Stable Equivalence of Representation-Finite Trivial Extension Algebras

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Let $k$ denote a fixed algebraically closed field. By algebra, we shall always mean a finite-dimensional associative $k$-algebra with an identity. Unless otherwise specified, modules are finitely generated right modules. We shall use, without further reference, results about Auslander–Reiten sequences and the Auslander–Reiten quiver such as can be found in [6], and about tilted algebras for which we refer to [3] and [7]. For an algebra $C$, we shall denote by $\tau_C$ the Auslander–Reiten translation $DTr$ in the category $\text{mod } C$ of finitely generated right $C$-modules. The trivial extension $T(C)$ of $C$ by its minimal injective cogenerator $DC = \text{Hom}_k(C, k)$ is defined to be the algebra whose additive structure is that of the group $C \oplus DC$, with the multiplication defined by

$$(x, f)(y, g) = (xy, xg + fy)$$

for $x, y \in C$ and $f, g \in C(DC)$. It is well known that $T(C)$ is a self-injective algebra. Thus a connected trivial extension algebra $T(C)$ is representation-finite if and only if the stable part of its Auslander–Reiten quiver is isomorphic to $\mathcal{Z}A/G$, where $A$ is one of the Dynkin diagrams $A_n$, $D_n$, $E_6$, $E_7$, or $E_8$ (called the Cartan class of $T(C)$), and $G$ an admissible group of automorphisms of $\mathcal{Z}A[10]$. Tachikawa has proved in [11] that if $C$ is

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hereditary and representation-finite, then $T(C)$ is representation-finite. Later, Hughes and Waschbüsch [9] (see also [8, 5]) have proved that if $C$ is tilted of Dynkin type $\Delta$, then $T(C)$ is representation-finite of Cartan class $\Delta$ and conversely, if $T(C)$ is representation-finite of Cartan class $\Delta$, there exists a tilted algebra $C'$ of Dynkin type $\Delta$ such that $T(C) \cong T(C')$. In fact, it was shown in [11] that $T(C)$ is representation-finite of Cartan class $\Delta$ if and only if $C$ is an iterated tilted algebra of Dynkin type $\Delta$. The objective of the present paper is to determine when two representation-finite trivial extension algebras are stably equivalent [2]. In [12], Tachikawa has proved that the trivial extensions of two hereditary algebras of the same type are stably equivalent using a generalisation of the reflection functors of Bernstein, Gelfand, and Ponomarev [4]. We shall now prove the following:

**Theorem.** Two basic, connected representation-finite trivial extension algebras are stably equivalent if and only if they have the same Cartan class.

We shall start by proving:

**Lemma.** Let $A$ be a basic, connected representation-finite hereditary algebra of type $\Delta$, $T_\Delta$ be a multiplicity-free tilting module and $B = \text{End } T_\Delta$. Then $T(A)$ and $T(B)$ are stably equivalent.

**Proof.** Observe that there exists a complete slice $\mathcal{S}$ of the Auslander–Reiten quiver $\Gamma_A$ of $A$ (which is entirely contained in the torsion class $\mathcal{S}(T_A) = \{ M_A \mid \text{Ext}_A^1(T, M) = 0 \}$ of $\text{mod } A$), and a complete slice $\mathcal{S}'$ of the Auslander–Reiten quiver $\Gamma_B$ of $B$ (which is entirely contained in the torsion-free class $\mathcal{S}(T_B) = \{ N_B \mid \text{Tor}_B^1(N, T) = 0 \}$ of $\text{mod } B$) such that the functors $\text{Hom}_A(T, -): \text{mod } A \to \text{mod } B$ and $- \otimes_B T: \text{mod } B \to \text{mod } A$ restrict to mutually inverse equivalences of the full subcategories $\mathcal{S}$ and $\mathcal{S}'$. Indeed, it suffices to take for $\mathcal{S}$ the complete slice of $\Gamma_A$ consisting of the isomorphism classes of the indecomposable injective $A$-modules, and then to let $\mathcal{S}'$ be the complete slice of $\Gamma_B$ consisting of the isomorphism classes of the $B$-modules of the form $\text{Hom}_A(B T_A, I_A)$, where $I_A$ is an indecomposable injective $A$-module [7]. If we now use the canonical embeddings of $\text{mod } A$ into $\text{mod } T(A)$ and of $\text{mod } B$ into $\text{mod } T(B)$, $\mathcal{S}$ and $\mathcal{S}'$ embed, respectively, as sections (in the sense of [3, Definition (2.5)]) of the stable Auslander–Reiten quivers of $T(A)$ and $T(B)$. That is to say, the set of modules on each slice yields a set of representatives of the $\tau$-orbits of the nonprojective indecomposable modules, and the set of irreducible morphisms with both domain and codomain on the slice yields a set of representatives of the $\sigma$-orbits of the irreducible morphisms in the corresponding stable categories. We shall deduce from this remark the definition of a stable equivalence functor $F: \text{mod } T(A) \to \text{mod } T(B)$ as follows: for every indecomposable $T(A)$-module $M$ on $\mathcal{S}$ (thus $M$ is also
an $A$-module), and for every irreducible morphism $f: M \to N$ of mod $T(A)$ with both $M$ and $N$ on $\mathcal{F}$ (thus $f$ is also an $A$-homomorphism), we put

$$F(M) = \Hom_A(T, M)$$

considered as a $(B)$-module,

and

$$F(f) = \Hom_A(T, f)$$

considered as a $(B)$-homomorphism

(in other words, $F|_{\mathcal{F}} = \Hom_A(T, -)$). Now, any indecomposable nonprojective $T(A)$-module $M$ can be uniquely written in the form $M \simeq \tau_{T(B)}^{-s} M_0$, where $M_0 \in \mathcal{F}$ and $0 \leq s \leq m_d$. (Here, $m_d$ denotes the smallest integer $m$ such that the residual class in the mesh category $k(\mathbb{Z}A)$ of a path in $\mathbb{Z}A$ of length greater than or equal to $m$ vanishes. Thus $m_{\Delta_n} = n$, $m_{\Omega_n} = 2n - 3$, $m_{\varepsilon_6} = 11$, $m_{\varepsilon_7} = 17$, and $m_{\varepsilon_8} = 29$). We then define

$$F(M) = \tau_{T(B)}^{-s} F(M_0) = \tau_{T(B)}^{-s} \Hom_A(T, M_0).$$

In the same way, every irreducible morphism $f: M \to N$ in mod $T(A)$ can be uniquely written in the form $f \simeq \sigma_{T(A)}^{-t} f_0$, where $f_0: M_0 \to N_0$, $M_0$, $N_0 \in \mathcal{F}$, and $0 \leq t \leq 2m_d$, and we define

$$F(f) = \sigma_{T(B)}^{-t} F(f_0) = \sigma_{T(B)}^{-t} \Hom_A(T, f_0).$$

This defines $F$ on the nonprojective indecomposable $T(A)$-modules, and on the irreducible morphisms of mod $T(A)$. Let now $f: M \to N$ be an arbitrary homomorphism between the indecomposable nonprojective $T(A)$-modules $M$ and $N$ which is not an isomorphism. We shall define $F(f)$ by induction on the length of a minimal path from $M$ to $N$ in the stable Auslander–Reiten quiver of $T(A)$. This length is equal to one if and only if $f$ is an irreducible morphism, and in this case, $F(f)$ is already defined above. If $f$ is not irreducible, let us consider an Auslander–Reiten sequence of mod $T(A)$ ending with $N$:

$$0 \to \tau_{T(A)}N \xrightarrow{u} E \xrightarrow{v} N \to 0$$

then there exists a $T(A)$-homomorphism $w: M \to E$ such that $f = vw$. The middle term $E$ of the previous sequence can be written as $E = P \oplus (\oplus_{i=1}^r E_i)$, where $P_{T(A)}$ is projective, and each of the $E_i (1 \leq i \leq r)$ is indecomposable nonprojective, then $v$ and $w$ can be written as

$$v = \begin{bmatrix} p & v_1 & v_2 & \cdots & v_r \end{bmatrix}$$

and

$$w = \begin{bmatrix} q \\ w_1 \\ w_2 \\ \vdots \\ w_r \end{bmatrix}.$$
where \( p: P \to N \) and the \( v_i: E_i \to N \) \((1 \leq i \leq r)\) are irreducible morphisms, and \( q: M \to P, \ w_i: M \to E_i \) \((1 \leq i \leq r)\). Thus,

\[
f = pq + \sum_{i=1}^{r} v_i w_i
\]

and we define

\[
F(f) = \sum_{i=1}^{r} F(v_i) F(w_i),
\]

where the \( F(v_i) \) and the \( F(w_i) \) have already been defined by induction. It is not hard to see that \( F(f) \) is well defined.

We have thus defined \( F \) on the indecomposable nonprojective \( T(A) \)-modules and the morphisms between them in \( \text{mod} \ T(A) \). We extend to arbitrary modules and morphisms in \( \text{mod} \ T(A) \) by additivity, thus completing the definition of \( F \). We construct in the same way a functor \( F': \text{mod} \ T(B) \to \text{mod} \ T(A) \): for every indecomposable \( T(B) \)-module \( M' \) on \( \mathcal{S}' \) (thus \( M' \) is a \( B \)-module) and every irreducible morphism \( f': M' \to N' \) in \( \text{mod} \ T(B) \) with \( M' \) and \( N' \) on \( \mathcal{S}' \) (thus \( f' \) is also a \( B \)-homomorphism), we let

\[
F(M') = M' \otimes_B T
\]

considered as a \( T(A) \)-module,

and

\[
F(f') = f' \otimes_B T
\]

considered as a \( T(A) \)-homomorphism

(in other words, \( F \mid_{\mathcal{S}'} = \otimes_B T \)), and we extend this definition to the whole stable category \( \text{mod} \ T(B) \) just as we did for \( F \).

Let us now prove that \( F \) and \( F' \) are quasi-inverse functors: any indecomposable nonprojective \( T(A) \)-module \( M \) can be uniquely written in the form \( M = \tau_{T(A)}^{-s} M_0 \) where \( M_0 \in \mathcal{S} \) and \( 0 \leq s \leq m_\beta \), thus

\[
(F' \circ F)(M) = (F' \circ F)(\tau_{T(A)}^{-s} M_0)
\]

\[
= F'(F(\tau_{T(A)}^{-s} M_0))
\]

\[
= F'(\tau_{T(B)}^{-1} \text{Hom}_A(T, M_0))
\]

\[
= \tau_{T(A)}^{s} \left\{ \text{Hom}_A(T, M_0) \otimes_B T \right\}
\]

\[
\cong \tau_{T(A)}^{s} M_0
\]

\[
= M.
\]
Observe that the isomorphism used is natural [7]. In the same way, we can show that for every morphism \( f: M \to N \) in \( \text{mod} \ T(A) \), we have

\[(F' \circ F)(f) \cong f.
\]

Thus \( F' \circ F \cong \text{id}_{\text{mod} \ T(A)} \). Similarly, \( F \circ F' \cong \text{id}_{\text{mod} \ T(B)} \) and the proof of the lemma is now complete.

**Remark.** Let \( i \) be a sink in the ordinary quiver of \( A \), \( e_i \) be the corresponding primitive idempotent, and \( T_A = (1 - e_i) A \oplus \tau^{-1}_A (e_i A) \). Then \( \text{Hom}_A(T, -) \) is a reflection functor [4], \( B \) is also hereditary, and our functor \( F: \text{mod} \ T(A) \to \text{mod} \ T(B) \) reduces to the functor \( S_i^+ \) of [12]. Observe also that, by definition, the functors \( F \) and \( F' \) commute with the Auslander–Reiten translations in \( \text{mod} \ T(A) \) and \( \text{mod} \ T(B) \), respectively.

**Proof of the Theorem.** Since the necessity follows directly from [10], we shall only prove the sufficiency: let \( R_1 \) and \( R_2 \) be two basic connected representation-finite trivial extension algebras of the same Cartan class \( A \). Then there exist two tilted algebras \( B_1 \) and \( B_2 \) of type \( A \) such that \( R_i \cong T(B_i) \) \( (i = 1, 2) \) [9]. Moreover, there exist two hereditary algebras \( A_1 \) and \( A_2 \) of type \( A \) and two tilting modules \( T_A^{(1)} \) and \( T_A^{(2)} \), respectively, on \( A_1 \) and \( A_2 \) such that \( B_i \cong \text{End} \ T_{A_i}^{(i)} \) \( (i = 1, 2) \). The previous lemma implies that \( \text{mod} \ R_i \cong \text{mod} \ T(B_i) \cong \text{mod} \ T(A_i) \) \( (i = 1, 2) \). Now \( A_1 \) and \( A_2 \) are hereditary algebras of the same type, thus it follows from [12] (or directly from our lemma and the above remark) that \( \text{mod} \ T(A_1) \cong \text{mod} \ T(A_2) \), and hence \( \text{mod} \ R_1 \cong \text{mod} \ R_2 \).

**Remark 1.** In view of the result in [1], our theorem may be restated as follows: if \( C_1 \) and \( C_2 \) are basic connected iterated tilted algebras of Dynkin type, their trivial extensions \( T(C_i) \) and \( T(C_2) \) are stably equivalent if and only if \( C_1 \) and \( C_2 \) are of the same type.

**Remark 2.** It follows from our theorem that any trivial extension of Cartan class \( A_n \) is stably equivalent to a Nakayama algebra. Also, any representation-finite trivial extension algebra is stably equivalent to an algebra with radical cube equal to zero. This was already proved in [12] for trivial extensions of hereditary algebras.

**REFERENCES**


