \mid = \mid V_2 \mid = \frac{1}{2} \mid V \mid = p^t$, where $t = 1$ or 2.

Let $\mathscr{H}$ be a p-Sylow subgroup of $\mathscr{G}$.

LEMMA 2.1. *$\mathscr{H}$ is transitive on V_1 and V_2.*

PROOF: Let $\mid \mathscr{G} \mid = p^\alpha k$, where p does not divide k. Then $\mid \mathscr{H} \mid = p^\alpha$. Let $v \in V_i$, $i = 1$ or 2. Since $\mathscr{G}$ is transitive on V_i, we have

$$(\mathscr{G} : \mathscr{G}_v) = \mid \mathscr{G}(v) \mid = \mid V_i \mid = p^t.$$

Therefore, $\mid \mathscr{G}_v \mid = p^{\alpha - t} k$. Now $\mathscr{H}_v$ is a p-group and $\mathscr{H}_v \subset \mathscr{G}_v$, so $\mid \mathscr{H}_v \mid \leqslant p^{\alpha - t}$. Therefore,

$$\mid \mathscr{H}(v) \mid = (\mathscr{H} : \mathscr{H}_v) = \mid \mathscr{H} \mid / \mid \mathscr{H}_v \mid \geqslant p^t.$$

Hence we must have $\mathscr{H}(v) = V_i$ as required.

LEMMA 2.2. *Let $v_1 \in V_1$ and $v_2 \in V_2$, with $\{v_1, v_2\} \in E$. Suppose there is a subset $S = \{\sigma_1, \ldots, \sigma_{p^t}\}$ of elements of $\mathscr{G}$ such that the points $\sigma_1(v_i), \ldots, \sigma_{p^t}(v_i)$ are distinct for $i = 1$ and for $i = 2$. Then the subgroup of $\mathscr{G}$ generated by S is non-Abelian.*

PROOF: Suppose not. Let $i = 1$ or 2. Suppose $\sigma_j^{-1}(v_i) = \sigma_k^{-1}(v_i)$ for some j and k, with $1 \leqslant j < k \leqslant p^t$. Then, since S generates an Abelian subgroup,

$$\sigma_k(v_i) = \sigma_k \sigma_j \sigma_j^{-1}(v_i) = \sigma_j \sigma_k \sigma_k^{-1}(v_i) = \sigma_j(v_i),$$

contradicting our hypotheses. Hence, we have

$$V_i = \{\sigma_1(v_i), \ldots, \sigma_{p^t}(v_i)\} = \{\sigma_1^{-1}(v_i), \ldots, \sigma_{p^t}^{-1}(v_i)\}.$$

Therefore the function $\eta \colon V \to V$ defined by

$$\eta(\sigma_i(v_1)) = \sigma_i^{-1}(v_2)$$

and

$$\eta(\sigma_i(v_2)) = \sigma_i^{-1}(v_1)$$

is a permutation of V.

Now we show that η is an automorphism of G. To see this, let

$$\{\sigma_i(v_1), \sigma_j(v_2)\} \in E.$$

Then, since S generates an Abelian subgroup,

$$\begin{aligned}\{\eta\sigma_i(v_1), \eta\sigma_j(v_2)\} &= \{\sigma_i^{-1}(v_2), \sigma_j^{-1}(v_1)\} \\ &= \{\sigma_i^{-1}\sigma_j^{-1}\sigma_j(v_2), \sigma_i^{-1}\sigma_j^{-1}\sigma_i(v_1)\} \in E\end{aligned}$$

because $\sigma_i^{-1}\sigma_j^{-1}$ is an automorphism of G. Hence, $\eta \in \mathscr{G}$. But this is a contradiction because every element of G maps V_1 onto V_1, while η maps V_1 onto V_2.

We are now ready to resume the proof of Theorem 2. First assume that $t = 1$ (i.e., $|V| = 2p$). Now $\mathscr{H}$ is a p-group and $\mathscr{H}$ is nontrivial by Lemma 2.1, so $\mathscr{H}$ has a non-trivial center. Let $\mathscr{K}$ be a cyclic subgroup of order p in the center of $\mathscr{H}$. Let $\{v_1, v_2\} \in E$, with $v_1 \in V_1$, $v_2 \in V_2$. For $i = 1$ or 2, we have

$$|\mathscr{K}(v_i)| = (\mathscr{K} : \mathscr{K}_{v_i}) = 1 \text{ or p}.$$

Therefore, either $\mathscr{K}(v_i) = \{v_i\}$ or $\mathscr{K}(v_i) = V_i$. If $\mathscr{K}(v_1) = V_1$ and $\mathscr{K}(v_2) = V_2$, we may take $S = \mathscr{K}$ in Lemma 2.2, and we have a contradiction. Hence, either $\mathscr{K}(v_1) = \{v_1\}$ or $\mathscr{K}(v_2) = \{v_2\}$, and without loss of generality we may assume $\mathscr{K}(v_1) = \{v_1\}$. Suppose that $\mathscr{K}(v_2) = \{v_2\}$. Let $u \in V_1$. By Lemma 2.1, $u = \sigma(v_1)$ for some $\sigma \in \mathscr{H}$. Since $\mathscr{K}$ is a central subgroup of $\mathscr{H}$,

$$\mathscr{K}(u) = \mathscr{K}\sigma(v_1) = \sigma\mathscr{K}(v_1) = \sigma(\{v_1\}) = \{u\}.$$

Similarly, if $u \in V_2$, then $\mathscr{K}(u) = \{u\}$. Therefore, the only permutation in $\mathscr{K}$ is the identity permutation, contradicting the fact that $\mathscr{K}$ has order p.

The only remaining possibility is that $\mathscr{K}(v_1) = \{v_1\}$ and $\mathscr{K}(v_2) = V_2$. Now $\{v_1, v_2\} \in E$ and the permutations in $\mathscr{K}$ preserve adjacency, so $\{v_1, u\} \in E$ for every $u \in V_2$. This implies that $d \geqslant p = |V|/2$. By Corollary 1.1, G is point-symmetric, contradicting our assumption to the contrary.

Now assume that $t = 2(|V| = 2p^2)$. Let $\mathscr{Z}$ be the center of $\mathscr{H}$. Suppose that for every $\sigma \in \mathscr{Z}$ which is not the identity, we have $\sigma(v) \neq v$ for every $v \in V$. Let $\{v_1, v_2\} \in E$, with $v_i \in V_i$, $i = 1, 2$. We have

$$|\mathscr{Z}(v_i)| = (\mathscr{Z} : \mathscr{Z}_{v_i}) = |\mathscr{Z}|,$$

since $\mathscr{Z}_{v_i} = \{1\}$. Now $\mathscr{Z}$ is a non-trivial p-group and $|\mathscr{Z}(v_i)| \leqslant |V_i| = p^2$, so $|\mathscr{Z}| = p$ or p^2. If $|\mathscr{Z}| = p^2$ we could take $S = \mathscr{Z}$ in Lemma 2.2 and obtain a contradiction, since $\mathscr{Z}$, the subgroup of $\mathscr{G}$ generated by $\mathscr{Z}$, is Abelian. Hence, $|\mathscr{Z}| = p$.

Let $i = 1$ or 2. Let $\mathscr{K}_i$ be the subgroup of $\mathscr{H}$ generated by $\mathscr{Z}$ and $\mathscr{H}_{v_i}$. By our assumption on $\mathscr{Z}$, $\mathscr{Z} \cap \mathscr{H}_{v_i} = \{1\}$. Now $\mathscr{Z}$ is a normal subgroup of $\mathscr{H}$, so

$$\mathscr{K}_i/\mathscr{Z} = \mathscr{Z}\mathscr{H}_{v_i}/\mathscr{Z} \cong \mathscr{H}_{v_i}/\mathscr{Z} \cap \mathscr{H}_{v_i} \cong \mathscr{H}_{v_i}.$$

Hence,

$$|\mathscr{K}_i| = |\mathscr{Z}|\,|\mathscr{K}_i/\mathscr{Z}| = |\mathscr{Z}|\,|\mathscr{H}_{v_i}| = p\,|\mathscr{H}_{v_i}|.$$

By Lemma 2.1,

$$|\mathscr{H}| = (\mathscr{H} : \mathscr{H}_{v_i})|\mathscr{H}_{v_i}| = |\mathscr{H}(v_i)|\,|\mathscr{H}_{v_i}| = p^2\,|\mathscr{H}_{v_i}|.$$

Therefore, $|\mathscr{K}_i| = |\mathscr{H}|/p$.

Since $\mathscr{K}_1$ and $\mathscr{K}_2$ are subgroups of $\mathscr{H}$, $1 \in \mathscr{K}_1 \cap \mathscr{K}_2$. Therefore,

$$\begin{aligned}|\mathscr{K}_1 \cup \mathscr{K}_2| &= |\mathscr{K}_1| + |\mathscr{K}_2| - |\mathscr{K}_1 \cap \mathscr{K}_2| \leqslant |\mathscr{K}_1| + |\mathscr{K}_2| - 1\\ &= \frac{2}{p}|\mathscr{H}| - 1 \leqslant |\mathscr{H}| - 1.\end{aligned}$$

Let $\sigma \in \mathscr{H} - (\mathscr{K}_1 \cup \mathscr{K}_2)$. Since $|\mathscr{Z}| = p$, $\mathscr{Z}$ is a cyclic group generated by an element τ. Let

$$S = \{\sigma^i\tau^j \mid 0 \leqslant i, j < p\}.$$

Let $k = 1$ or 2. Suppose $\sigma^i\tau^j(v_k) = \sigma^{i'}\tau^{j'}(v_k)$ for some $\sigma^i\tau^j, \sigma^{i'}\tau^{j'} \in S$. Then $\sigma^{i-i'}\tau^{j-j'}(v_k) = v_k$, since τ commutes with σ. Therefore, $\sigma^{i-i'}\tau^{j-j'} \in \mathscr{H}_{v_k}$, so $\sigma^{i-i'} = (\sigma^{i-i'}\tau^{j-j'})\tau^{j'-j} \in \mathscr{K}_k$. The order of σ is a power of p, so if p does not divide $i - i'$ there is an l such that $\sigma = (\sigma^{i-i'})^l \in \mathscr{K}_k$. This is impossible, so p divides $i - i'$. But $|i - i'| < p$, so $i = i'$. Hence, $\tau^j(v_k) = \tau^{j'}(v_k)$, so $\tau^{j-j'}(v_k) = v_k$. By our assumption on $\mathscr{Z}$, $\tau^{j-j'} = 1$. Now τ has order p and $|j - j'| < p$, so $j = j'$. We now conclude that the set S satisfies the hypotheses of Lemma 2.2. The subgroup of $\mathscr{G}$ generated by S is the subgroup generated by σ and τ. Since σ and τ commute, this subgroup is Abelian, contradicting the conclusion of Lemma 2.2.

We have now shown that the assumption we made about $\mathscr{Z}$ is false, i.e., there is a $\sigma \in \mathscr{Z}$ with $\sigma \neq 1$ and a point $v_1 \in V$ such that $\sigma(v_1) = v_1$. Now σ has order p^α for some $\alpha \geqslant 1$. Replacing σ by $\sigma^{p^{\alpha-1}}$ we may assume that σ has order p. Let $\mathscr{Y}$ be the subgroup of $\mathscr{Z}$ generated by σ.

Without loss of generality we may assume that $v_1 \in V_1$. Choose $v_2 \in V_2$

so $\{v_1, v_2\} \in E$. Let $u \in V_1$. By Lemma 2.1 there is a $\tau \in \mathscr{H}$ such that $u = \tau(v_1)$. Since $\sigma \in \mathscr{Z}$ we have

$$\sigma(u) = \sigma\tau(v_1) = \tau\sigma(v_1) = \tau(v_1) = u,$$

so σ leaves every element of V_1 fixed. If $\sigma(v_2) = v_2$ a similar argument would show that σ leaves every element of V_2 fixed. This would imply that $\sigma = 1$. But $\sigma \neq 1$, so $\sigma(v_2) \neq v_2$. Hence, $|\mathscr{Y}(v_2)| > 1$. Now

$$|\mathscr{Y}(v_2)| = (\mathscr{Y} : \mathscr{Y}_{v_2}) = 1 \text{ or } p,$$

so $|\mathscr{Y}(v_2)| = p$.

Let

$$\mathscr{K} = \{\tau \in \mathscr{H} \mid \tau(v_2) \in \mathscr{Y}(v_2)\}.$$

If $\tau, \eta \in \mathscr{K}$, then $\tau(v_2) = \sigma^i(v_2)$ and $\eta(v_2) = \sigma^j(v_2)$ for some i and j. Therefore,

$$\tau\eta(v_2) = \tau\sigma^j(v_2) = \sigma^j\tau(v_2) = \sigma^j\sigma^i(v_2) \in \mathscr{Y}(v_2),$$

so $\tau\eta \in \mathscr{K}$. Therefore, $\mathscr{K}$ is a subgroup of $\mathscr{H}$. We need several facts about $\mathscr{K}$. These facts are proved in the following lemmas.

LEMMA 2.3. *The subgroup $\mathscr{K}$ is normal in $\mathscr{H}$, and $(\mathscr{H} : \mathscr{K}) = p$.*

PROOF: Since $\mathscr{H}$ is a p-group, any subgroup of index p is normal. (See, for example, [1, Theorem IV, p. 122].) Hence, it suffices to show that $(\mathscr{H} : \mathscr{K}) = p$. We have $\mathscr{Y} \subset \mathscr{K}$, so $\mathscr{Y}(v_2) \subset \mathscr{K}(v_2) \subset \mathscr{Y}(v_2)$. Therefore, $|\mathscr{K}(v_2)| = |\mathscr{Y}(v_2)| = p$. Now $\mathscr{H}_{v_2} \subset \mathscr{K}$, so $\mathscr{K}_{v_2} = \mathscr{K} \cap \mathscr{H}_{v_2} = \mathscr{H}_{v_2}$. Hence,

$$\begin{aligned} p^2 = |\mathscr{H}(v_2)| &= (\mathscr{H} : \mathscr{H}_{v_2}) = (\mathscr{H} : \mathscr{K})(\mathscr{K} : \mathscr{H}_{v_2}) \\ &= (\mathscr{H} : \mathscr{K})|\mathscr{K}(v_2)| = (\mathscr{H} : \mathscr{K})\, p, \end{aligned}$$

so $(\mathscr{H} : \mathscr{K}) = p$.

LEMMA 2.4. *Let $\{u_1, u_2\} \in E$ with $u_1 \in V_1$ and $u_2 \in V_2$. Let $\tau \in \mathscr{K}$. Then $\{u_1, \tau(u_2)\} \in E$ and $\{\tau(u_1), u_2\} \in E$.*

PROOF: By Lemma 2.1 there is an $\eta \in \mathscr{H}$ such that $\eta(v_2) = u_2$. Since $\mathscr{K}$ is normal, $\eta^{-1}\tau\eta \in \mathscr{K}$. Therefore, $\eta^{-1}\tau\eta(v_2) = \sigma^i(v_2)$ for some i. Hence, we have

$$\tau(u_2) = \eta\eta^{-1}\tau\eta(v_2) = \eta\sigma^i(v_2) = \sigma^i\eta(v_2) = \sigma^i(u_2).$$

Since σ leaves every point in V_1 fixed, we have $\sigma^i(u_1) = u_1$. Therefore,

$$\{u_1, \tau(u_2)\} = \{\sigma^i(u_1), \sigma^i(u_2)\} \in E.$$

Now

$$\{\tau(u_1), u_2\} = \{\tau(u_1), \tau(\tau^{-1}(u_2))\} \in E$$

because $\{u_1, \tau^{-1}(u_2)\} \in E$ by what we have already shown.

LEMMA 2.5. *Let $u \in V$. Then $|\mathscr{K}(u)| = p$.*

PROOF: We have $p^2 \geqslant |\mathscr{K}(u)| = (\mathscr{K} : \mathscr{K}_u) = p^\alpha$ for some α, so $|\mathscr{K}(u)| = 1, p$, or p^2. If $|\mathscr{K}(u)| = 1$, then $\mathscr{K} \subset \mathscr{H}_u$. This is impossible since $(\mathscr{H} : \mathscr{H}_u) = p^2$ while $(\mathscr{H} : \mathscr{K}) = p$. Suppose $|\mathscr{K}(u)| = p^2$. Choose $u' \in V$, so $\{u, u'\} \in E$. By Lemma 2.4, $\{v, u'\} \in E$ for every $v \in \mathscr{K}(u)$. This implies that $d \geqslant p^2 = 1/2 \mid V \mid$. By Corrollary 1.1, G is point-symmetric, contradicting our assumption to the contrary. Hence, $|\mathscr{K}(u)| = p$.

LEMMA 2.6. *Let $\eta \in \mathscr{H}$ and let $u \in V$. If $\eta\mathscr{K}(u) \cap \mathscr{K}(u)$ is non-empty, then $\eta \in \mathscr{K}$.*

PROOF: Since $\mathscr{K}$ is normal, $\eta\mathscr{K} = \mathscr{K}\eta$, so

$$\eta\mathscr{K}(u) = \mathscr{K}\eta(u) = \mathscr{K}(\eta(u)).$$

The orbits under $\mathscr{K}$ of any two points are either equal or disjoint, so

$$\eta\, K(u) = \mathscr{K}(\eta(u)) = \mathscr{K}(u).$$

Let $\mathscr{L}$ be the subgroup of $\mathscr{H}$ generated by η and $\mathscr{K}$. Both η and $\mathscr{K}$ map the set $\mathscr{K}(u)$ into itself, so $\mathscr{L}$ maps $\mathscr{K}(u)$ into itself. Therefore, $\mathscr{L}(u) = \mathscr{K}(u)$. We have $\mathscr{K} \subset \mathscr{L} \subset \mathscr{H}$ and $(\mathscr{H} : \mathscr{K}) = p$ which is prime, so either $\mathscr{L} = \mathscr{H}$ or $\mathscr{L} = \mathscr{K}$. Now

$$|\mathscr{H}(u)| = p^2 > p = |\mathscr{K}(u)| = |\mathscr{L}(u)|,$$

so $\mathscr{L} \neq \mathscr{H}$. Therefore, $\mathscr{L} = \mathscr{K}$, so $\eta \in \mathscr{L} = \mathscr{K}$.

LEMMA 2.7. *Let $\eta \in \mathscr{H} - \mathscr{K}$ and let $i = 1$ or 2. Then V_i is the disjoint union of the sets*

$$\mathscr{K}(v_i), \eta\mathscr{K}(v_i), \ldots, \eta^{p-1}\mathscr{K}(v_i).$$

PROOF: By Lemma 2.5 and the normality of $\mathscr{K}$,

$$|\eta^j \mathscr{K}(v_i)| = |\mathscr{K}\eta^j(v_i)| = p.$$

There are p subsets and $|V_i| = p^2$, so it suffices to show that $\eta^j \mathscr{K}(v_i)$ and $\eta^k \mathscr{K}(v_i)$ are disjoint for $0 \leqslant j < k \leqslant p - 1$. Suppose

$$u \in \eta^j \mathscr{K}(v_i) \cap \eta^k \mathscr{K}(v_i).$$

Then

$$\eta^{-j}(u) \in \mathscr{K}(v_i) \cap \eta^{k-j} \mathscr{K}(v_i).$$

By Lemma 2.6, $\eta^{k-j} \in \mathscr{K}$. Now $0 < k - j < p$ and the order of η is a power of p, so there is an l such that $\eta = (\eta^{k-j})^l$. This contradicts the hypothesis that $\eta \notin \mathscr{K}$.

We are now ready to complete the proof of Theorem 2. By Lemma 2.3 there is an $\eta \in \mathscr{H} - \mathscr{K}$. For each i with $0 \leqslant i < p$ let α_i be a 1-1 map of $\eta^i\mathscr{K}(v_1)$ onto $\eta^{-i}\mathscr{K}(v_2)$ and let β_i be a 1-1 map of $\eta^i\mathscr{K}(v_2)$ onto $\eta^{-i}\mathscr{K}(v_1)$. Such maps exist because, by Lemma 2.5 and the normality of $\mathscr{K}$, the sets $\eta^i\mathscr{K}(v_j)$, $-p < i < p, j = 1$ or 2 all contain exactly p elements. Applying Lemma 2.7 to the elements η and η^{-1} we see that we may define a permutation φ of V by setting $\varphi(u) = \alpha_i(u)$ for $u \in \eta^i\mathscr{K}(v_1)$ and $\varphi(u) = \beta_i(u)$ for $u \in \eta^i\mathscr{K}(v_2)$.

Now we show that φ is an automorphism of G. Let $\{u_1, u_2\} \in E$ with $u_1 \in V_1$ and $u_2 \in V_2$. Then $u_1 = \eta^i\tau_1(v_1)$ and $u_2 = \eta^j\tau_2(v_2)$ for some i, j, τ_1, τ_2 with $0 \leqslant i, j < p$, and $\tau_1, \tau_2 \in \mathscr{K}$. We have $\varphi(u_1) = \eta^{-i}\tau_3(v_2)$ and $\varphi(u_2) = \eta^{-j}\tau_4(v_1)$ for some $\tau_3, \tau_4 \in \mathscr{K}$. Now

$$\{\eta^{-j}\tau_1(v_1), \eta^{-i}\tau_2(v_2)\} = \{\eta^{-i-j}(u_1), \eta^{-i-j}(u_2)\} \in E.$$

Since $\mathscr{K}$ is normal, $\eta^{-j}\tau_4\tau_1^{-1}\eta^j$ and $\eta^{-i}\tau_3\tau_2^{-1}\eta^i$ are in $\mathscr{K}$. Therefore, by Lemma 2.4,

$$\begin{aligned}\{\varphi(u_1), \varphi(u_2)\} &= \{\eta^{-j}\tau_4(v_1), \eta^{-i}\tau_3(v_2)\} \\ &= \{(\eta^{-j}\tau_4\tau_1^{-1}\eta^j)(\eta^{-j}\tau_1)(v_1), (\eta^{-i}\tau_3\tau_2^{-1}\eta)(\eta^{-i}\tau_2)(v_2)\} \in E.\end{aligned}$$

Hence, $\varphi \in \mathscr{G}$. But this is a contradiction since every element of $\mathscr{G}$ maps V_1 onto V_1 while φ maps V_1 onto V_2.

3. Regular Line-Symmetric Graphs Which Are Not Point-Symmetric

In this section we give some methods of constructing regular line-symmetric graphs that are not point-symmetric. To avoid endless repetition of the phrase "regular and line-symmetric but not point-symmetric" we use the following definition. A graph is said to be *admissible* if it is regular and line-symmetric but not point-symmetric. The *degree* of an admissible graph is its degree of regularity.

We will be particularly interested in the number of points that an admissible graph may have. Observe that if G is an admissible graph with v points, then the graph consisting of r disjoint copies of G is an admissible graph with rv points. In view of this trivial construction, it would be more pertinent to ask how many points a *connected* admissible graph may have. The corollary to the following result shows that the additional requirement of connectedness does not change the number of points that an admissible graph may have.

THEOREM 3. *Let $G = (V, E)$ be an admissible graph of degree d with v points. Let r be a positive integer. Then there is an admissible graph $\tilde{G}$ of degree rd with rv points. Furthermore, if G is connected, then $\tilde{G}$ is connected.*

PROOF: Let $R = \{1, 2, ..., r\}$. We define $\tilde{G} = (\tilde{V}, \tilde{E})$ by setting $\tilde{V} = V \times R$ and

$$\tilde{E} = \{\{(u, i), (v, j)\} | \{u, v\} \in E \text{ and } i, j \in R\}.$$

Clearly $\tilde{G}$ is regular of degree rd and $\tilde{G}$ has rv points. Furthermore, if G is connected, then so is $\tilde{G}$. Let σ be an automorphism of G. For each $v \in V$ let τ_v be a permutation of R. Then the permutation η of $\tilde{V}$ defined by

$$\eta((v, i)) = (\sigma(v), \tau_v(i))$$

is an automorphism of $\tilde{G}$. From this observation and the fact that G is line-symmetric, it follows that $\tilde{G}$ is line-symmetric.

It remains to show that $\tilde{G}$ is not point-symmetric. Suppose the contrary. Let $v_1, v_2 \in V$. Then there is an automorphism σ of $\tilde{G}$ such that

$$\sigma((v_1, 1)) = (v_2, 1).$$

I claim that there is a permutation τ of V such that $\tau(v_1) = v_2$ and for each $u \in V$ there are numbers $i, j \in R$ with

$$\sigma(u, i) = (\tau(u), j).$$

To see this, for each $u \in V - \{v_1\}$ let

$$S_u = \{u' \in V - \{v_2\} | \sigma((u, i)) = (u', j) \text{ for some } i, j \in R\}.$$

Let $u_1, ..., u_k \in V - \{v_1\}$ and let $S = S_{u_1} \cup \cdots \cup S_{u_k}$. Then

$$\sigma((\{u_1, ..., u_k\} \times R) \cup \{(v_1, 1)\}) \subset (S \cup \{v_2\}) \times R.$$

Since σ is a 1-1 map of $V \times R$ onto itself, this implies that

$$r \mid S \cup \{v_2\}| = |(S \cup \{v_2\}) \times R| \geqslant rk + 1.$$

Hence,

$$|S| = |S \cup \{v_2\}| - 1 \geqslant k - 1 + \frac{1}{r} > k - 1,$$

so $|S| \geqslant k$. Therefore, the family $\{S_u\}_{u \in V - \{v_1\}}$ of subsets of $V - \{v_2\}$ satisfies the condition in Hall's Theorem [3, Theorem 1.1, p. 48] for the existence of a system of distinct representatives. Consequently, for each $u \in V - \{v_1\}$ there is an element $\tau(u) \in V - \{v_2\}$ with $\tau(u) \in S_u$ and $\tau(u) \neq \tau(u')$ for $u \neq u'$. If we set $\tau(v_1) = v_2$, then τ is the required permutation of V.

Let $\{u_1, u_2\} \in V$. Let $i_1, j_1, i_2, j_2 \in R$ be such that

$$\sigma((u_1, i_1)) = (\tau(u_1), j_1) \text{ and } \sigma((u_2, i_2)) = (\tau(u_2), j_2).$$

Now $\{(u_1, i_1), (u_2, i_2)\} \in \tilde{E}$, so $\{(\tau(u_1), j_1), (\tau(u_2), j_2), j_2)) \in \tilde{E}$ and therefore $\{\tau(u_1), \tau(u_2)\} \in E$. Hence, τ is an automorphism of G. But $\tau(v_1) = v_2$ and v_1 and v_2 are arbitrary points in V. This implies that G is point-symmetric, contradicting the hypothesis that G is admissible.

COROLLARY 3.1. *If there is an admissible graph with v points, then there is a connected admissible graph with v points.*

PROOF: Let G be an admissible graph with v points. Then each connected component of G is admissible and all components of G are isomorphic. Let G' be a connected component of G and let v' be the number of points in G'. If G has r components, then $v = rv'$. The conclusion follows by applying Theorem 3 to the connected admissible graph G'.

THEOREM 4. *Let $\mathscr{A}$ be an Abelian group with "+" as the binary operation. Let T be an automorphism of $\mathscr{A}$. Let $r > 1$ be an integer and let $a \in \mathscr{A}$. Suppose that $T^r(a) = \pm a$, $T^i(a) \neq a$ for $0 < i < r$, and $T^i(a) \neq -a$ for $0 \leqslant i < r$. Then there is an admissible graph G with $2r\,|\mathscr{A}|$ points and degree $2r$.*

PROOF: Define a set V by

$$V = \{0, 1\} \times \{0, 1, 2, \ldots, r - 1\} \times \mathscr{A}.$$

Let E be the set of all two element subsets of V that are of the form

$$\{(0, i, x), (1, j, x)\} \text{ or } \{(0, i, x), (1, j, x + T^i(a))\}.$$

Let $G = (V, E)$. Then G has $2r\,|\mathscr{A}|$ points and G is regular of degree $2r$.

We define permutations σ_y, τ, η, and ρ of V as follows

$$\sigma_y((\epsilon, i, x)) = (\epsilon, i, x + y) \quad \text{for} \quad y \in \mathscr{A},$$

$$\tau((0, i, x)) = (0, i, x),$$

$$\tau((1, i, x)) = \begin{cases} (1, i + 1, x), & \text{if} \quad i < r - 1, \\ (1, 0, x), & \text{if} \quad i = r - 1, \end{cases}$$

$$\eta((0, i, x)) = (0, i, -x - T^i(a)),$$

$$\eta((1, i, x)) = (1, i, -x),$$

$$\rho((1, i, x)) = (1, i, T(x)),$$

$$\rho((0, i, x)) = \begin{cases} (0, i + 1, T(x)), & \text{if} \quad i < r - 1, \\ (0, 0, T(x)), & \text{if} \quad i = r - 1 \text{ and } T^r(a) = a, \\ (0, 0, T(x) - a), & \text{if} \quad i = r - 1 \text{ and } T^r(a) = -a. \end{cases}$$

It is easy to verify that these permutations are automorphisms of G. Furthermore, by repeated applications of these automorphisms any line of G may be transformed into any other line. Hence, G is line-symmetric.

For each $u \in V$ let

$$L(u) = \{v \in V \,|\, \{u, v\} \in E\}.$$

If α is an automorphism of G, then $L(\alpha(u)) = \alpha(L(u))$. Suppose G is point-symmetric. Then there is an automorphism α of G such that $\alpha((1, 0, 0)) = (0, 0, 0)$. Let

$$\alpha((1, 1, 0)) = (\epsilon, i, x).$$

We have

$$L((1, 0, 0)) = L((1, 1, 0)),$$

so

$$L((0, 0, 0)) = L((\epsilon, i, x)).$$

Now $(1, 0, 0) \in L((0, 0, 0)) = L((\epsilon, i, x))$ so we must have $\epsilon = 0$. Clearly, $L((0, i, x)) = L((0, j, y))$ if and only if $\{x, x + T^i(a)\} = \{y, y + T^j(a)\}$. Therefore, $\{x, x + T^i(a)\} = \{0, a\}$. There are two possibilities. First, we may have $x = a$ and $x + T^i(a) = 0$. This implies that $T^i(a) = -a$, contradicting our hypotheses. The only remaining possibility is $x = 0$ and $x + T^i(a) = a$. This implies that $T^i(a) = a$, which is possible only if $i = 0$. We have now arrived at the conclusion that

$$\alpha((1, 1, 0)) = (0, 0, 0) = \alpha((1, 0, 0)).$$

This contradicts the fact that α is a permutation of V. Hence, G is not point-symmetric, so it is admissible.

The following theorem summarizes what is known about the number of points that an admissible graph may have.

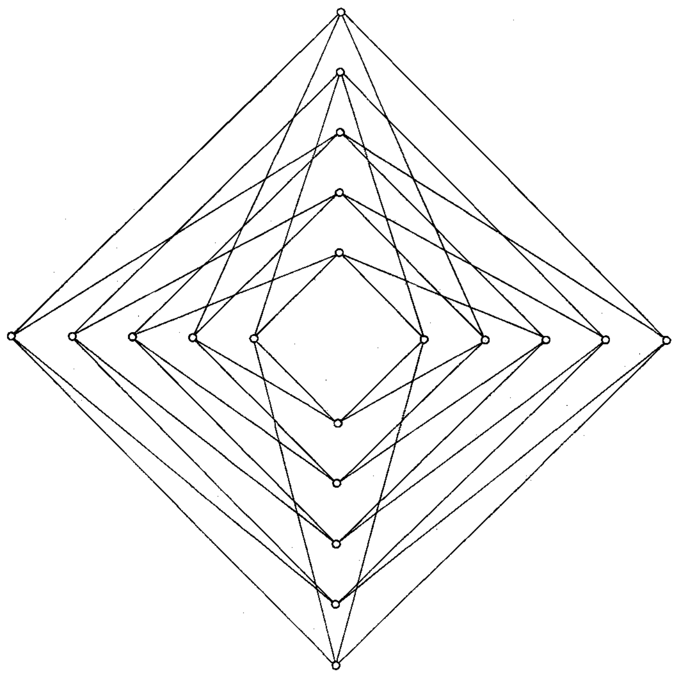

FIGURE 1. An admissible graph with 20 vertices.

THEOREM 5. *Let v be a positive integer. There are no admissible graphs with v points if v satisfies one of the following conditions:*

(3.1) *v is odd;*
(3.2) *$v = 2p$ or $2p^2$, where p is prime;*
(3.3) *$v < 30$ and 4 does not divide v;*
(3.4) *$v < 20$.*

There is an admissible graph with v points if v satisfies one of the following conditions:

(3.5) *v is divisible by $2p^3$, where p is an odd prime;*
(3.6) *v is divisible by $2pq$, where p and q are odd primes, and p divides $q - 1$;*

(3.7) *v is divisible by $2pq^2$, where p and q are primes, q is odd, and p divides $q + 1$;*

(3.8) *$v \geqslant 20$ and 4 divides v.*

PROOF: Assume (3.1). The conclusion follows from Corollary 1.1. Assume (3.2). The conclusion follows from Theorem 2. Assume (3.3) but not (3.1) or (3.2). Then $v = 2$ and the conclusion is obvious. Assume (3.4) but not (3.2) or (3.3). Then $v = 12$ or 16. The only proof we have in these cases consists of examining all line-symmetric graphs with v vertices that satisfy the conclusion of Theorem 1. This argument is too lengthy to be included here.

Conditions (3.5) to (3.7) are all of the form "v is divisible n," where n has some specified form. By Theorem 3 it suffices to construct an admissible graph with v points when $v = n$. The constructions will be based on Theorem 4. As usual, Z_n will denote the cyclic group of order n.

(3.5) Let $\mathscr{A} = Z_p \times Z_p$, where the two copies of Z_p have generators g_1 and g_2. Let T be the automorphism of $\mathscr{A}$ defined by $T(g_1) = g_1 + g_2$, $T(g_2) = g_2$. Let $a = g_1$ and $r = p$. Now $T^i(g_1) = g_1 + ig_2$. Hence, $T^p(g_1) = g_1 + pg_2 = g_1$, $T^i(g_1) = g_1 + ig_2 \neq g_1$ for $0 < i < p$, and $T^i(g_1) = g_1 + ig_2 \neq -g_1$ for all i because p is odd.

(3.6) Let $\mathscr{A} = Z_q$ with generator g. Since q is prime and p divides $q - 1$, there is an integer x such that $x^p \equiv 1 \pmod q$ but $x^i \not\equiv 1 \pmod q$ for $0 < i < p$. Let T be the automorphism of $\mathscr{A}$ defined by $T(g) = xg$. Let $a = g$ and $r = p$. Now $T^i(g) = x^i g$, so $T^i(g) = ng$ if and only if $x^i \equiv n \pmod q$. Hence, $T^p(g) = g$ but $T^i(g) \neq g$ for $0 < i < p$. Suppose $T^i(g) = -g$ for some i. Then $x^i \equiv -1 \pmod q$. Since p is odd, this implies that

$$-1 \equiv (-1)^p \equiv (x^i)^p \equiv (x^p)^i \equiv 1^i \equiv 1 \pmod q.$$

This is impossible because q is odd.

(3.7) Let $\mathscr{A} = Z_q \times Z_q$ where the two copies of Z_q have generators g_1 and g_2. First suppose that $p = 2$. Let T be the automorphism of $\mathscr{A}$ defined by $T(g_1) = g_2$ and $T(g_2) = g_1$. Let $a = g_1$ and $r = 2$. Then $T^2(g_1) = g_1$, $T(g_1) = g_2 \neq \pm g_1$ and $T^0(g_1) = g_1 \neq -g_1$ since q is odd.

Now suppose that p is odd. The group of automorphisms of $\mathscr{A}$ is just the group of all 2×2 non-singular matrices with coefficients in the finite field Z_q. This group contains $(q-1)^2 q(q+1)$ elements. Now p is prime and p divides $q + 1$, so there is an automorphism T of $\mathscr{A}$ such that $T^p = 1$ but $T^i \neq 1$ for $0 < i < p$. Let $r = p$. Since $T \neq 1$, there is an $a \in \mathscr{A}$ with $T(a) \neq a$. We have $T^p(a) = a$. Suppose $T^i(a) = a$ for some i with $0 < i < p$. Then $T^{\lambda i + \mu p}(a) = a$ for all integers λ and μ. Now i and p are relatively prime so we can choose λ and μ so that $\lambda i + \mu p = 1$. This contradicts our assumption that $T(a) \neq a$. Finally, suppose $T^i(a) = -a$ for some i. Now p is odd so

$$-a = (-1)^p a = (T^i)^p(a) = (T^p)^i(a) = a.$$

Since q is odd, this is possible only if $a = 0$. But $T(0) = 0$, so this contradicts our assumption that $T(a) \neq a$.

Now assume that (3.8) holds. We consider four cases:

(i) $v = 4p$, where p is prime and $p \equiv 1 \pmod 4$.

Let $\mathscr{A} = Z_p$ with generator g. Since $p \equiv 1 \pmod 4$, there is an integer x with $x^2 \equiv -1 \pmod p$. Let T be the automorphism of $\mathscr{A}$ defined by $T(g) = xg$. Let $a = g$ and $r = 2$. Then $T^2(g) = x^2 g = -g$. If $T(g) = \pm g$ then $x \equiv \pm 1 \pmod p$, so $-1 \equiv x^2 \equiv (\pm 1)^2 \equiv 1 \pmod p$. Furthermore, if $g = T^0(g) = -g$, then $1 \equiv -1 \pmod p$. But $1 \not\equiv -1 \pmod p$ because $p \geqslant 5$.

(ii) $v = 4p$, where p is prime, $p \equiv -1 \pmod 4$, and $p \geqslant 7$. We cannot use Theorem 4 in this case, so we explicitly construct an admissible graph with v vertices.

Let g be a generator of the cyclic group Z_p. Let $V = \{0, 1\} \times \{0, 1\} \times Z_p$. Let E be the set of all two element subsets of V of the form

$$\{(0, 0, x), (1, \epsilon, x + i^2 g)\}$$

or $\{(0, 1, x), (1, \epsilon, x - i^2 g)\}$, where $\epsilon = 0$ or 1, $x \in Z_p$, and i is an integer with $1 \leqslant i \leqslant (p-1)/2$. Let $G = (V, E)$. Clearly G is regular of degree $p - 1$.

The following permutations of V are automorphisms of G:

$$\sigma((\epsilon, \delta, x) = (\epsilon, \delta, x + g) \qquad \epsilon, \delta = 0, 1, \quad x \in \mathscr{A}$$

$$\left.\begin{aligned} \tau((0, 0, x)) &= (0, 1, -x) \\ \tau((0, 1, x)) &= (0, 0, -x) \\ \tau((1, \epsilon, x)) &= (1, \epsilon, -x) \end{aligned}\right. \qquad x \in \mathscr{A}, \quad \epsilon = 0, 1,$$

$$\left.\begin{aligned} \eta((0, \epsilon, x)) &= (0, \epsilon, x) \\ \eta((1, 0, x)) &= (1, 1, x) \\ \eta((1, 1, x)) &= (1, 0, x) \end{aligned}\right. \qquad x \in \mathscr{A}, \quad \epsilon = 0, 1,$$

$$\rho_i((\epsilon, \delta, x)) = (\epsilon, \delta, i^2 x) \qquad \epsilon, \delta = 0, 1, \quad x \in \mathscr{A}$$

for $1 \leqslant i \leqslant (p-1)/2$. To see that ρ_i is an automorphism of G, we observe that if $1 \leqslant i, j \leqslant (p-1)/2$, then $ij \equiv \pm k \pmod p$ for some k with $1 \leqslant k \leqslant (p-1)/2$ so $i^2 j^2 g = k^2 g$. By successive applications of the above automorphisms, any line of G can be transformed into any other line of G. Hence, G is line-symmetric.

As in the proof of Theorem 4, for $u \in V$ let

$$L(u) = \{v \in V \mid \{u, v\} \in E\}.$$

Suppose G is point-symmetric. Then there is an automorphism α of G such that $\alpha((1, 0, 0)) = (0, 0, 0)$. Let $\alpha((1, 1, 0)) = (\gamma, \delta, ng)$, where $\gamma, \delta = 0$ or 1 and $0 \leqslant n < p$. Now $L((1, 0, 0)) = L((1, 1, 0))$, so $L((0, 0, 0)) = L((\gamma, \delta, ng))$. Therefore, $\gamma = 0$. Suppose $n = 0$. Then $\delta = 1$ since α is $1 - 1$. Hence,

$$(1, 0, -g) \in L((0, 1, 0)) = L((0, 0, 0)).$$

This implies that $-1 \equiv i^2 \pmod p$ for some integer i. This is impossible since -1 is not a quadratic residue mod p. ($p \equiv -1 \pmod 4$). Therefore, $0 < n < p$.

If n is a quadratic residue mod p, there is an i with $1 \leqslant i \leqslant (p - 1)/2$ and $i^2 n \equiv 1 \pmod p$. If n is a non-residue, there is an i with $1 \leqslant i \leqslant (p - 1)/2$ and $i^2 n \equiv -1 \pmod p$. In either case we can choose i so $\rho_i((0, \delta, ng)) = (0, \delta, \epsilon g)$, where $\epsilon = \pm 1$. Now $\rho_i((0, 0, 0)) = (0, 0, 0)$, so $L((0, 0, 0)) = L((0, \delta, \epsilon g))$. Suppose $\delta = 0$. If $\epsilon = 1$, then $(1, 0, g) \in L((0, 0, 0)) = L((0, 0, g))$. This implies that $g = g + i^2 g$ for some i with $1 \leqslant i \leqslant (p - 1)/2$. But $g = g + i^2 g$ only if $i^2 \equiv 0 \pmod p$ and this is clearly impossible. If $\epsilon = -1$, then $(1, 0, 0) \in L((0, 0, -g)) = L((0, 0, 0))$, which is impossible. Hence, $\delta = 1$. If $\epsilon = 1$ then $(1, 0, g) \in L((0, 0, 0)) = L((0, 1, g))$, which is impossible. Therefore, $\epsilon = -1$.

We have now arrived at the conclusion that $L((0, 0, 0)) = L((0, 1, -g))$. This implies that for every i with $1 \leqslant i \leqslant (p - 1)/2$ there is a j with $1 \leqslant j \leqslant (p - 1)/2$ and $j^2 \equiv -i^2 - 1 \pmod p$. In other words, if r is a quadratic residue mod p, then so is $-r - 1$. Now $p \geqslant 7$, so $1 = 1^2$, $4 = 2^2$, and $9 = 3^2$ are all quadratic residues, mod p. Hence, so are $-2 = -1 - 1$, $-5 = -4 - 1$, and $-10 = -9 - 1$. Therefore, $10 = (-2)(-5)$ is a quadratic residue mod p. But if 10 and -10 are both residues, then -1 must be a residue. This contradicts the fact that $p \equiv -1 \pmod 4$ and completes the proof that G is admissible.

By (i), (ii), and Theorem 3, an admissible graph with v points exists whenever v satisfies (3.8) and v is divisible by a prime $p \geqslant 5$. It remains to consider the cases $v = 2^a \cdot 3^b$, $a \geqslant 2$, $2^a \cdot 3^b \geqslant 20$. If $b \geqslant 2$, then $2 \cdot 2 \cdot 3^2$ divides v. Now 2 and 3 are prime, 3 is odd, and 2 divides $3 + 1$, so v satisfies (3.7). If $b = 1$, then 24 divides v. If $b = 0$, then 32 divides v. By Theorem 3 it now suffices to consider the cases $v = 32$ and $v = 24$.

(iii) $v = 32$. Let $\mathscr{A} = Z_8$ with generator g. Let T be the automorphism of $\mathscr{A}$ defined by $T(g) = 3g$. Let $a = g$ and $r = 2$. Then

$$T^2(g) = 9g = g,\ T(g) = 3g \neq \pm g, \text{ and } g = T^0(g) \neq -g.$$

(iv) $v = 24$. Let $A = \{0, 1\} \times \{0, 1\} \times \{1, 2, 3\}$. Let B be the set of all subsets of A that are of the form

$$\{(0, \delta, i), (1, \delta, i), (0, \epsilon, j), (1, \epsilon, j)\},$$

where $\delta, \epsilon = 0, 1$ and $1 \leqslant i < j \leqslant 3$. Let $V = A \cup B$. Let

$$E = \{\{a, b\} \mid a \in A, b \in B, \text{ and } a \in b\}.$$

Let $G = (V, E)$. Then G is a graph with 24 points, which is regular of degree 4.

Let σ and τ be permutations of $\{0, 1\}$ and $\{1, 2, 3\}$, respectively. For each $i \in \{1, 2, 3\}$, let ρ_i be a permutation of $\{0, 1\}$. Define a permutation α of V by

$$\alpha((\delta, \epsilon, i)) = (\sigma(\delta), \rho_i(\epsilon), \tau(i))$$

for $(\delta, \epsilon, i) \in A$ and

$$\alpha(\{a_1, a_2, a_3, a_4\}) = \{\alpha(a_1), \alpha(a_2), \alpha(a_3), \alpha(a_4)\}$$

for $\{a_1, a_2, a_3, a_4\} \in B$. Then α is an automorphism of G. Furthermore, given any two lines in G, there is an automorphism of this form sending one into the other. Hence, G is line-symmetric.

Suppose G is point-symmetric. Then there is an automorphism α of G with $\alpha((0, 0, 1)) \in B$. Now

$$L((0, 0, 1)) = L((1, 0, 1)),$$

so

$$L(\alpha(0, 0, 1) = L(\alpha(1, 0, 1)).$$

Hence,

$$\alpha((1, 0, 1)) \in B.$$

But for $b \in B$, $L(b) = b$, so

$$\alpha((0, 0, 1)) = L(\alpha((0, 0, 1))) = L(\alpha((1, 0, 1))) = \alpha((1, 0, 1)).$$

This contradicts the fact that α is 1-1 and completes the proof that G is admissible.

4. Open Problems

We conclude with a list of questions about admissible graphs which have not been answered.

(4.1) For which integers v is there an admissible graph with v points?

(4.2) Is there an admissible graph with 30 points? (This is the smallest value of v for which (4.1) is open.)

(4.3) Is there an admissible graph with $2pq$ points, where p and q are odd primes, $p < q$, and p does not divide $q - 1$? (This is the simplest class of values of v for which (4.1) is open.)

Is there an admissible graph with $v = 2v'$ points and degree d when

(4.4) $d \geqslant v/4$?

(4.5) d is prime?

(4.6) d and v' are relatively prime?

(4.7) d is prime and d does not divide v'? (None of the admissible graphs that we have constructed satisfies any of the conditions (4.4) to (4.7).)

(4.8) For which pairs of integers v and d is there a connected admissible graph with v points and degree d?

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