The Moore–Penrose inverse of a factorization

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Received 13 June 2000; accepted 10 January 2003

Submitted by G. de Oliveira

Abstract

In this paper, we consider the product of matrices $PAQ$, where $A$ is von Neumann regular and there exist $P'$ and $Q'$ such that $P'PA = A = AQ'Q$. We give necessary and sufficient conditions in order to $PAQ$ be Moore–Penrose invertible, extending known characterizations. Finally, an application is given to matrices over separative regular rings.

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Keywords: Matrices over rings; von Neumann regularity; Moore–Penrose invertibility; Factorization; Separative regular rings

1. Introduction

Let $R$ be an arbitrary ring with unity 1, $\mathcal{M}(R)$ the set of (finite) matrices over $R$, $\mathcal{M}_{m\times n}(R)$ the subset of $m \times n$ matrices and $\mathcal{M}_m(R)$ the ring of $m \times m$ matrices over $R$. Let $^*$ be an involution, see [11], on the matrices over $R$. Given an $m \times n$ matrix $T$ over a ring $R$, then $T$ is (von Neumann) regular if there exists an $n \times m$ matrix $T^-$ such that

$$TT^- = T.$$

$T^-$ is called a von Neumann inverse of $T$ and the set of all von Neumann inverses of $T$ will be denoted by $T[1]$. That is,

$$T[1] = \{X \in \mathcal{M}_{n\times m}(R) : TXT = T\}.$$

$T^{(1)}$ denotes an arbitrary element of $T[1]$.

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doi:10.1016/S0024-3795(03)00391-4
$T$ is said to be Moore–Penrose invertible with respect to $^*$ if there exists a (unique) $n \times m$ matrix $T^\dagger$ such that:

\[
TT^\dagger T = T, \\
T^\dagger TT^\dagger = T^\dagger, \\
(TT^\dagger)^* = TT^\dagger, \\
(T^\dagger T)^* = T^\dagger T.
\]

In addition, we will consider the following sets

\[
T\{1, 3\} = T\{1\} \cap \{X \in M_{m \times n}(R) : (TX)^* = TX\}, \\
T\{1, 4\} = T\{1\} \cap \{X \in M_{m \times n}(R) : (XT)^* = XT\}.
\]

An arbitrary element of $T\{1, 3\}$ (resp. $T\{1, 4\}$) will be denoted by $T_{1,3}$ (resp. $T_{1,4}$).

If $m = n$, then the group inverse of $T$ exists if there is a (unique) $T^\#$ such that

\[
TT^\#T = T, \\
T^\#TT^\# = T^\#, \\
TT^\# = T^\#T.
\]

In [13], the group inverse of $T = PAQ$ in which $A$ is regular and for which there exist $P', Q'$ such that $P'PA = A = AQQ'$, was characterized by means of classical invertibility. From that general result followed that, if $T$ is regular, then $T^\#$ exists if and only if $T^2I + I - TT^\#$ is invertible if and only if $T^\#T^2 + I - T^\#T$ is invertible.

In the present paper we make an analogue of that group inverse result concerning the Moore–Penrose inverse. This will also give an alternative proof of the main result from [9], as well as a more general formula for the computation of the Moore–Penrose inverse of a matrix, extending results from [4,9,12]. As an application we derive the Moore–Penrose inverse of matrices over separative regular rings, using recent results that appear in [1].

For further notations and definitions, we refer to [2].

2. Results

In [12], the Moore–Penrose inverse w.r.t. an involution $^*$ of a matrix product $PAQ$, in which $A$ was regular and $A = A^*$, was considered. Also in [4], the Moore–Penrose inverse w.r.t. an involution $^*$ of a matrix product $PAQ$, in which $A$ had a Moore–Penrose inverse, was considered. More recently, see [9], these results were generalized up to $PAQ$ with $P$ and $Q$ invertible and $A$ regular. Here we consider now the more general factorization $PAQ$ for which $A$ is regular and there exist matrices $P', Q'$ such that $P'PA = A = AQQ'$.

The following lemma will play an important role in the coming proofs.

This lemma is part of a more general result presented in [10] which reflects the independence of the invertibility of $TT^*TT^{(1)} + I - TT^{(1)}$ and of $T^{(1)}TT^*T + I - T^{(1)}T$ to the choice of $T^{(1)} \in T[1]$.

Theorem 2. Let $A \in \mathcal{M}_{m \times n}(R)$ be a (von Neumann) regular matrix and $P \in \mathcal{M}_{p \times m}(R), Q \in \mathcal{M}_{n \times q}(R)$. The following conditions are equivalent:

1. $AQQ^*A^*P^*PAA^{-1} + I_m - AA^{-1}$ is invertible for all $A^{-1} \in A[1]$.
3. $(PAQ)^\dagger$ exists w.r.t. $*$ and there exist $P', Q'$ such that $P'PA = A = AQQ'$.

Moreover,

$$(PAQ)^\dagger = (PU^{-1}AQ)^* = (PAV^{-1}Q)^*,$$

where $U = AQQ^*A^*P^*PAA^{-1} + I_m - AA^{-1}$ and $V = A^1AQ^*A^*P^*PA + I_n - A^{-1}A$.

Proof. (1) $\iff$ (2). Let $A^{-1} \in A[1]$. If $U$ is invertible then $AQQ^*A^*P^*AA^{-1}$ is invertible in the ring $AA^{-1}\mathcal{M}_mAA^{-1}$. That is, there exists $X \in AA^{-1}\mathcal{M}_mAA^{-1}$ for which

$$AQQ^*A^*P^*PAA^{-1}X = AA^{-1} = XAQQ^*A^*P^*PAA^{-1}.$$

Then

$$A^{-1}AQQ^*A^*P^*PAA^{-1}X = A^{-1}A = A^{-1}XAQQ^*A^*P^*PA,$$

which implies $A^{-1}XA \in A^{-1}A.\mathcal{M}_nA^{-1}A$ is an inverse of $A^{-1}AQ^*A^*P^*PA$ in $A^{-1}A.\mathcal{M}_nA^{-1}A$. Therefore, $A^{-1}AQ^*A^*P^*PA + I_n - A^{-1}A$ is an invertible matrix, and applying the lemma we obtain the implication. The converse is analogous.

(3) $\Rightarrow$ (1). In the first place,

$$\left[PAQ(PAQ)^* + I_p - PAQ(PAQ)^\dagger\right]^{-1}$$

$$= \left[PAQ(PAQ)^*PAQ(PAQ)^\dagger + I_p - PAQ(PAQ)^\dagger\right]^{-1}$$

$$= (PAQ)^\dagger(PAQ)^\dagger + I_p - PAQ(PAQ)^\dagger.$$

Thus, there exists $(PAQ)^{(1)} \in (PAQ)|1$ (namely $(PAQ)^\dagger$) such that

$$PAQ(PAQ)^*PAQ(PAQ)^{(1)} + I_p - PAQ(PAQ)^{(1)}$$

$$= (PAQ)^\dagger(PAQ)^\dagger + I_p - PAQ(PAQ)^\dagger.$$
is invertible, which implies, using an extended version of the lemma, see [10], its invertibility for all \((PAQ)^{(1)}(PAQ)[1]\). It is clear that \(Q' A^{-} P'\) is a von Neumann inverse of \(PAQ\). Then
\[
PAQ(PAQ)^{\ast}PAQ(Q' A^{-} P') + I_{p} - PAQ(Q' A^{-} P')
\]
is invertible, i.e.,
\[
K = PAQQ^{\ast}A^{\ast}P^{\ast}PAA^{-}P' + I_{p} - PAA^{-}P'
\]
is invertible. Setting \(E = PAA^{-}P'\), and since \(E^{2} = E\) and \(K\) is invertible, then
\[
W = PAQQ^{\ast}A^{\ast}P^{\ast}PAA^{-}P' = EKE
\]
is invertible in the ring \(E.\mathcal{M}_{p}(R)E\). So, there exists a \(X \in E.\mathcal{M}_{p}(R)E\) such that
\[
E = WX, \quad (1)
\]
\[
E = XW. \quad (2)
\]
By (1), and as \(EX = X\),
\[
PAA^{-}P' = E
= WX
= WEX
= (WPAA^{-})P'X
= (PAQQ^{\ast}A^{\ast}P^{\ast}PAA^{-})P'X.
\]
Multiplying on the left by \(P'\) and on the right by \(PAA^{-}\), we have
\[
(AQQ^{\ast}A^{\ast}P^{\ast}PAA^{-})P'XPAA^{-} = AA^{-}
\]
and therefore
\[
[(AA^{-})AQ^{\ast}A^{\ast}P^{\ast}P(AA^{-})][(AA^{-})P'XP(AA^{-})] = AA^{-}. \quad (3)
\]
By (2), and as \(XE = X\),
\[
PAA^{-}P' = E
= WX
= XEW
= XPAA^{-}P'W
= XP(AQQ^{\ast}A^{\ast}P^{\ast}PAA^{-})P'.
\]
Multiplying on the left by \(AA^{-} P'\) and on the right by \(PAA^{-}\),
\[
[(AA^{-})P'XP(AA^{-})][(AA^{-})AQ^{\ast}A^{\ast}P^{\ast}P(AA^{-})] = AA^{-}. \quad (4)
\]
Combining (3) and (4), it follows that \(AQ^{\ast}A^{\ast}P^{\ast}PAA^{-}\) is invertible in the ring \(AA^{-}.\mathcal{M}_{m}(R)AA^{-}\) and therefore \(AQ^{\ast}A^{\ast}P^{\ast}PAA^{-} + I_{m} - AA^{-}\) is an invertible matrix. The lemma gives now the desired result.
(1) ⇒ (3). We remark that the invertibility of $U$ is equivalent to the invertibility of $V$, for any (not necessarily the same) choice of $A^{(1)}$ in $U$ and in $V$ (see [10]).

Since $UA = AQ(PAQ)^*PA$ then

$$PAQ = (PU^{-1}AQ)(PAQ)^*PAQ.$$ 

Similarly, $AV = AQ(PAQ)^*PA$ implies

$$PAQ = PAQ(PAQ)^*(PAV^{-1}Q).$$ 

Therefore $PAQ$ is Moore–Penrose invertible (see [11]) with

$$(PAQ)^\dagger = (PAV^{-1}Q)^* = (PAQ)^*(PAV^{-1}Q)^*.$$ 

From $AV = AQ(PAQ)^*PA$ we derive the equality $A* = (PAV^{-1})^*PAQQ^*A^*$, which gives

$$(PAQ)^\dagger = (PU^{-1}AQ)^* = (PAV^{-1}Q)^*,$$

since $AV = UA$. □

**Remark 3.** When $P, Q$ are the identity matrices in the previous result, we obtain the following.

Given a regular $m \times n$ matrix $T$ over $R$, the following conditions are equivalent:

1. $TT^*TT^{(1)} + I_m - TT^{(1)}$ is invertible, for all $T^{(1)} \in T[1]$.
2. $T^{(1)}TT^* + I_n - T^{(1)}T$ is invertible, for all $T^{(1)} \in T[1]$.
3. The Moore–Penrose inverse $T^\dagger$ exists w.r.t. $*$. 

In that case,

$$T^\dagger = T^*(TT^*TT^{(1)} + I_m - TT^{(1)})^{-1}$$

$$= \left(T^{(1)}TT^* + I_n - T^{(1)}T\right)^{-1}T^*.$$ 

If $R$ is regular ring and $*$ is a general involution on $M(R)$, then using the Remark 3 and the fact that any matrix over a regular ring is regular, we can characterize Moore–Penrose invertibility of any matrix over $R$ by using classical invertibility.

**Remark 4.** If the involution $*$ on matrices over a regular ring is $*$-reducing, i.e., all matrices are $*$-cancelable, then it is known that all matrices are Moore–Penrose invertible, see Lemma 3(iv) in [11], which means now that all the expressions $TT^*TT^{(1)} + I_m - TT^{(1)}$ and $T^{(1)}TT^* + I_n - T^{(1)}T$ are invertible for all matrices $T$ over a regular ring with a $*$-reducing involution $*$ on the matrices over the regular ring.
We now generalize the following well known result for matrices over the complexes, see [3].

If \( ^{+} \) denotes the conjugate transpose of a complex matrix then the Moore–Penrose inverse of a complex matrix \( T \) w.r.t. \( ^{+} \) always exists and is given by

\[
T^\dagger = \left[ T^\dagger (TT^\dagger)^{-} \right] \left[ (T^\dagger T)^{-} T^\dagger \right] = T^{1.4} TT^{1.3}.
\]

**Theorem 5.** If \( PAQ \in \mathcal{M}(R) \) is a matrix product for which there exist matrices \( P' \) and \( Q' \) such that \( P'PA = A = AQ' \), then the Moore–Penrose inverse of \( PAQ \) exists if and only if \( (PA)^{1.3} \) and \( (AQ)^{1.4} \) exist, in which case

\[
(PAQ)^\dagger = (AQ)^{1.4} A(PA)^{1.3}.
\]

**Proof.** Assume, in the first place, \( (PA)^{1.3} \) and \( (AQ)^{1.4} \) exist. Then

\[
AQ = AQ(AQ)^{1.4} A = AQ(AQ)^*(AQ)^{1.4} * ,
\]

and hence

\[
PAQ = PAQ(PAQ)^* (P')^*(AQ)^{1.4} * .
\]

Analogously,

\[
PA = PA(PA)^{1.3} PA = ((PA)^{1.3})^* (PA)^* PA
\]

and hence

\[
PAQ = ((PA)^{1.3})^* (AG)^* (PAQ)^* PAQ.
\]

Applying [11, Lemma 3],

\[
(PAQ)^\dagger = (AQ)^{1.4} P'(PAQ) Q'(PA)^{1.3} = (AQ)^{1.4} A(PA)^{1.3}.
\]

Conversely, assume \( (PAQ)^\dagger \) exists. Then,

\[
PAQ = PAQ(PAQ)^* ((PAQ)^\dagger)^* ,
\]

which implies that \( AQ = AQ(AQ)^* X \) is a consistent matrix equation. We will show that \( X^* \in AQ[1, 4] \). It follows from \( AQ = AQ(AQ)^* X \) and \( (AQ)^* = X^*(AQ)(AQ)^* \) that

\[
AQ = AQX^* AQ(AQ)^* X = AQX^* A.
\]

Also, the idempotent

\[
X^* AQ = X^* AQX^* A Q
= X^* AQX^* AQ(AQ)^* X
= X^* AQ(AQ)^* X
\]

is symmetric.

Similar arguments show that \( (PA)^{1.3} \) exists if \( (PAQ)^\dagger \) exists. \( \square \)
It has to be remarked that, in the previous Theorem, there are no conditions on the matrix $A$. If, however, $A$ is supposed to have a Moore–Penrose inverse $A^\dagger$ then we obtain the following known result from [4].

**Corollary 6.** Let $A$ be a $m \times n$ matrix over $R$ with Moore–Penrose inverse $A^\dagger$, and $P, Q$ as in the previous Theorem. Then the following conditions are equivalent:

1. $PAQ$ is Moore–Penrose invertible.
2. $PA$ and $AQ$ are both Moore–Penrose invertible.
3. $(PA)^*PA + I_n - A^\dagger A$ and $AQ(AQ)^* + I_m - AA^\dagger$ are invertible matrices.

Then,

$$(PAQ)^\dagger = (AQ)^\dagger A(PA)^\dagger$$

$$= (AQ)^* \left[ AQ(AQ)^* + I_m - AA^\dagger \right]^{-1} \times A \left[ (PA)^*PA + I_n - A^\dagger A \right]^{-1} (PA)^*$$

**Proof.** (1) $\Rightarrow$ (2). By one hand, we have shown that the Moore–Penrose invertibility of $PAQ$ implies

$$(PA)[1, 3], (AQ)[1, 4] \neq \emptyset.$$  

By another hand, the existence of $A^\dagger$ implies

$$(PA)[1, 4], (AQ)[1, 3] \neq \emptyset,$$

since $A^\dagger P^\dagger \in (PA)[1, 4]$ and $Q'A^\dagger \in (AQ)[1, 3]$. Thus, $PA, QA$ are both Moore–Penrose invertible.

(2) $\Rightarrow$ (1) is obvious from the previous theorem.

(2) $\Rightarrow$ (3). If $(PA)^\dagger$ and $(AQ)^\dagger$ exist, then the invertibility of $A^{(1)}AA^*P^*PA + I_n - A^{(1)}A$ and of $AQ(AQ)^* + I_m - AA^{(1)}$ hold, for all $A^{(1)} \in A[1]$. Taking $A^{(1)} = A^\dagger$, the implication follows.

(3) $\Rightarrow$ (2). If

$$(PA)^*PA + I_n - A^\dagger A = A^\dagger A(PA)^*PA + I_n - A^\dagger A$$

is invertible, then $A^{(1)}(PA)^*PA + I_n - A^{(1)}A$ is invertible for any von Neumann inverse $A^{(1)}$ of $A$, which gives the existence of $(PA)^\dagger$.

An analogue argument gives the Moore–Penrose invertibility of $AQ$ from the invertibility of $AQ(AQ)^* + I_m - AA^\dagger$. □
3. Application to matrices over separative regular rings

It is known that square matrices over a unit regular ring admit a diagonal reduction by invertible matrices (see [7]). This means that for these cases the Moore–Penrose inverse can be characterized by Theorem 2, but also by Theorem 2 in [9].

A recent result, see [1], states that “every square matrix over a separative regular ring also admits a diagonal reduction by invertible matrices”. Therefore, the Moore–Penrose inverse can be characterized in these cases in the same way as for matrices over unit regular rings.

We now consider also “non-square” matrices over separative regular rings and show how Theorems 2 and 5 can be applied to characterize the Moore–Penrose inverse.

If $T_{m \times n} \in \mathcal{M}_{m \times n}(R)$, with $m < n$, then we can complete it to a square matrix by adding zeros, and it follows from [1] that there exist invertible matrices $P, Q$ and a diagonal matrix $D$ such that

$$
\begin{bmatrix}
T_{m \times n} \\
0_{(n-m) \times n}
\end{bmatrix} = PDQ.
$$

(5)

Therefore

$$
T_{m \times n} = \left( \begin{bmatrix} I_m & 0_{m \times (n-m)} \end{bmatrix} P \right) D Q = \tilde{P} D Q,
$$

(6)

where $\tilde{P} = \left[ I_m \ 0_{m \times (n-m)} \right] P$, and we are in the conditions of Theorem 2 since

$$
P' = p^{-1} \begin{bmatrix} I_m \\
0_{(n-m) \times m}
\end{bmatrix}
$$

is a matrix such that

$$
P' \left( \begin{bmatrix} I_m & 0_{m \times (n-m)} \end{bmatrix} P \right) D = D.
$$

We therefore can apply Theorem 2 to the factorization (6).

By Theorem 5, $T^*$ exists if and only if $(DQ)^{1,4}$ and $(\tilde{P}D)^{1,3}$ exist, in which case

$$
T^* = (DQ)^{1,4} D (\tilde{P}D)^{1,3} = (DQ)^* (DQ^* D^* (1) D (D^* \tilde{P}^* \tilde{P} D)^{(1)} (\tilde{P} D)^*)^*.
$$

This follows from the fact that matrices over regular rings are always regular and from the following general facts:

1. $XX^*$ regular and $X^{1,4}$ exists implies $X^*(XX^*)^{(1)} \in X[1, 4]$.
2. $X^*X$ regular and $X^{1,3}$ exists implies $(X^*X)^{(1)} X^* \in X[1, 3]$.

Indeed, if $X^{1,4} \in X[1, 4]$ then $X = XX^*(X^{1,4})^*$ and $X^* = X^{1,4}XX^*$. These imply

$$
X^*(XX^*)^{(1)} X = X^{1,4} XX^*(XX^*)^{(1)} XX^*(X^{1,4})^* = X^{1,4} XX^*(X^{1,4})^*.
$$
which is symmetric. Multiplying on the left by $X$, it is clear that $X^*(XX^*)^{(1)}$ is also a von Neumann inverse of $X$.

The remaining implication is proved similarly.

**Remark 7.** If $m > n$ then $\begin{bmatrix} T_{m \times n} & 0_{m \times (m-n)} \end{bmatrix}$ is square and an analogous characterization can be made.

**Acknowledgments**

I want to thank Professor R. Puystjens for his valuable comments and for suggesting the application to matrices over separative regular rings. I also thank the referee for suggestions to improve the readability of the paper.

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