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New Extrapolation Estimates

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Given a sublinear operator T satisfying that

for every measurable set A and every $1 < p \leqslant p_0$, with C independent of A and p, we show that

$$\sup_{r>0} \frac{\int_{1/r}^{\infty} \lambda^{\nu}_{Tf}(y)\, dy}{1+\log^+ r} \lesssim \int_{\mathcal{M}} |f(x)|\,(1+\log^+ |f(x)|)\, d\mu(x).$$

This estimate allows us to improve Yano's extrapolation theorem and also to obtain that for every $f \in L \log L(\mu)$,

$$\overline{\lim_{r\to\infty}} \frac{\int_{1/r}^{\infty} \lambda^{\nu}_{Tf}(y)\, dy}{\log r} \lesssim \|f\|_1.$$

Other types of extrapolation results are also given. © 2000 Academic Press

1. INTRODUCTION

In 1951, Yano (see [Y, Z]), using the ideas of Titchmarsh [T], proved that for every sublinear operator satisfying

$$\left(\int_{\mathcal{N}} |Tf(x)|^p \, d\nu(x)\right)^{1/p} \leqslant \frac{C}{p-1}\left(\int_{\mathcal{M}} |f(x)|^p \, d\mu(x)\right)^{1/p},$$

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where $\mathcal{N}$ and $\mathcal{M}$ are two finite measure spaces, T: $L\log L(\mu) \to L^1(\nu)$ is bounded.

The purpose of this work is to show, using a different argument, that under a weaker condition on the operator T, namely

$$\left(\int_{\mathcal{N}} |T\chi_A(x)|^p \, d\nu(x)\right)^{1/p} \leqslant \frac{C}{p-1} \mu(A)^{1/p}$$

for every measurable set $A \subset \mathcal{M}$ and every $1 < p \leqslant p_0$, with C independent of A and p,

$$\sup_{r>0} \frac{\int_{1/r}^{\infty} \lambda_{Tf}^{\nu}(y)\, dy}{1+\log^+ r} \leqslant K \int_{\mathcal{M}} |f(x)| \, (1+\log^+ |f(x)|) \, d\mu(x),$$

where λ_{Tf}^{ν} is the distribution function of Tf with respect to ν, and μ and ν are two σ-finite measures. This estimate allows us to improve Yano's theorem. Also, under the above condition on T, we obtain that, for every $f \in L\log L(\mu)$,

$$\overline{\lim_{r\to\infty}} \frac{\int_{1/r}^{\infty} \lambda_{Tf}^{\nu}(y)\, dy}{\log r} \leqslant C \, \|f\|_1.$$

In the setting of weak extrapolation results we have to mention the work of Sjölin [Sj], who was able to obtain endpoint estimates for a sublinear operator T satisfying the restricted weak-type estimate

$$\sup_{y\geqslant 0} y \lambda_{T\chi_A}^{\nu}(y)^{1/p} \leqslant \frac{C}{p-1} \mu(A)^{1/p}, \tag{1}$$

with μ and ν finite measures. Some years later, Soria [So] improved the above extrapolation result by showing that if T satisfies that $\sup_{y>0} y\lambda_{T\chi_A}^{\nu}(y) \leqslant C/(p-1)\, \mu(A)^{1/p}$, for every measurable set A, every $1 < p \leqslant p_0$, and ν an arbitrary σ-finite measure, then T applies the space B_{φ}^* boundedly into $L^{1,\infty}$ with $\varphi(t) = t(1+\log^+ 1/t)$ (see Section 4 for the definition of B_{φ}^*).

In this paper, we shall prove an extrapolation estimate for an operator T satisfying (1) in a general measure space.

Also, in the 1990s, the extrapolation theory was extended to the setting of compatible couples of Banach spaces in the work of Jawerth and Milman (see [JM1, JM2]). See also the work of Sobukawa [S].

Constants such as C will denote universal constants (independent of f and p and, whenever it makes sense, independent also of r) and may change from one occurrence to the next. As usual, the symbol $f \approx g$ will indicate the existence of a universal positive constant C such that $(1/C)\, f \leqslant g \leqslant Cf$, while the symbol $f \lesssim g$ means that $f \leqslant Cg$. We shall write $\|g\|_p$ to denote either $\|g\|_{L^p(\mu)}$ or $\|g\|_{L^p(\nu)}$, and $\lambda_g^\nu(y) = \nu(\{x \in \mathcal{N};\, |g(x)| > y\})$ is the distribution function of g with respect to the measure ν (see [BS]). Throughout this paper $(\mathcal{N}, \nu)$ and $(\mathcal{M}, \mu)$ are two σ-finite measure spaces.

Finally, let us mention that the theory developed in this paper can be easily extended to the case of a sublinear operator T satisfying

$$\|Tf\|_p \leqslant \frac{1}{\varphi(p)} \|f\|_p,$$

for every $1 < p \leqslant p_0$, where φ is essentially an increasing function such that $\varphi(p)$ tends to zero as p tends to 1. In particular, we can take $\varphi(p) = (p-1)^\alpha$ with $\alpha > 0$.

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2. SOME PREVIOUS RESULTS AND A GENERAL PRINCIPLE

For simplicity, throughout this paper we shall assume that T is sublinear operator in the sense that $|T(\lambda f)| = |\lambda|\, |Tf|$ and

$$\left| T\left(\sum_{j=0}^{\infty} f_j \right)(x) \right| \leqslant \sum_{j=0}^{\infty} |Tf_j(x)|, \qquad \text{a.e. } x.$$

As usual, if T is sublinear in the classical sense, we can adapt out proofs by first considering bounded functions and then extending the result by some density argument.

Proposition 2.1. *If a function f satisfies that, for every $1 < p \leqslant p_0$,*

$$\|Tf\|_{L^p(\nu)} \leqslant \frac{1}{p-1} \|f\|_{L^p(\mu)},$$

then, for every $r \geqslant e^{1/(p_0 - 1)}$,

$$\frac{\int_{\mathcal{N}} (|Tf(x)| - 1/r)_+ \, d\nu(x)}{\log r} = \frac{\int_{1/r}^{\infty} \lambda_{Tf}^\nu(y)\, dy}{\log r}$$

$$\leqslant K \int_{\mathcal{M}} |f(x)|^{1 + 1/\log r}\, d\mu(x), \qquad (2)$$

where $K = \sup_{p \geqslant 1} (e/p)(p-1)^{1-p}$. If in addition $f \in L^1 \cap (\bigcup_{p>1} L^p)$, we have that

$$\overline{\lim_{r \to \infty}} \frac{\int_{1/r}^{\infty} \lambda_{Tf}^\nu(y)\, dy}{\log r} \leqslant K \int_{\mathcal{M}} |f(x)|\, d\mu(x). \qquad (3)$$

Proof. For every $r>0$ and $1<p\leqslant p_0$,

$$\int_{1/r}^{\infty} \lambda_{Tf}^{v}(y)\,dy = \int_{1/r}^{\infty} y^{p-1}\lambda_{Tf}^{v}(y)\frac{1}{y^{p-1}}\,dy \leqslant r^{p-1}\int_{1/r}^{\infty} y^{p-1}\lambda_{Tf}^{v}(y)\,dy$$

$$\leqslant r^{p-1}\frac{1}{p}\|Tf\|_p^p \leqslant \frac{1}{p}\left(\frac{1}{p-1}\right)^{p-1}\frac{r^{p-1}}{p-1}\|f\|_p^p$$

$$\leqslant \frac{K}{e}\frac{r^{p-1}}{p-1}\|f\|_p^p.$$

Taking now $p=1+1/\log r$ with $r\geqslant e^{1/(p_0-1)}$, we obtain (2).

The last part follows immediately from applying the dominated convergence theorem. ∎

From now on, we shall assume that the estimates on the operator T hold for $1<p\leqslant 2$; that is, we shall work with $p_0=2$. The general case follows with the obvious modifications.

Lemma 2.2. *For every* $0\leqslant r\leqslant s\leqslant\infty$,

$$\int_r^s \lambda_{\sum_k f_k}^{v}(y)\,dy \leqslant \inf_{\sum a_k=1}\sum_k \int_{a_k r}^{s}\lambda_{f_k}^{v}(y)\,dy.$$

Proof. For every $\sum a_k=1$, we have that

$$\int_r^s \lambda_{\sum_{k=0}^{\infty} f_k}^{v}(y)\,dy = \int_{\mathcal{N}}\left(\min\left(\left|\sum_{k=0}^{\infty} f_k(x)\right|, s\right)-r\right)_+ dv(x)$$

$$\leqslant \int_{\mathcal{N}}\left(\sum_{k=0}^{\infty}\min(|f_k(x)|, s)-r\right)_+ dv(x)$$

$$\leqslant \sum_{k=0}^{\infty}\int_{\mathcal{N}}(\min(|f_k(x)|, s)-a_k r)_+\,dv(x)$$

$$=\sum_{k=0}^{\infty}\int_{a_k r}^{s}\lambda_{f_k}^{v}(y)\,dy,$$

from which the result follows. ∎

Definition 2.3. We say that a function d is a dyadic function and we write $d\in D$ if $d=\sum_{k\in\mathbb{Z}}2^k\chi_{A_k}$ with A_k pairwise disjoint measurable sets. Similarly, we say that d is a dyadic$^+$ function and we write $d\in D^+$ if $d=\sum_{k=0}^{\infty}2^k\chi_{A_k}$ with A_k as before.

LEMMA 2.4. *Let f be a positive function. Then*:

(a) $f=\sum_{j=0}^{\infty} d_j$ *with* $d_j \in D$ *and* $d_j \leqslant f/2^j$.

(b) f *can be written as* $f=B+S_d$, *where* $0 \leqslant B \leqslant \min(1, f)$ *and* $S_d = \sum_{j=0}^{\infty} d_j$ *with* $d_j \in D^+$ *and* $d_j \leqslant f/2^j$.

Proof. The proof of (a) can be found in [So] (see also [Sj]) and the proof of (b) is a modification of (a) as follows: Let $f_0=f$, $\underline{f_0}=f\chi_{\{f\leqslant 1\}}$, $\overline{f_0}=f\chi_{\{f\geqslant 1\}}$ and let us write

$$\overline{f_0}=\overline{f_0}-\sum_{k=0}^{\infty} 2^k\chi_{E_{k,0}}+\sum_{k=0}^{\infty} 2^k\chi_{E_{k,0}}=f_1+\sum_{k=0}^{\infty} 2^k\chi_{E_{k,0}}=f_1+d_0,$$

where $E_{k,0}=\{x: 2^k\leqslant f_0(x)<2^{k+1}\}$. Then $f=\underline{f_0}+d_0+f_1$ and it holds that $d_0\leqslant\overline{f_0}\leqslant f$, $d_0\in D^+$, and $f_1\leqslant f/2$.

Analogously, we write

$$f_1=\underline{f_1}+\overline{f_1}=\underline{f_1}+\left(\overline{f_1}-\sum_{k=0}^{\infty} 2^k\chi_{E_{k,1}}\right)+\sum_{k=0}^{\infty} 2^k\chi_{E_{k,1}}=\underline{f_1}+f_2+d_1,$$

where $E_{k,1}=\{x: 2^k\leqslant f_1(x)<2^{k+1}\}$.

Following with this construction, we obtain that $f=B+S_d$, where $B=\sum_j \underline{f_j}$, and one can easily see that the required properties on B and S_d are satisfied. ∎

This lemma allows us to formulate some general principles which are very useful to obtain the boundedness of sublinear operators in (essentially) quasi-normed lattice spaces.

THEOREM 2.5 (General Principles). *Let T be a sublinear operator. Let* $E=\{f; \|f\|_E<\infty\}$, *where* $\|\cdot\|_E$ *is a positively homogeneous functional satisfying the lattice property* ($|f|\leqslant|g| \Rightarrow \|f\|_E\leqslant\|g\|_E$), *and let* $(F, \|\cdot\|_F)$ *be a quasi-normed space. Then*

$$\|Tf\|_F\leqslant C\,\|f\|_E, \tag{4}$$

for every $f\in E$, *if and only if one of the following conditions holds*:

(i) *Equation* (4) *holds for every function* $d\in D$. *Equivalently, there exists a constant* $C>0$ *such that, for every* $d\in D$ *with* $\|d\|_E=1$, $\|Td\|_F\leqslant C$.

(ii) *Equation* (4) *holds for every function* $d\in D^+$ *and every* B *such that* $\|B\|_\infty\leqslant 1$.

(iii) *Equation* (4) *holds for every function* $d \in D^+$ *with* $\|d\|_E \leqslant 1$ *and there exists a constant* $C > 0$ *such that* $\|TB\|_F \leqslant C$, *for every* B *such that* $\|B\|_\infty \leqslant 1$ *and* $\|B\|_E \leqslant 1$.

Proof. We shall only prove (iii) since the proofs of (i) and (ii) follow the same pattern. The necessary condition is clear. Let us then prove the sufficient condition. Since E satisfies the lattice property, we only need to show (4) for positive functions f, and since both $\|\cdot\|_E$ and $\|\cdot\|_F$ are positively homogeneous, it is enough to show that $\|Tf\|_F \leqslant C$ for every $\|f\|_E = 1$. By the previous lemma, we write $f = B + \sum_{j=0}^{\infty} d_j$, and we have that $\|B\|_E \leqslant 1$ and $\|d_j\|_E \leqslant 1$, for every $j \geqslant 0$. Let $\alpha > 0$ be such that $\|\cdot\|_F^\alpha$ is subadditive (see [BL]). Then

$$\|Tf\|_F^\alpha \leqslant \|TB\|_F^\alpha + \sum_{j=0}^{\infty} \|Td_j\|_F^\alpha \lesssim \left(1 + \sum_{j=0}^{\infty} \|d_j\|_E^\alpha\right)$$

$$\lesssim \left(1 + \sum_{j=0}^{\infty} \|2^{-j} f\|_E^\alpha\right) \leqslant C,$$

and the result follows. ∎

3. RESTRICTED STRONG-TYPE EXTRAPOLATION

THEOREM 3.1. *Let* T *be a sublinear operator satisfying*

$$\|Tf\|_{L^p(\nu)} \leqslant \frac{1}{p-1} \|f\|_{L^p(\mu)}, \tag{5}$$

for every $f \in L^p(\mu)$ *and every* $1 < p \leqslant 2$. *Then, for every* $f \in L \log L(\mu)$,

$$\sup_{r>0} \frac{\int_{1/r}^{\infty} \lambda_{Tf}^{\nu}(y)\, dy}{1 + \log^+ r} \lesssim \int_{\mathscr{M}} |f(x)|\, (1 + \log^+ |f(x)|)\, d\mu(x). \tag{6}$$

Proof. Let $f \in L \log L(\mu)$ and let us write

$$f = \sum_{k=0}^{\infty} f_k = f\chi_{\{|f(x)| \leqslant 1\}} + \sum_{k=1}^{\infty} f\chi_{\{2^{k-1} \leqslant |f(x)| < 2^k\}}.$$

Then, by Lemma 2.2, we have that, for $r \geqslant e$,

$$\frac{\int_{1/r}^{\infty} \lambda_{Tf}^{v}(y)\,dy}{\log r} \leqslant \sum_{k=0}^{\infty} \frac{\log r2^{k+1}}{\log r} \frac{\int_{1/(r2^{k+1})}^{\infty} \lambda_{Tf_k}^{v}(y)\,dy}{\log r2^{k+1}}$$

$$\lesssim \sum_{k=0}^{\infty} (1+k) \frac{\int_{1/(r2^{k+1})}^{\infty} \lambda_{Tf_k}^{v}(y)\,dy}{\log r2^{k+1}}$$

Now, by Proposition 2.1, we get that, for $r \geqslant e$, and $k \geqslant 1$,

$$\frac{\int_{1/(r2^{k+1})}^{\infty} \lambda_{Tf_k}^{v}(y)\,dy}{\log r2^{k+1}} \lesssim \int_{\mathscr{M}} |f_k(x)|^{1+1/\log(r2^{k+1})}\,d\mu(x)$$

$$\lesssim \int_{\{2^{k-1} \leqslant |f(x)| < 2^k\}} |f(x)|\, 2^{k/\log(r2^{k+1})}\,d\mu(x)$$

$$\lesssim \int_{\{2^{k-1} \leqslant |f(x)| < 2^k\}} |f(x)|\,d\mu(x),$$

and similarly for $k=0$. Therefore,

$$\sup_{r \geqslant e} \frac{\int_{1/r}^{\infty} \lambda_{Tf}^{v}(y)\,dy}{\log r} \lesssim \sum_k (1+k) \int_{\{2^{k-1} \leqslant |f(x)| < 2^k\}} |f(x)|\,d\mu(x)$$

$$\approx \int_{\mathscr{M}} |f(x)|(1+\log^{+}|f(x)|)\,d\mu(x),$$

from which the result follows. ∎

Remark 3.2. (1) It is important to observe that if

$$\int_{1}^{\infty} \lambda_{Tf}(y)\,dy \lesssim \int_{\mathscr{M}} |f(x)|(1+\log^{+}|f(x)|)\,d\mu(x), \tag{7}$$

then

$$\int_{1/r}^{\infty} \lambda_{Tf}(y)\,dy = (1/r) \int_{1}^{\infty} \lambda_{T(rf)}(z)\,dy$$

$$\lesssim \int_{\mathscr{M}} |f(x)|(1+\log^{+}(r\,|f(x)|))\,d\mu(x)$$

$$\lesssim (1+\log^{+} r) \int_{\mathscr{M}} |f(x)|(1+\log^{+}(|f(x)|))\,d\mu(x),$$

and therefore (6) and (7) are equivalent.

(2) A second important remark for our purposes is that we have only used condition (5) on the functions f_k.

Our next result improves Yano's extrapolation theorem.

THEOREM 3.3. *If T is a sublinear operator satisfying that for every measurable set A and every $1<p\leqslant 2$,*

$$\|T\chi_A\|_p \leqslant \frac{1}{p-1}\mu(A)^{1/p}, \tag{8}$$

then $T\colon L\log L(\mu)\to L^1(v)+L^\infty(v)$ is bounded.

Proof. Since $L^1(v)+L^\infty(v)$ and $L\log L(\mu)$ are normed spaces, we can apply Theorem 2.5(iii) and hence it is enough to show that $\|Td\|_{L^1(v)+L^\infty(v)} \lesssim \|d\|_{L\log L(\mu)}$ for every function $d\in D^+$ with $\|d\|_{L\log L(\mu)}\leqslant 1$ and that for every B such that $\|B\|_{L\log L}\leqslant 1$ and $\|B\|_\infty\leqslant 1$, $\|TB\|_{L^1(v)+L^\infty(v)}\leqslant C$.

Now, for the first case, let $d=\sum_{k=0}^\infty 2^k\chi_{A_k}$ with $\|d\|_{L\log L}\leqslant 1$ and let $j\leqslant 0$ be such that $2^{j-1}\leqslant\|d\|_{L\log L}\leqslant 2^j$. We have that $d_k=d\chi_{\{2^k\leqslant d<2^{k+1}\}}=2^k\chi_{A_k}$, and by Theorem 3.1 and Remark 3.2(2) we obtain that

$$\int_1^\infty \lambda^v_{Td}(y)\,dy \lesssim \int_{\mathcal{M}} |d(x)|(1+\log^+(|d(x)|))\,d\mu(x). \tag{9}$$

Now, since $d/2^j\in D^+$, we get

$$\int_{2^j}^\infty \lambda^v_{Td}(y)\,dy \lesssim \int_{\mathcal{M}} |d(x)|\left(1+\log^+\left(\frac{|d(x)|}{2^j}\right)\right)d\mu(x).$$

Consequently, the function $(|Td(x)|-2^j)_+\in L^1$ and

$$\begin{aligned}|Td(x)| &= (|Td(x)|\,\chi_{\{|Td(x)|<2^j\}}+2^j\chi_{\{|Td(x)|\geqslant 2^j\}})\\ &\quad+(|Td(x)|-2^j)\,\chi_{\{|Td(x)|\geqslant 2^j\}}\in L^\infty(v)+L^1(v).\end{aligned} \tag{10}$$

Moreover,

$$\begin{aligned}\|Td\|_{L^1(v)+L^\infty(v)} &\lesssim 2^j+\int_{\mathcal{M}}|d(x)|\left(1+\log^+\left(\frac{|d(x)|}{2^j}\right)\right)d\mu(x)\\ &\lesssim \|d\|_{L\log L(\mu)}.\end{aligned}$$

To prove the second part, let B be such that $\|B\|_{L\log L}\leqslant 1$ and $\|B\|_\infty\leqslant 1$. By interpolation, we have that T is of strong-type (p,p) for every $1<p<2$ and therefore

$$\int_1^\infty \lambda^\nu_{TB}(y)\,dy\leqslant \int_1^\infty \lambda^\nu_{TB}(y)\,y^{p-1}\,dy\lesssim \int_{\mathcal{M}} |B(x)|^p\,d\mu(x)$$

$$\leqslant \int_{\mathcal{M}} |B(x)|\,d\mu(x)\leqslant \|B\|_{L\log L}\leqslant 1. \tag{11}$$

and by the same argument used in (10), we obtain the result. ∎

COROLLARY 3.4. *If T is a sublinear operator satisfying* (8), *then T satisfies* (6).

Proof. We can assume that $f\geqslant 0$. Then let us write $f=B+\sum_{j=0}^\infty d_j$ as in Lemma 2.4. By Lemma 2.2,

$$\int_3^\infty \lambda^\nu_{Tf}(y)\,dy\leqslant \int_1^\infty \lambda^\nu_{TB}(y)\,dy+\sum_{j=0}^\infty \int_{1/2^j}^\infty \lambda^\nu_{Td_j}(y)\,dy.$$

Now, since $B\leqslant f$ we obtain, by (11), that

$$\int_1^\infty \lambda^\nu_{TB}(y)\,dy\leqslant \int_{\mathcal{M}} f(x)\,d\mu(x).$$

Also, to estimate the second term, we observe that, by (9) and the fact that $2^jd_j\leqslant f$,

$$\int_{1/2^j}^\infty \lambda^\nu_{Td_j}(y)\,dy=\int_{1/2^j}^\infty \lambda^\nu_{T(2^jd_j)}(2^jy)\,dy=2^{-j}\int_1^\infty \lambda^\nu_{T(2^jd_j)}(y)\,dy$$

$$\lesssim 2^{-j}\int_{\mathcal{M}} f(x)(1+\log^+(f(x)))\,d\mu(x).$$

Summing in $j\geqslant 0$ and applying Remark 3.2(1) we obtain the result. ∎

Let us now show that (3) also holds, for every $f\in L\log L$.

THEOREM 3.5. *Let T be a sublinear operator satisfying* (8). *Then, for every $f\in L\log L(\mu)$,*

$$\overline{\lim_{r\to\infty}}\frac{\int_{1/r}^\infty \lambda^\nu_{Tf}(y)\,dy}{\log r}\lesssim \|f\|_1 \qquad \textit{and} \qquad \underline{\lim_{y\to 0}}\; y\lambda^\nu_{Tf}(y)\lesssim \|f\|_1.$$

Proof. If $\varepsilon > 0$,

$$\frac{\int_{1/r}^{\infty} \lambda^{\nu}_{T(\varepsilon f)}(y)\, dy}{\log r} = \frac{\int_{1/r}^{\infty} \lambda^{\nu}_{Tf}(y/\varepsilon)\, dy}{\log r} = \varepsilon \frac{\int_{1/(\varepsilon r)}^{\infty} \lambda^{\nu}_{Tf}(y)\, dy}{\log r}$$

$$= \varepsilon \frac{\log(\varepsilon r)}{\log r} \frac{\int_{1/(\varepsilon r)}^{\infty} \lambda^{\nu}_{Tf}(y)\, dy}{\log(\varepsilon r)},$$

and, if we apply (6) to the function εf, with $f \in L \log L$, we obtain that, for every $r \geqslant e$,

$$\frac{\log(\varepsilon r)}{\log r} \frac{\int_{1/(\varepsilon r)}^{\infty} \lambda^{\nu}_{Tf}(y)\, dy}{\log(\varepsilon r)} \lesssim \int_{\mathcal{M}} |f(x)|(1 + \log^{+}(\varepsilon\, |f(x)|))\, d\mu(x).$$

From this estimate and letting first r tend to infinity and then ε tend to zero, we obtain the first inequality. The second follows immediately from L'Hôpital's rule. ▮

Remark 3.6. (i) If for a ν-measurable function g we define

$$q(g) = \overline{\lim_{r \to \infty}} \frac{\int_{1/r}^{\infty} \lambda^{\nu}_{g}(y)\, dy}{\log r},$$

we have, by Lemma 2.2, that q is a seminorm and thus Theorem 3.5 says that if $S = \{g;\, q(g) < \infty\}$, T can be extended to a bounded operator from $L^1(\mu)$ into the completion of S with respect to q.

(ii) If we take $T = M$ to be the Hardy–Littlewood maximal operator, then it is known (see [CS]) that $y\lambda_{Mf}(y)$ is equivalent to a decreasing function. As a consequence, we have that $\underline{\lim}_{y \to 0}\, y\lambda^{\nu}_{Tf}(y) \approx \sup_{y > 0}\, y\lambda^{\nu}_{Tf}(y)$ and the second inequality in Theorem 3.5 is nothing but the weak-type estimate $(1, 1)$ for M.

Finally, let us just mention that we cannot expect to get the weak-type $(1, 1)$ estimate for T since it is known (see [K]) that there are operators T satisfying (1) for which the weak-type $(1, 1)$ estimate does not hold.

4. RESTRICTED WEAK-TYPE EXTRAPOLATION

In this section, we shall assume that our sublinear operator T satisfies the following restricted weak-type condition: there exists a constant $C > 0$ such that, for every measurable set A and every $1 < p \leqslant 2$,

$$\sup_{y > 0} \lambda^{\nu}_{T\chi_A}(y)^{1/p}\, y \leqslant \frac{C}{p - 1} \mu(A)^{1/p}. \tag{12}$$

First, we observe that condition (12) is equivalent to having that

$$\sup_{y>0} \Big(\sup_{1<p\leqslant 2} y^p(p-1)\Big) \lambda^{v}_{T\chi_A}(y) \lesssim \mu(A),$$

and hence, taking $p=1+1/(1+\log^+(1/y))$,

$$\sup_{y>0} \frac{y}{1+\log^+(1/y)} \lambda^{v}_{T\chi_A}(y) \lesssim \mu(A). \tag{13}$$

Now, the space $L^1(v)+L^\infty(v)$ is characterized as the set of measurable functions such that

$$\|f\|_{L^1(v)+L^\infty(v)} = \int_0^1 f^*_v(s)\, ds = \int_0^\infty \min(1, \lambda^v_f(y))\, dy < \infty.$$

This last equality leads us to define a weak-type version of this space as follows: let $W(L^1+L^\infty)$ (weak-(L^1+L^∞)) be the set of measurable functions such that

$$\|f\|_{W(L^1+L^\infty)} = \sup_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \min(1, \lambda^v_f(y))\, dy < \infty.$$

Observe that $\|f\|_{W(L^1+L^\infty)} \approx \sup_{r>0} \int_r^{2r} \min(1, \lambda^v_f(y))\, dy$ and that this last expression is a quasi-norm. Also, if the measure v is finite, then $L^1(v)+L^\infty(v)=L^1(v)$ and the space $W(L^1+L^\infty)$ is the weak-type space $L^{1,\infty}$.

Let us also recall (see [So]) that the space B_{φ^*} mentioned in the Introduction was defined as the set of measurable functions such that

$$\|f\|_{B_{\varphi^*}} = \int_0^\infty \varphi(\lambda_f(y)) \left[1+\log\left(\frac{\|f\|_\varphi}{y\varphi(\lambda_f(y))}\right)\right] dy < \infty,$$

where $\|f\|_\varphi = \int_0^\infty \varphi(\lambda_f(y))\, dy$.

THEOREM 4.1. *If T satisfies* (12) *and* $\varphi(t)=t(1+\log^+ 1/t)$, *then*

$$T\colon B_{\varphi^*} \to W(L^1+L^\infty)$$

is bounded.

Proof. Since $W(L^1+L^\infty)$ and B_{φ^*} are quasi-normed lattices, we can apply our first general principle, Theorem 2.5(i). That is, we have to see that $\|Td\|_{W(L^1+L^\infty)} \leqslant C$ for every $d\in D$ such that $\|d\|_{B_{\varphi^*}}=1$.

Hence let $d=\sum_{k\in\mathbb{Z}} 2^k\chi_{A_k}\in D$ and let us assume that $\|d\|_{B_{\varphi^*}}=1$. If we write $E_k = \cup_{j=k+1}^\infty A_k$, we obtain that $d=\sum_{k\in\mathbb{Z}} 2^k\chi_{E_k}$. Then, to show that there exists a constant C such that $\|Td\|_{W(L^1+L^\infty)} \leqslant C$, it is enough to prove that $\sup_{j\geqslant 0}\int_{2^j}^{2^{j+1}} \lambda^v_{Td}(y)\,dy \leqslant C$. By Lemma 2.2 and (13) we have that

$$\sup_{j\geqslant 0}\int_{2^j}^{2^{j+1}} \lambda^v_{(\sum_{k\in\mathbb{Z}} 2^k\,|T\chi_{E_k}|)}(y)\,dy$$

$$\leqslant \sup_{j\geqslant 0}\inf_{\sum_k a_k=1}\sum_k 2^k \int_{a_k 2^j/2^k}^{2^{j+1}/2^k} (\lambda^v_{|T\chi_{E_k}|})(y)\,dy$$

$$\lesssim \sup_{j\geqslant 0}\inf_{\sum_k a_k=1}\sum_k 2^k\mu(E_k)\int_{a_k 2^j/2^k}^{2^{j+1}/2^k} \frac{(1+\log^+(1/y))}{y}\,dy.$$

A simple computation shows that if we take

$$a_k = \frac{2^k\mu(E_k)(1+\log^+(1/\mu(E_k)))}{\sum_j 2^j\mu(E_j)(1+\log^+(1/\mu(E_j)))},$$

then the above expression is equivalent to $\|d\|_{B_{\varphi^*}} = 1$. ∎

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