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JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Journal of Computational and Applied Mathematics 199 (2007) 68–79

www.elsevier.com/locate/cam

Optimally space-localized band-limited wavelets on $\mathbb{S}^{q-1}$

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Received 26 November 2004; received in revised form 11 March 2005

Abstract

The localization of a function can be analyzed with respect to different criteria. In this paper, we focus on the uncertainty relation on spheres introduced by Goh and Goodman [Uncertainty principles and asymptotic behavior, Appl. Comput. Harmon. Anal. 16 (2004) 69–89], where the localization of a function is measured in terms of the product of two variances, the variance in *space* domain and the variance in *frequency* domain. After deriving an explicit formula for the variance in space domain of a function in the space $W^s_{n,q}$ of spherical polynomials of degree at most $n+s$ which are orthogonal to all spherical polynomials of degree at most n, we are able to identify—up to rotation and multiplication by a constant—the polynomial in $W^s_{n,q}$ with minimal variance in space-domain, or in other words, to determine the optimally space-localized polynomial in $W^s_{n,q}$.

MSC: 94A17

Keywords: Uncertainty principles; Spherical harmonics; Optimal localization

1. Introduction

It is well-known that uncertainty principles provide a way of measuring a trade-off between *space* and *frequency localizations* of a given function. The starting point of this work is the uncertainty principle on spheres introduced by Goh and Goodman [3]. As pointed out there, under certain assumptions on the parity and the codomain of the functions, this general uncertainty principle reduces to other known uncertainty relations studied before by other authors [1,2,9,12]. In fact, Breitenberger [1] introduced the uncertainty principle on the circle. Later, Narcowich and Ward [9] initiated the study of the uncertainty principle on higher dimensional spheres. Then the following articles by Freeden et al. [2] and Rösler and Voit [12] investigated ramifications of [9]. In particular, [12] answered a question concerning optimality posed in [9].

Throughout this paper we will focus on the space $W^s_{n,q}$ of polynomials of degree at most $n+s$ ($s \in \mathbb{N}$) on the $(q-1)$-dimensional sphere $\mathbb{S}^{q-1} \subseteq \mathbb{R}^q$, which are orthogonal to the space $V_{n,q}$ of polynomials on $\mathbb{S}^{q-1}$ of degree less or equal to n. Our aim is to derive an explicit formula for the *optimally space-localized* polynomials in $W^s_{n,q}$ according to the uncertainty relation introduced in [3], or in other words, to determine the polynomials in $W^s_{n,q}$ with minimal variance in space-domain. While the problem of characterizing the optimally space-localized spherical polynomials of

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0377-0427/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.cam.2005.03.096

degree at most n has already been treated before in Mhaskar et al. [7], the optimally space-localized polynomials in $W^s_{n,q} = V_{n+s,q} \ominus V_{n,q}$ have only been presented in their explicit form for $q = 2$ in [10] and $q = 3$ in [6].

The obtained optimally space-localized polynomials in $W^s_{n,q}$ and in $V_{n,q}$ as described in [7] are positive definite functions on the sphere, since the coefficients in their Gegenbauer expansion are all nonnegative. Along the same lines as in [9], these functions can now be used to construct a polynomial multiresolution-analysis on the sphere. Hereby, the scaling spaces $V_{n,q}$ and the wavelet spaces $W^s_{n,q}$ are spanned by *translates* of an optimally space-localized polynomial in $V_{n,q}$ or $W^s_{n,q}$, respectively, where by *translates* we mean that the localized functions, which as we will see are zonal polynomials on the sphere, are centered at the nodes of a fundamental system of the corresponding space.

The article is organized as follows. After introducing in Section 2 the necessary notation, we review in Section 3 the uncertainty principle for functions on $\mathbb{S}^{q-1}$ and determine an explicit formula for the variance in space-domain of any polynomial in $W^s_{n,q}$. This explicit formula allows us to characterize the optimally space-localized function in $W^s_{n,q}$. Finally, the cases of $q = 2$ and 3 are reviewed in the light of the new result.

2. Fundamentals

Let $\mathbb{R}^q$ be the q-dimensional Euclidean space with inner product and norm defined as usual by $x \cdot y = \sum_{k=1}^q x_k\, y_k$ and $\|x\|_2 = \sqrt{x \cdot x}$ and let $\mathbb{S}^{q-1} = \{x \in \mathbb{R}^q : \ \|x\|_2 = 1\}$ be the unit sphere in $\mathbb{R}^q$. If $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q\}$ denotes the canonical orthonormal basis of $\mathbb{R}^q$, then any point $\xi_q \in \mathbb{S}^{q-1}$ may be represented as

$$\xi_q = t\boldsymbol{e}_q + \sqrt{1-t^2}\, \xi_{q-1}, \quad t \in [-1, 1], \tag{2.1}$$

where ξ_{q-1} is a unit vector in the space spanned by $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{q-1}\}$ and $t = \boldsymbol{e}_q \cdot \xi_q$. Let $\mathrm{d}\omega_q$ denote the surface element of $\mathbb{S}^{q-1}$. For given functions $F, G : \mathbb{S}^{q-1} \longrightarrow \mathbb{C}$, we introduce the inner product and norm

$$\langle F, G\rangle = \int_{\mathbb{S}^{q-1}} F(\xi_q)\overline{G(\xi_q)}\, \mathrm{d}\omega_q(\xi_q), \quad \|F\| = \sqrt{\langle F, F\rangle}.$$

As usual, we let $L^2(\mathbb{S}^{q-1})$ denote the Hilbert space of all measurable functions $F : \mathbb{S}^{q-1} \longrightarrow \mathbb{C}$ satisfying $\|F\| < \infty$.

In view of the above parameterization of $\mathbb{S}^{q-1}$, the surface element $\mathrm{d}\omega_q$ can be written as $\mathrm{d}\omega_q = (1-t^2)^{(q-3)/2}\,\mathrm{d}t\ \mathrm{d}\omega_{q-1}$, yielding a recursive formula for the computation of the surface of the $(q-1)$-dimensional sphere, see e.g. Müller [8]:

$$\Omega_q := |\mathbb{S}^{q-1}| = \int_{\mathbb{S}^{q-1}} \mathrm{d}\omega_q(\xi_q) = \int_{\mathbb{S}^{q-2}} \int_{-1}^{1} (1-t^2)^{(q-3)/2}\,\mathrm{d}t\,\mathrm{d}\omega_{q-1}(\xi_{q-1}) = \frac{2\pi^{q/2}}{\Gamma(q/2)}.$$

An important subspace of $L^2(\mathbb{S}^{q-1})$ is the space $\mathrm{Harm}_m(\mathbb{S}^{q-1})$ of *spherical harmonics* of degree m, $m \in \mathbb{N}_0$. It can be shown that

$$N(q,m) := \dim \mathrm{Harm}_m(\mathbb{S}^{q-1}) = \begin{cases} \dfrac{(2m+q-2)\Gamma(m+q-2)}{\Gamma(m+1)\Gamma(q-1)}, & m \geqslant 1, \\ 1, & m = 0. \end{cases}$$

A seminal result in the theory of spherical harmonics is the so-called *addition theorem*, which provides the following closed expression for the reproducing kernel of $\mathrm{Harm}_m(\mathbb{S}^{q-1})$: Given an arbitrary real-valued $L^2(\mathbb{S}^{q-1})$-orthonormal basis $\{S_l(q,\cdot) : l = 1, \ldots, N(q,m)\}$ of $\mathrm{Harm}_m(\mathbb{S}^{q-1})$, we have

$$\sum_{l=1}^{N(q,m)} S_l(q,\xi)\, S_l(q,\eta) = \frac{N(q,m)}{\Omega_q} C_m^{(q-2)/2}(\xi \cdot \eta), \quad \xi, \eta \in \mathbb{S}^{q-1}, \tag{2.2}$$

where $C_m^{(q-2)/2}$ denotes the Gegenbauer polynomial of index $(q-2)/2$ and degree m normalized according to the condition $C_m^{(q-2)/2}(1) = 1$. For the proof of this fundamental result, we refer to Müller [8, Theorem 2]. Let

$$G_m^j(t) := (1-t^2)^{j/2} \frac{\mathrm{d}^j}{\mathrm{d}t^j} C_m^{(q-2)/2}(t), \quad j = 0, \ldots, m, \ \ m \in \mathbb{N}_0. \tag{2.3}$$

In particular, for $j=0$ we have $G_m^0 = C_m^{(q-2)/2}$. Starting with the three-term recurrence relation for the Gegenbauer polynomials (see e.g. [11, Section 3])

$$tC_m^{(q-2)/2}(t) = \lambda_m C_{m+1}^{(q-2)/2}(t) + (1-\lambda_m)C_{m-1}^{(q-2)/2}(t), \quad m \in \mathbb{N}_0, \tag{2.4}$$

with $\lambda_m = (m+q-2)/(2m+q-2)$, $C_{-1}^{(q-2)/2} \equiv 0$ and $C_0^{(q-2)/2} \equiv 1$, it is not difficult to derive a three-term recurrence relation for the functions G_m^j.

Proposition 1. *Let $j \in \mathbb{N}_0$. The functions G_m^j, $m \geqslant j$, $m \in \mathbb{N}_0$, defined as in* (2.3) *satisfy the three-term recurrence relation*

$$tG_m^j(t) = \alpha_m^j G_{m+1}^j(t) + \beta_m^j G_{m-1}^j(t), \quad m = j, j+1, \ldots, \tag{2.5}$$

where $G_{j-1,q}^j \equiv 0$, $G_j^j(t) = (1-t^2)^{j/2} j!\, a_j$, a_j being the leading coefficient of $C_j^{(q-2)/2}$, and the recurrence coefficients are given by

$$\alpha_m^j := \frac{(m+q-2)(m-j+1)}{(m+1)\,(2m+q-2)}, \quad \beta_m^j := \frac{m(m+j+q-3)}{(2m+q-2)\,(m+q-3)}. \tag{2.6}$$

Proof. For simplicity of notation, let $C_m := C_m^{(q-2)/2}$ and let $C_m^{(j)}$ denote the *j*th derivative of C_m. On the one hand, taking *j*th derivatives on both sides of Eq. (2.4) leads to

$$jC_m^{(j-1)}(t) + tC_m^{(j)}(t) = \lambda_m\, C_{m+1}^{(j)}(t) + (1-\lambda_m)\, C_{m-1}^{(j)}(t). \tag{2.7}$$

On the other hand, considering the following equality for the Gegenbauer polynomials C_m, (see e.g. [4, Chapter 5])

$$tC_m'(t) - \frac{m}{m+q-3}\, C_{m-1}'(t) = mC_m(t).$$

and taking $(j-1)$st derivatives on both sides of this equation, we come up with

$$tC_m^{(j)}(t) - \frac{m}{m+q-3}\, C_{m-1}^{(j)}(t) = (m-j+1)C_m^{(j-1)}(t),$$

or in other words,

$$C_m^{(j-1)}(t) = \frac{1}{m-j+1}\, tC_m^{(j)}(t) - \frac{m}{(m+q-3)\,(m-j+1)} C_{m-1}^{(j)}(t).$$

Substituting this expression for $C_m^{(j-1)}$ into relation (2.7), yields

$$\begin{aligned} tC_m^{(j)}(t) &= \frac{(m-j+1)(m+q-2)}{(m+1)(2m+q-2)}\, C_{m+1}^{(j)}(t) \\ &\quad + \frac{m-j+1}{m+1}\left(\frac{m}{2m+q-2} + \frac{mj}{(m-j+1)(m+q-3)}\right) C_{m-1}^{(j)}. \end{aligned}$$

Finally, using the factorization

$$(m+q-3)(m-j+1) + j(2m+q-2) = (m+1)(m+j+q-3)$$

and multiplying the above equation by $(1-t^2)^{j/2}$, we obtain the desired recurrence relation. □

According to Müller [8, Lemma 14], we know that

$$\int_{-1}^{1} G_m^j(t)\, G_l^j(t)\, (1-t^2)^{(q-3)/2}\, \mathrm{d}t = (\kappa_m^j)^2\, \delta_{m,l}, \tag{2.8}$$

where the normalization constant $(\kappa_m^j)^2$ is given by

$$(\kappa_m^j)^2 := \frac{\Omega_q}{\Omega_{q-1}} \frac{m!}{(m-j)!} \frac{\Gamma(m+j+q-2)}{\Gamma(m+q-2)\,N(q,m)}. \tag{2.9}$$

In particular, we denote $\kappa_m := \kappa_m^0$. The following proposition provides a relation between the recurrence coefficients α_m^j, β_m^j and the norms $(\kappa_m^j)^2$.

Proposition 2. *Let α_m^j, β_m^j and $(\kappa_m^j)^2$ be defined as above. Then*

$$\alpha_m^j\,(\kappa_{m+1}^j)^2 = \beta_{m+1}^j(\kappa_m^j)^2 \quad \textit{for all } m \in \mathbb{N}. \tag{2.10}$$

Proof. The proof follows directly from using the definitions of α_m^j, β_m^j, $(\kappa_m^j)^2$ and $N(q,m)$ and applying the property $\Gamma(n+1) = n\,\Gamma(n)$ of the Gamma function. □

Making use of Proposition 2, the three-term recurrence relation for the normalized functions $g_m^j := G_m^j/\kappa_m^j$ reads

$$t g_m^j(t) = \gamma_m^j\, g_{m+1}^j(t) + \gamma_{m-1}^j\, g_{m-1}^j(t), \quad m = j, j+1, \ldots, \tag{2.11}$$

where the recurrence coefficients γ_m^j are given by

$$\gamma_m^j = \frac{\alpha_m^j\,\kappa_{m+1}^j}{\kappa_m^j} = \left(\frac{(m-j+1)(m+j+q-2)}{(2m+q-2)(2m+q)}\right)^{1/2}, \quad m \geqslant j.$$

Let $n \in \mathbb{N}$ and let $g_m^j(\cdot; n+1)$ be the functions defined by the three-term recurrence relation

$$t g_m^j(t; n+1) = \gamma_{m+n+1}^j g_{m+1}^j(t; n+1) + \gamma_{m+n}^j g_{m-1}^j(t; n+1), \quad m = j, j+1, \ldots, \tag{2.12}$$

where we have shifted the subindex of the recurrence coefficients in (2.11) by $n+1$. By the derivation of the recurrence relation (2.5) and the construction of the functions $g_m^j(\cdot; n+1)$, it is clear that

$$(1-t^2)^{j/2} \frac{\mathrm{d}^j g_m^0(t; n+1)}{\mathrm{d}t^j} = g_m^j(t; n+1) \quad \text{for } t \in \mathbb{R}. \tag{2.13}$$

Let $V_{n,q}$ denote the space of spherical polynomials of degree less or equal to n on $\mathbb{S}^{q-1}$. As it is well known, $V_{n,q} = \bigoplus_{m=0}^n \mathrm{Harm}_m(\mathbb{S}^{q-1})$. Let $s \in \mathbb{N}$. Throughout this work we will focus on the space $W_{n,q}^s := V_{n+s,q} \ominus V_{n,q}$ of spherical polynomials of degree at most $n+s$ which are orthogonal to $V_{n,q}$. We will call this space *wavelet space* and denote its dimension with $M_s := \sum_{m=n+1}^{n+s} N(q,m)$. Using the parameterization (2.1) of $\mathbb{S}^{q-1}$ in local coordinates, an orthogonal basis for $W_{n,q}^s$ is given by the spherical harmonics

$$S_{m,j,k}(q, \xi_q) = G_m^j(t) S_{j,k}(q-1, \xi_{q-1}),$$

where $m = n+1, \ldots, n+s$, $j = 0, \ldots, m$ and $k = 1, \ldots, N(q-1, j)$ and

$$\{S_{j,k}(q-1, \xi_{q-1}) :\; j = 0, \ldots, m,\; k = 1, \ldots, N(q-1, j)\}$$

is an orthonormal basis of $V_{m,q-1} = \bigoplus_{j=0}^m \mathrm{Harm}_j(\mathbb{S}^{q-2})$.

The goal of this paper is to compute the optimally *space-localized* polynomial in $W_{n,q}^s$. For this purpose, the localization is measured in terms of the generalized uncertainty relation on $\mathbb{S}^{q-1}$ studied in [3]. We consider the expansion of $P \in W_{n,q}^s$ in the basis of spherical harmonics $S_{m,j,k}$ and derive an explicit formula for the variance in space-domain.

3. An uncertainty principle on the $\mathbb{S}^{q-1}$

As it is shown in [3, Corollary 5.1], the variances in space and momentum domain of a function $F \in \mathscr{C}^1(\mathbb{S}^{q-1})$ with $\int_{\mathbb{S}^{q-1}} \xi_q |F(\xi_q)|^2 \,\mathrm{d}\omega_q(\xi_q) \neq \mathbf{0}$ can be computed as

$$\mathrm{var}_S(F) = \left(\left(\frac{\int_{\mathbb{S}^{q-1}} |F(\xi_q)|^2 \,\mathrm{d}\omega_q(\xi_q)}{\| \int_{\mathbb{S}^{q-1}} \xi_q |F(\xi_q)|^2 \,\mathrm{d}\omega_q(\xi_q) \|_2} \right)^2 - 1 \right)^{1/2},$$

and

$$\mathrm{var}_M(F) = \left(\frac{\int_{\mathbb{S}^{q-1}} |\nabla_{\mathbb{S}^{q-1}} F(\xi_q)|^2 \,\mathrm{d}\omega_q(\xi_q)}{\int_{\mathbb{S}^{q-1}} |F(\xi_q)|^2 \,\mathrm{d}\omega_q(\xi_q)} \right)^{1/2},$$

where $\nabla_{\mathbb{S}^{q-1}}$ is the usual surface gradient on $\mathbb{S}^{q-1}$. With this notation, the uncertainty relation presented in [3] reads

$$U(F) := \mathrm{var}_S(F)\,\mathrm{var}_M(F) \geqslant \frac{(q-1)}{2}.$$

In the following lemma, we derive an explicit formula for the variance in space domain of any polynomial $P \in W^s_{n,q}$ with $\int_{\mathbb{S}^{q-1}} \xi_q |P(\xi_q)|^2 \,\mathrm{d}\omega_q \neq \mathbf{0}$.

Lemma 1. *Let $n, s \in \mathbb{N}$ and let $\{S_{m,j,k}(q,\xi_q) = G^j_m(t) S_{j,k}(q-1,\xi_{q-1})\}_{m,j,k}$ be the above-mentioned orthogonal basis of $W^s_{n,q}$. Given $P \in W^s_{n,q}$, let*

$$P(\xi_q) = \sum_{m=n+1}^{n+s} \sum_{j=0}^{m} \sum_{k=1}^{N(q-1,j)} a_{m,j,k} G^j_m(t) S_{j,k}(\xi_{q-1})$$

be its expansion in the basis $\{S_{m,j,k}\}$. If

$$2 \sum_{m=n+1}^{n+s-1} \sum_{j=0}^{m} \sum_{k=1}^{N(q-1,j)} \alpha^j_m (\kappa^j_{m+1})^2 Re\{a_{m,j,k}\overline{a_{m+1,j,k}}\} \neq 0,$$

then the variance in space-domain of P can be computed as

$$\mathrm{var}_S(P) = \left(\left(\frac{\sum_{m=n+1}^{n+s} \sum_{j=0}^{m} \sum_{k=1}^{N(q-1,j)} |a_{m,j,k}|^2 \,(\kappa^j_m)^2}{2\sum_{m=n+1}^{n+s-1} \sum_{j=0}^{m} \sum_{k=1}^{N(q-1,j)} \alpha^j_m (\kappa^j_{m+1})^2 Re\{a_{m,j,k}\overline{a_{m+1,j,k}}\}} \right)^2 - 1 \right)^{1/2}. \tag{3.1}$$

The constants α^j_m and κ^j_m are defined as in (2.6) *and* (2.9), *respectively.*

Proof. Making use of the orthonormality of $\{S_{j,k}\}$ and the orthogonality of $\{G^j_m\}$ according to (2.8), it is straightforward to check that

$$\|P\|^2 = \int_{\mathbb{S}^{q-1}} |P(\xi_q)|^2 \,\mathrm{d}\omega_q(\xi_q) = \int_{\mathbb{S}^{q-2}} \int_{-1}^{1} |P(\xi_{q-1}, t)|^2 (1-t^2)^{(q-3)/2} \,\mathrm{d}t \,\mathrm{d}\omega_{q-1}(\xi_{q-1})$$

$$= \sum_{m=n+1}^{n+s} \sum_{j=0}^{m} \sum_{k=1}^{N(q-1,j)} |a_{m,j,k}|^2 \,(\kappa_{m,j})^2.$$

Let us now compute $\xi_S(P) := \int_{\mathbb{S}^{q-1}} \xi_q |P(\xi_q)|^2 \,\mathrm{d}\omega_q(\xi_q)$. We can assume without loss of generality that the first $(q-1)$ components of $\xi_S(P)$ are equal to zero. Indeed, if $\xi_S(P) \notin \mathrm{span}\{\boldsymbol{e}_q\}$, then we can find an orthogonal transformation

$Q \in SO(q)$ such that $Q \cdot \xi_S(P) = \|\xi_S(P)\|_2 \boldsymbol{e}_q$. Accordingly, we are in a position to construct a polynomial $\widehat{P}(\cdot) := P(Q^* \cdot (\cdot)) \in W^s_{n,q}$ with the desired center in $\boldsymbol{e}_q$-direction, namely

$$\xi_S(\widehat{P}) = \int_{\mathbb{S}^{q-1}} \eta_q \, |P(Q^* \cdot \eta_q)|^2 \, d\omega_q(\eta_q) \overset{\eta_q = Q\cdot\mu_q}{=} \int_{Q^*(\mathbb{S}^{q-1})} Q \cdot \mu_q |P(\mu_q)|^2 |\det Q| \, d\omega_q(\mu_q)$$

$$= Q \cdot \int_{\mathbb{S}^{q-1}} \mu_q |P(\mu_q)|^2 \, d\omega_q(\mu_q) = Q \cdot \xi_S(P) = \|\xi_S(P)\|_2 \boldsymbol{e}_q.$$

Thus, let us assume that $\int_{\mathbb{S}^{q-1}} \xi_q |P(\xi_q)|^2 \, d\omega_q(\xi_q) = \| \int_{\mathbb{S}^{q-1}} \xi_q |P(\xi_q)|^2 \, d\omega_q(\xi_q) \|_2 \boldsymbol{e}_q$. The qth component of $\xi_S(P)$ is then given by

$$\int_{\mathbb{S}^{q-2}} \int_{-1}^{1} t \left| \sum_{m=n+1}^{n+s} \sum_{j=0}^{m} \sum_{k=1}^{N(q-1,j)} a_{m,j,k} G_m^j(t) \, S_{j,k}(\xi_{q-1}) \right|^2 (1-t^2)^{(q-3)/2} \, dt \, d\omega_{q-1}(\xi_{q-1})$$

$$= \sum_{m,m'=n+1}^{n+s} \sum_{j=0}^{\min(m,m')} \sum_{k=1}^{N(q-1,j)} a_{m,j,k} \overline{a_{m',j,k}} \int_{-1}^{1} t G_m^j(t) G_{m'}^j(t) (1-t^2)^{(q-3)/2} \, dt.$$

Applying the three-term recurrence relation (2.5), we have that

$$\int_{-1}^{1} t G_m^j(t) \, G_{m'}^j(t) (1-t^2)^{(q-3)/2} \, dt = \alpha_m^j (\kappa_{m+1}^j)^2 \delta_{m+1,m'} + \beta_m^j (\kappa_{m-1}^j)^2 \delta_{m-1,m'}.$$

Substituting this equation into the above expression, performing an index shift in m' and making use of relation (2.10), we finally obtain the desired expression for $\| \int_{\mathbb{S}^{q-1}} \xi_q |P(\xi_q)|^2 \, d\omega_q(\xi_q) \|^2$, namely

$$\sum_{m=n+1}^{n+s-1} \sum_{j=0}^{m} \sum_{k=1}^{N(q-1,j)} \alpha_m^j (\kappa_{m+1}^j)^2 a_{m,j,k} \overline{a_{m+1,j,k}} + \beta_{m+1}^j (\kappa_m^j)^2 a_{m+1,j,k} \overline{a_{m,j,k}}$$

$$= 2 \sum_{m=n+1}^{n+s-1} \sum_{j=0}^{m} \sum_{k=1}^{N(q-1,j)} \alpha_m^j (\kappa_{m+1}^j)^2 \operatorname{Re}\{a_{m,j,k} \overline{a_{m+1,j,k}}\}. \qquad \square$$

3.1. Optimally space-localized wavelet functions

With the help of Eq. (3.1), we are now in a position to compute the optimally space-localized polynomials in $W^s_{n,q}$.

Theorem 1. *Let $n, s \in \mathbb{N}$ and let $x^s_{\max}$ denote the largest zero of the associated Gegenbauer polynomial $g_s(\cdot; n+1)$ defined as in* (2.12) *for $j = 0$. Then*

$$\operatorname{var}_S(P^*) = \min\{\operatorname{var}_S(P) \colon P \neq 0, \;\; P \in W^s_{n,q}\} = \frac{\sqrt{1 - (x^s_{\max})^2}}{x^s_{\max}},$$

and an optimally space-localized spherical polynomial—up to rotation and multiplication by a constant—is given by

$$P^*(\cdot) = \frac{1}{\Omega_q} \sum_{m=n+1}^{n+s} \sqrt{N(q,m)} \, g_{m-(n+1)}(x^s_{\max}; n+1) C_m^{(q-2)/2}(\boldsymbol{e}_q \cdot (\cdot)). \tag{3.2}$$

Proof. According to the explicit expression for the variance in space-domain presented in Eq. (3.1), minimizing $\mathrm{var}_S(P)$ over all $P \in W_{n,q}^s$ is equivalent to maximizing the reciprocal quotient

$$\frac{2\sum_{m=n+1}^{n+s-1}\sum_{j=0}^{m}\sum_{k=1}^{N(q-1,j)}\alpha_m^j(\kappa_{m+1}^j)^2\,\mathrm{Re}\,\{a_{m,j,k}\,\overline{a_{m+1,j,k}}\}}{\sum_{m=n+1}^{n+s}\sum_{j=0}^{m}\sum_{k=1}^{N(q-1,j)}|a_{m,j,k}|^2(\kappa_m^j)^2}$$

over all coefficient vectors $\boldsymbol{a}=(a_{m,j,k})_{m=n+1,\dots,n+s;\,j=0,\dots,m;\,k=1,\dots,N(q-1,j)}$ in $\mathbb{C}^{M_s}\backslash\mathbf{0}$.

Let $x_{m,j,k} := a_{m,j,k}\kappa_m^j$. Using this notation and changing the summation order of the sums both in denominator and numerator, the above quotient can be written as

$$\frac{2\sum_{j=0}^{n+s-1}\sum_{k=1}^{N(q-1,j)}\sum_{m=\max(j,n+1)}^{n+s-1}\gamma_m^j\,\mathrm{Re}\,\{x_{m,j,k}\,\overline{x_{m+1,j,k}}\}}{\sum_{j=0}^{n+s}\sum_{k=1}^{N(q-1,j)}\sum_{m=\max(j,n+1)}^{n+s}|x_{m,j,k}|^2}$$

with $\gamma_m^j=\alpha_m^j\kappa_{m+1}^j/\kappa_m^j$. This quotient of sums can now be interpreted as a Rayleigh quotient $\boldsymbol{x}^*\mathbf{M}_{n,s}\boldsymbol{x}/(\boldsymbol{x}^*\boldsymbol{x})$, where $\boldsymbol{x}$ is a vector with its components ordered in the way given by the summation indices j, k, m, i.e.

$$\begin{aligned}
&(x_{n+1,0,1},\dots,x_{n+s,0,1},x_{n+1,1,1},\dots,x_{n+s,1,1},\dots,x_{n+1,1,N(q-1,1)},\dots,x_{n+s,1,N(q-1,1)},\\
&\quad x_{n+1,2,1},\dots,x_{n+s,2,1},\dots,x_{n+1,2,N(q-1,2)},\dots,x_{n+s,2,N(q-1,2)},\dots,\\
&\quad x_{n+1,n+1,1},\dots x_{n+s,n+1,1},\dots,x_{n+1,n+1,N(q-1,n+1)},\dots,x_{n+s,n+1,N(q-1,n+1)},\\
&\quad x_{n+2,n+2,1},\dots,x_{n+s,n+2,1},\dots,x_{n+2,n+2,N(q-1,n+2)},\dots,x_{n+s,n+2,N(q-1,n+2)},\\
&\quad \dots,x_{n+s,n+s,1},\dots,x_{n+s,n+s,N(q-1,n+s)}),
\end{aligned}$$

and $\mathbf{M}_{n,s}$ is the block diagonal matrix

$$\begin{aligned}
\mathbf{M}_{n,s}=\mathrm{diag}\Bigg(&\mathbf{J}_{n+1,s}^0,\underbrace{\mathbf{J}_{n+1,s}^1,\dots,\mathbf{J}_{n+1,s}^1}_{N(q-1,1)\text{ times}},\underbrace{\mathbf{J}_{n+1,s}^2,\dots,\mathbf{J}_{n+1,s}^2}_{N(q-1,2)\text{ times}},\dots,\underbrace{\mathbf{J}_{n+1,s}^{n+1},\dots,\mathbf{J}_{n+1,s}^{n+1}}_{N(q-1,n+1)\text{ times}}\\
&\times\underbrace{\mathbf{J}_{n+2,s}^{n+2},\dots,\mathbf{J}_{n+2,s}^{n+2}}_{N(q-1,n+2)\text{ times}},\underbrace{\mathbf{J}_{n+3,s}^{n+3},\dots,\mathbf{J}_{n+3,s}^{n+3}}_{N(q-1,n+3)\text{ times}},\dots,\underbrace{\mathbf{J}_{n+s,s}^{n+s},\dots,\mathbf{J}_{n+s,s}^{n+s}}_{N(q-1,n+s)\text{ times}}\Bigg)
\end{aligned}$$

with $\mathbf{J}_{n+1,s}^j\in\mathbb{R}^{s\times s}$, $j=0,\dots,n+1$, given by

$$\mathbf{J}_{n+1,s}^j=\begin{pmatrix}
0 & \gamma_{n+1}^j & 0 & & & \\
\gamma_{n+1}^j & 0 & \gamma_{n+2}^j & 0 & & \\
 & \gamma_{n+2}^j & 0 & \gamma_{n+3}^j & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & \gamma_{n+s-2}^j & 0 & \gamma_{n+s-1}^j \\
 & & & & \gamma_{n+s-1}^j & 0
\end{pmatrix}, \tag{3.3}$$

and $\mathbf{J}^j_{j,s} \in \mathbb{R}^{(n+s-j+1)\times(n+s-j+1)}$, $j = n+2, \ldots, n+s$ given by

$$\mathbf{J}^j_{j,s} = \begin{pmatrix} 0 & \gamma^j_j & 0 & & & \\ \gamma^j_j & 0 & \gamma^j_{j+1} & 0 & & \\ & \gamma^j_{j+1} & 0 & \gamma^j_{j+2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \gamma^j_{n+s-2} & 0 & \gamma^j_{n+s-1} \\ & & & & \gamma^j_{n+s-1} & 0 \end{pmatrix}. \tag{3.4}$$

The matrices (3.4) arise from deleting the first $j-(n+1)$ columns and rows of the matrices (3.3). Note that for $j=n+s$ we have $\mathbf{J}^{n+s}_{n+s-1,s} = 0 \in \mathbb{R}^{1\times 1}$.

Let $\lambda_{\max}(\mathbf{M}_{n,s})$, $\lambda_{\max}(\mathbf{J}^j_{n+1,s})$, $j=0,\ldots,n+1$, and $\lambda_{\max}(\mathbf{J}^j_{j,s})$, $j=n+2,\ldots,n+s$, denote the largest eigenvalues of the symmetric matrices $\mathbf{M}_{n,s}$, $\mathbf{J}^j_{n+1,s}$ and $\mathbf{J}^j_{j,s}$, respectively. Thus, by the Theorem of Rayleigh-Ritz (see Horn and Johnson [5, p. 176]), we know that

$$\max_{\boldsymbol{x}\neq \mathbf{0}} \frac{\boldsymbol{x}^*\mathbf{M}_{n,s}\boldsymbol{x}}{\boldsymbol{x}^*\boldsymbol{x}} \leqslant \lambda_{\max}(\mathbf{M}_{n,s}) = \max\left\{ \max_{j=0,\ldots,n+1}\{\lambda_{\max}(\mathbf{J}^j_{n+1,s})\}, \max_{j=n+2,\ldots,n+s}\{\lambda_{\max}(\mathbf{J}^j_{j,s})\}\right\}$$

with equality if and only if $\boldsymbol{x}$ is an eigenvector of $\mathbf{M}_{n,s}$ with eigenvalue $\lambda_{\max}(\mathbf{M}_{n,s})$. We have dropped the absolute value in the first equality since, as we will see later, the eigenvalues of $\mathbf{J}^j_{n+1,s}$ and $\mathbf{J}^j_{j,s}$ are distributed symmetrically around the origin. Hence, our problem reduces to finding the largest eigenvalue of $\mathbf{M}_{n,s}$. Note now that the tridiagonal matrices $\mathbf{J}^j_{n+1,s}$ and $\mathbf{J}^j_{j,s}$ are the so-called *Jacobi matrices* corresponding to the functions $g^j_m(\cdot;n+1)$, $j=0,\ldots,n+1$ and $g^j_m(\cdot;j)$, $j=n+2,\ldots,n+s$. Rewriting the three-term recurrence relation (2.12) in compact matrix notation as

$$t\boldsymbol{g}^j_m(t;n+1) = \mathbf{J}^j_{n+1,s}\boldsymbol{g}^j_m(t;n+1) + \gamma^j_{n+s} g^j_s(t;n+1)\boldsymbol{e}_s, \quad j=0,\ldots,n+1,$$

and

$$t\boldsymbol{q}^j_m(t;j) = \mathbf{J}^j_{j,s}\boldsymbol{q}^j_m(t;j) + \gamma^j_{n+s} g^j_{n+s-j+1}(t;j)\boldsymbol{e}_{n+s-j+1}, \quad j=n+2,\ldots,n+s,$$

with

$$\boldsymbol{g}^j_m(t;n+1) = (g^j_0(t;n+1), g^j_1(t;n+1), \ldots, g^j_{s-1}(t;n+1))^{\mathrm{T}} \in \mathbb{R}^s,$$

and

$$\boldsymbol{q}^j_m(t;j) = (g^j_0(t;j), g^j_1(t;j), \ldots, g^j_{n+s-j}(t;j))^{\mathrm{T}} \in \mathbb{R}^{(n+s-j+1)},$$

we draw the conclusion that the zeros of the functions $g^j_s(\cdot;n+1)$, $j=0,\ldots,n+1$, and of the functions $g^j_{n+s-j+1}(\cdot;j)$, $j=n+2,\ldots,n+s$, or according to (2.13) the zeros of the jth derivatives of $g_s(\cdot;n+1)$, $j=0,\ldots,n+1$, and of the jth derivatives of $g_{n+s-j+1}(\cdot;j)$, $j=n+2,\ldots,n+s$, with exception of the zero at $t=\pm 1$, constitute the spectrum of the corresponding matrices $\mathbf{J}^j_{n+1,s}$ and $\mathbf{J}^j_{j,s}$. Since these functions are orthogonal with respect to a symmetric measure, we know that the eigenvalues of the matrices $\mathbf{J}^j_{n+1,s}$ and $\mathbf{J}^j_{j,s}$ are symmetrically distributed around the origin. On the one hand, by virtue of Rolle's Theorem, the zeros of the derivatives of the associated Gegenbauer polynomial $g_s(\cdot;n+1)$ are located in between the zeros of $g_s(\cdot;n+1)$ and consequently

$$\max_{j=0,\ldots,n+1}\{\lambda_{\max}(\mathbf{J}^j_{n+1,s})\} = \lambda_{\max}(\mathbf{J}^0_{n+1,s}) = x^s_{\max},$$

where $x^s_{\max}$ denotes the largest zeros of $g_s(\cdot;n+1)$. On the other hand, taking into account that the zeros of $g^j_{n+s-j+1}(\cdot;j)$, $j=n+2,\ldots,n+s$, lie in between the zeros of $g_{n+s-j+1}(\cdot;j)$, $j=n+2,\ldots,n+s$, we can conclude that

$$\lambda_{\max}(\mathbf{J}^0_{j,s}) \geqslant \lambda_{\max}(\mathbf{J}^j_{j,s}), \quad j=n+2,\ldots,n+s. \tag{3.5}$$

Since $\mathbf{J}^0_{j,s}$, $j = n+2, \ldots, n+s$, is the submatrix of $\mathbf{J}^0_{n+1,s}$, obtained by deleting the first $j-(n+1)$ columns and rows of $\mathbf{J}^0_{n+1,s}$, we can claim that the largest eigenvalue of $\mathbf{J}^0_{n+1,s}$ is greater than the largest eigenvalue of $\mathbf{J}^0_{j,s}$, $j = n+2, \ldots, n+s$, i.e.

$$\lambda_{\max}(J^0_{n+1,s}) \geqslant \lambda_{\max}(\mathbf{J}^0_{j,s}), \quad j = n+2, \ldots, n+s. \tag{3.6}$$

Combining inequalities (3.5) and (3.6), the largest eigenvalue is finally given by

$$\max\left\{\max_{j=0,\ldots,n+1}\{\lambda_{\max}(\mathbf{J}^j_{n+1,s})\}, \max_{j=n+2,\ldots,n+s}\{\lambda_{\max}(\mathbf{J}^j_{j,s})\}\right\} = \lambda_{\max}(\mathbf{J}^0_{n+1,s}) = x^s_{\max},$$

and its corresponding eigenvector is

$$\alpha\left(g_0(x^s_{\max}; n+1), g_1(x^s_{\max}; n+1), \ldots, g_{s-1}(x^s_{\max}; n+1), \underbrace{0, \ldots, 0}_{M_s - s \text{ times}}\right)^{\mathrm{T}}, \quad \alpha \in \mathbb{R}.$$

Let us assume $\alpha = 1/\sqrt{\Omega_q}$. With this choice, we obtain that

$$a_{m,j,k} = \begin{cases} \dfrac{g_{m-(n+1)}(x^s_{\max}; n+1)}{\sqrt{\Omega_q}\,\kappa^j_m} & \text{for } j=0,\ k=1,\ m=n+1,\ldots,n+s, \\ 0 & \text{otherwise.} \end{cases}$$

Making then use of (3.1), the minimal variance in space-domain attains the value

$$\mathrm{var}_S(P^*) = \left(\left(\frac{1}{x^s_{\max}}\right)^2 - 1\right)^{1/2} = \frac{\sqrt{1-(x^s_{\max})^2}}{x^s_{\max}}.$$

It now remains to prove that the choice of the just computed coefficients yields in fact the polynomial in (3.2). Here, the observation

$$\frac{S_{m,j,k}(q, \boldsymbol{e}_q)}{\kappa^j_m} = \frac{G^j_m(1) S_{j,k}(q-1, \mathbf{0})}{\kappa^j_m} = \begin{cases} \dfrac{1}{\sqrt{\Omega_{q-1}}\kappa_m}, & j = 0, \\ 0 & \text{otherwise,} \end{cases}$$

plays the key role, since it enables us to write the optimally space-localized polynomial in the requested form (note that $S_{0,1} \equiv 1/\sqrt{\Omega_{q-1}}$). Indeed, after an application of the addition theorem (2.2), the optimal polynomial P^* is given by

$$\begin{aligned} P^* &= \frac{1}{\sqrt{\Omega_q}} \sum_{m=n+1}^{n+s} g_{m-(n+1)}(x^s_{\max}; n+1)\sqrt{\Omega_{q-1}}\kappa_m \sum_{j=0}^{m}\sum_{k=1}^{N(q,j)} \frac{S_{m,j,k}}{\kappa^j_m}\frac{S_{m,j,k}(\boldsymbol{e}_q)}{\kappa^j_m} \\ &= \frac{\sqrt{\Omega_{q-1}}}{\sqrt{\Omega_q^3}} \sum_{m=n+1}^{n+s} g_{m-(n+1)}(x^s_{\max}; n+1)\kappa_m N(q,m) C_m^{(q-2)/2}(\boldsymbol{e}_q\cdot(\cdot)) \\ &= \frac{1}{\Omega_q} \sum_{m=n+1}^{n+s} g_{m-(n+1)}(x^s_{\max}; n+1)\sqrt{N(q,m)}\, C_m^{(q-2)/2}(\boldsymbol{e}_q\cdot(\cdot)), \end{aligned}$$

where in the last equality, we have used that $\kappa_m = (\Omega_q/\Omega_{q-1} N(q,m))^{1/2}$. This completes the proof. □

3.2. Remarks

For $q = 2$, the Gegenbauer polynomials $\{C^0_k\}_{k\in\mathbb{N}_0}$ are the Tschebyscheff polynomials of the first kind

$$T_k(x) = \cos(k \arccos x), \quad x \in [-1, 1].$$

Table 1
Uncertainty product of the polynomials $\psi^n_{n,\mathbf{b}^*} \in W^n_{n,3}$ with $n = 2^j$ for $j = 2, \ldots, 9$

n	$\mathrm{var}_S(\psi^n_{n,\mathbf{b}^*})$	$\mathrm{var}_M(\psi^n_{n,\mathbf{b}^*})$	$U(\psi^n_{n,\mathbf{b}^*})$
4	0.7210	7.0380	5.0742
8	0.3616	13.0886	4.7323
16	0.1858	25.1771	4.6771
32	0.0949	49.3476	4.6836
64	0.0481	97.6850	4.6968
128	0.0242	194.3581	4.7061
256	0.0122	387.7033	4.7115
512	0.0061	774.3934	4.7144

Using the three-term recurrence relation of the Tschebyscheff polynomials of the second kind $\{U_k\}_{k\in\mathbb{N}_0}$

$$xU_k(x) = \tfrac{1}{2}U_{k+1}(x) + \tfrac{1}{2}U_{k-1}(x), \quad k = 0, 1, \ldots,$$

with $U_{-1}(x) = 0$, $U_0(x) = 1$ and $U_1(x) = 2x$, we realize that the associated Tschebyscheff polynomials of the first kind $T_k(\cdot; n+1)$ are the classical Tschebyscheff polynomials of the second kind $U_k(\cos\theta) = \sin(k+1)\theta/\sin\theta$. The largest zero of $U_s(\cos\theta)$ is $\cos(\pi/(s+1))$ and hence by Theorem 1, the optimally space-localized polynomial in $W^s_{n,2}$ has the form

$$\frac{1}{2\pi}\sum_{k=n+1}^{n+s} U_{k-(n+1)}(x^s_{\max})\sqrt{\frac{2}{\pi}}\,T_k(\cos\theta) = \frac{1}{\sqrt{2\pi^3}\sin\pi/(s+1)}\sum_{k=n+1}^{n+s}\sin\frac{(k-n)\pi}{s+1}\cos k\theta.$$

Considering the index shifts $n+1 \to m$ and $n+s \to n$, the resulting polynomials coincide, up to a normalization constant, with the optimally-space localized even polynomial wavelets studied in [10].

For $q = 3$, the Gegenbauer polynomials $\{C^{1/2}_k\}_{k\in\mathbb{N}_0}$ are the Legendre polynomials $\{P_k\}_{k\in\mathbb{N}_0}$. As it is pointed out in [6, Theorem 5.8], the uncertainty product of a polynomial of the form

$$\psi^s_{n,\mathbf{b}} = \frac{1}{4\pi}\sum_{m=n+1}^{n+s}(2m+1)b_m P_m(\boldsymbol{e}_3\cdot(\cdot)), \tag{3.7}$$

with $\mathbf{b} \in \mathbb{C}^s\backslash\mathbf{0}$ such that $\sum_{m=n+1}^{n+s} m b_m b_{m-1} \neq 0$ can be computed as

$$1 \leqslant \left(\left(\frac{\sum_{m=n+1}^{n+s} b_m^2(2m+1)}{2\sum_{m=n+1}^{n+s} m b_m b_{m-1}}\right)^2 - 1\right)^{1/2}\left(\frac{\sum_{m=n+1}^{n+s} b_m^2\, m\,(m+1)\,(2m+1)}{\sum_{m=n+1}^{n+s} b_m^2\,(2m+1)}\right)^{1/2}.$$

In particular, note that by taking $b^*_m := p_{m-(n+1)}(x^s_{\max}; n+1)/\sqrt{(2m+1)}$, we arrive at the optimally-space localized polynomial in $W^s_{n,3}$. This localized function coincides with the one obtained in [6, Chapter 5], where the aim was to compute the polynomial in $W^s_{n,3}$ with minimal variance in space-domain according to the uncertainty principle on $\mathbb{S}^2$ introduced in [9].

On the other hand, for $\mathbf{b} = \mathbf{1} := (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^s$, we obtain the reproducing kernel of $W^s_{n,3}$, for which it is known that is optimally $L^2(\mathbb{S}^2)$-localized in the sense that

$$\left\|\frac{\psi^s_{n,\mathbf{1}}}{\psi^s_{n,\mathbf{1}}(\xi)}\right\| = \min\{\|Q\| : Q \in W^s_{n,q} \text{ and } Q(\xi) = 1\}.$$

Let us compute and compare the uncertainty products of the optimally space-localized and optimally $L^2(\mathbb{S}^2)$-localized polynomials in $W^s_{n,3}$. Note that for $\mathbf{b}=\mathbf{1} := (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^s$, the uncertainty product of the optimally $L^2(\mathbb{S}^2)$-localized

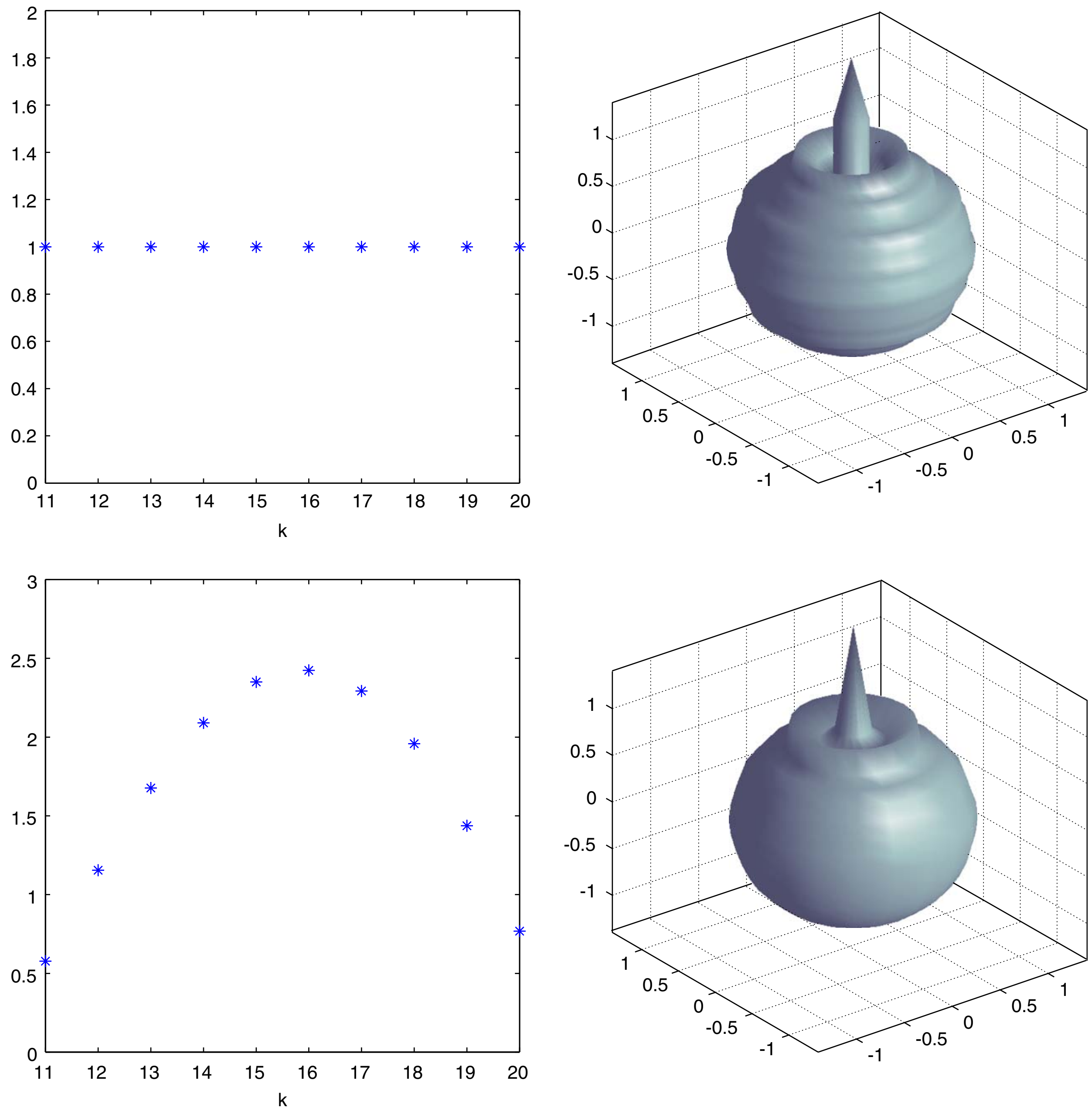

Fig. 1. On the right hand side we display plots of the wavelet function $\psi^{10}_{10,\mathbf{1}}$ and the optimal space localized wavelet function $\psi^{10}_{10,\mathbf{b}^*}$. On the left hand side we plot the values of the corresponding coefficient vectors $\mathbf{b}=\mathbf{1}$ and $\mathbf{b}^*$. Note that the kernel $\psi^{10}_{10,\mathbf{b}^*}$ exhibits a smoother behavior around the peak $\boldsymbol{e}_3$ than the function $\psi^{10}_{10,\mathbf{1}}$.

polynomials in $W^s_{n,3}$ attains the value

$$U(\psi^s_{n,\mathbf{1}}) = \mathrm{var}_S(\psi^s_{n,\mathbf{1}}) \cdot \mathrm{var}_F(\psi^s_{n\mathbf{1}}) = \frac{\sqrt{(2s-1)\,(n^2+\frac{1}{2}\,(1+s)^2+n\,(2+s))}}{(s-1)}.$$

In particular, for $s=2$ and n we obtain

$$U(\psi^2_{n,\mathbf{1}}) = \mathcal{O}(n) \quad \text{and} \quad U(\psi^n_{n,\mathbf{1}}) = \mathcal{O}(n^{1/2}).$$

Table 1 displays the values of the uncertainty product of the optimally space-localized wavelet in $W^n_{2^j,3}$ $(j=2,\ldots,9)$. Since we do not know an explicit formula for the zeros of the associated Legendre polynomial $p_s(\cdot; n+1)$ $(s \in \mathbb{N})$ or equivalently for the eigenvalues of the matrix $\mathbf{J}^0_{n+1,s}$ in (3.3) with $\gamma^0_m = m/\sqrt{4m^2-1}$, all the table values had to be computed numerically. As the variance in space domain decreases with n, the variance in momentum domain grows. However, the decay of $\mathrm{var}_S(\psi^n_{n,\mathbf{b}^*})$ seems to be stronger than the growth of $\mathrm{var}_M(\psi^n_{n,\mathbf{b}^*})$, so that in the end, the product of these two quantities, i.e. the uncertainty product, only shows a slight increase with n.

In Fig. 1, we illustrate the behavior of the coefficients b^*_m $(m=n+1,\ldots,2n)$ which lead to the optimally space-localized polynomial $\psi^{10}_{10,\mathbf{b}^*}$. In addition, we display the wavelet function arising from the mentioned choice of the coefficient vector $\mathbf{b}^*$. A direct comparison of the plots of the two wavelet functions $\psi^{10}_{10,\mathbf{1}}$ and $\psi^{10}_{10,\mathbf{b}^*}$ shows how the space localization of the polynomials (3.7) is improved by selecting the coefficient vector $\mathbf{b}^*$.

Acknowledgment

The author wants to thank the referees for their valuable remarks.

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