A characterization of Euler’s constant

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Abstract

We prove the following theorem. Let $\alpha$ and $\beta$ be real numbers. The inequality

$$
\Gamma(x^\alpha + y^\beta) \leq \Gamma\left(\Gamma(x) + \Gamma(y)\right)
$$

holds for all positive real numbers $x$ and $y$ if and only if $\alpha = \beta = -\gamma$. Here, $\Gamma$ and $\gamma = 0.57721 \ldots$ denote Euler’s gamma function and Euler’s constant, respectively.

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1. Introduction

The classical gamma function, introduced by L. Euler in 1729, is defined for $x > 0$ by

$$
\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt = \frac{1}{x} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{1}{k}\right)^x \left(1 + \frac{x}{k}\right)^{-1} \right\}.
$$

In view of its importance in many mathematical branches as well as in related fields, the $\Gamma$-function has been the subject of intensive research. The main properties of the gamma function and its relatives are collected in, for instance, [1, Chapter 6]. Remarkable historical comments on this subject can be found in [2,3,6,7,11]. We also refer the reader to Sándor’s detailed bibliography on the gamma function; see [10].

Our work has been motivated by an interesting research paper published by Monreal and Tomás [9] in 1998. In this article, the authors study several functional equations (in

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one and two variables) arising in computer graphics. One of these equations is

\[ f(x + y) = f(f(x) + f(y)). \]

We investigate the functional inequality

\[ \Gamma(x^\alpha + y^\beta) \leq \Gamma(\Gamma(x) + \Gamma(y)). \tag{1.1} \]

More precisely, we ask for all real parameters \( \alpha \) and \( \beta \) such that (1.1) is valid for all positive numbers \( x \) and \( y \). It turns out that the answer to this question leads to a new characterization of the famous Euler constant \( \gamma \). Indeed, in Section 3 we show that (1.1) is valid for all \( x, y > 0 \) if and only if \( \alpha = \beta = -\gamma \).

The constant \( \gamma \), introduced by Euler in 1734, is defined by the limit

\[ \gamma = \lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.57721 \ldots. \]

It appears in several mathematical areas like analysis and number theory. Numerous series and integral representations for \( \gamma \) are known in the literature. A famous open problem is to prove that \( \gamma \) is an irrational number. The connection between the gamma function and \( \gamma \) is given by the formula \( \Gamma'(1) = -\gamma \). Much information on Euler’s constant can be found in the survey paper [4] and in the monographs [5,8].

The numerical values in this paper have been calculated via the computer program Maple V, Release 5.1.

2. Lemmas

Throughout this paper, we denote by \( x_0 = 1.46163 \ldots \) the only positive zero of \( \psi = \Gamma'/\Gamma \). In order to prove our main result we need some inequalities for the gamma function. These inequalities are given in the following three lemmas.

Lemma 1. For all \( x > 0 \) we have

\[ x^{-\gamma} \leq \Gamma(x) \tag{2.1} \]

with equality holding if and only if \( x = 1 \).

Proof. We define for \( x > 0 \)

\[ g(x) = \log \Gamma(x) + \gamma \log x. \]

Then, we obtain

\[ g'(x) = \psi(x) + \frac{\gamma}{x}. \]

Using the integral representation

\[ \psi^{(n)}(x) = (-1)^{n+1} \int_{0}^{\infty} e^{-xt} \frac{t^n}{1 - e^{-t}} dt \quad (n \in \mathbb{N}; x > 0) \]
(see [1, p. 260]) with \( n = 1 \) leads to
\[
g''(x) = \psi'(x) - \gamma x^2 = \int_0^\infty e^{-xt} \frac{t}{1 - e^{-t}} dt - \gamma \int_0^\infty e^{-xt} t dt
\]
\[
= \int_0^\infty e^{-xt} \frac{1 - \gamma + \gamma e^{-t}}{1 - e^{-t}} dt > 0. \tag{2.2}
\]
Since \( g(1) = g'(1) = 0 \), we conclude from (2.2) that (2.1) holds, with equality if and only if \( x = 1 \). □

**Lemma 2.** Let \( c = (x_0/2)^{-1/\gamma} = 1.721 \ldots \) For all \( x \geq 3 \) we have
\[
\Gamma(x^{-\gamma}) < \Gamma\left(\Gamma(x) + \Gamma(c)\right). \tag{2.3}
\]

**Proof.** We define for \( x \geq 3 \)
\[
\Delta(x) = \log \left(\Gamma(x) + \Gamma(c)\right) - \log \Gamma(x^{-\gamma}).
\]
Differentiation yields
\[
x \Delta'(x) = A(x) + B(x), \tag{2.4}
\]
where
\[
A(x) = \Gamma(x + 1) \psi(x) \psi(\Gamma(x) + \Gamma(c)) \tag{2.5}
\]
and
\[
B(x) = \gamma x^{-\gamma} \psi(x^{-\gamma}).
\]
The three functions on the right-hand side of (2.5) are positive and strictly increasing on \([3, \infty)\). This implies
\[
A(x) \geq A(3). \tag{2.6}
\]
Let \( G(t) = t \psi(t) \). Using the series representation
\[
\psi^{(n)}(t) = (-1)^n n! \sum_{k=0}^{\infty} \frac{1}{(t+k)^{n+1}} \quad (n \in \mathbb{N}; t > 0)
\]
(see [1, p. 260]) with \( n = 1 \) and \( n = 2 \) yields
\[
G''(t) = 2 \psi'(t) + t \psi''(t) = 2 \sum_{k=1}^{\infty} \frac{k}{(t+k)^3} > 0.
\]
It follows that \( G \) is strictly convex on \((0, \infty)\). We set \( t = x^{-\gamma} \) and \( t_0 = 1/4 \). Then, \( G'(t_0) = 0.071 \ldots \) and
\[
G(t) \geq G(t_0) + (t - t_0)G'(t_0) \geq G(t_0) - t_0 G'(t_0).
\]
It follows that
\[
B(x) = \gamma G(x^{-\gamma}) \geq \gamma [G(t_0) - t_0 G'(t_0)]. \tag{2.7}
\]
Combining (2.4), (2.6), and (2.7) leads to
\[ x \Delta'(x) \geq A(3) + \gamma [G(t_0) - t_0 G'(t_0)] = 4.295 \ldots \]
Hence,
\[ \Delta(x) \geq \Delta(3) = 0.099 \ldots \]
This implies (2.3).

Lemma 3. Let \( c = (x_0/2)^{-1/\gamma} = 1.721 \ldots \). For all \( y \geq c \) we have
\[ \Gamma(3^{-\gamma} + y^{-\gamma}) < \Gamma \left( \Gamma(x_0) + \Gamma(y) \right) \] (2.8)

Proof. We define for \( y \geq c \)
\[ p(y) = \log \Gamma \left( \Gamma(x_0) + \Gamma(y) \right) - \log \Gamma(3^{-\gamma} + y^{-\gamma}) \]
Then,
\[ yp'(y) = v(y) + \gamma w(y^{-\gamma}) \] (2.9)
with
\[ v(y) = \Gamma(y + 1) \psi(y) \psi \left( \Gamma(x_0) + \Gamma(y) \right) \]
and
\[ w(z) = z \psi(3^{-\gamma} + z) = G(3^{-\gamma} + z) - 3^{-\gamma} \psi(3^{-\gamma} + z), \]
where \( G(x) = x \psi(x) \). The functions \( y \mapsto \Gamma(y + 1) \) and \( \psi \) are positive and strictly increasing on \([c, \infty)\). Since
\[ \Gamma(x_0) + \Gamma(y) \geq 2 \Gamma(x_0) = 1.77 \ldots > x_0, \]
it follows that \( y \mapsto \psi \left( \Gamma(x_0) + \Gamma(y) \right) \) is also positive and strictly increasing on \([c, \infty)\).
This implies that \( v \) is strictly increasing on \([c, \infty)\). The function \( w \) is strictly convex on \((0, \infty)\) with \( w'(0.4) = 0.033 \ldots \). It follows that \( w \) is strictly increasing on \([0.4, \infty)\).

In order to show that \( p' \) is positive on \([c, \infty)\), we consider two cases.
Case 1. \( c \leq y \leq 2 \).
Using the monotonicity of \( v \) and \( w \) we obtain from (2.9)
\[ yp'(y) \geq v(c) + \gamma w(1.8^{-\gamma}) = 0.003 \ldots, \quad \text{if } c \leq y \leq 1.8 \]
and
\[ yp'(y) \geq v(1.8) + \gamma w(2^{-\gamma}) = 0.030 \ldots, \quad \text{if } 1.8 \leq y \leq 2. \]
Case 2. \( y \geq 2 \).
We set \( z = y^{-\gamma} \) and \( z_1 = 3^{-\gamma} \). Then, \( w'(z_1) = 0.319 \ldots \) and
\[ w(z) \geq w(z_1) + (z - z_1) w'(z_1) \geq w(z_1) - z_1 w'(z_1). \]
It follows that
\[ yp'(y) \geq v(2) + \gamma [w(z_1) - z_1 w'(z_1)] = 0.047 \ldots. \]
Thus, \( p \) is strictly increasing on \([c, \infty)\). We obtain
\[
p(y) \geq p(c) = 0.029 \ldots \text{ for } y \geq c.
\]
This leads to (2.8). \qed

3. The main result

We are now in a position to present our main result.

**Theorem.** Let \( \alpha \) and \( \beta \) be real numbers. The inequality
\[
\Gamma(x^\alpha + y^\beta) \leq \Gamma(\Gamma(x) + \Gamma(y))
\]
holds for all positive real numbers \( x \) and \( y \) if and only if \( \alpha = \beta = -\gamma \).

**Proof.** First, we suppose that (3.1) is valid for all \( x, y > 0 \). Let
\[
F(x, y) = \Gamma(\Gamma(x) + \Gamma(y)) - \Gamma(x^\alpha + y^\beta).
\]
We set
\[
u(x) = F(x, x) \quad \text{and} \quad v(x) = F(x, 1/x).
\]
Then we have for \( x > 0 \)
\[
u(x) \geq 0 = u(1) \quad \text{and} \quad v(x) \geq 0 = v(1).
\]
It follows that
\[
0 = u'(1) = (\gamma - 1)(2\gamma + \alpha + \beta) \quad \text{and} \quad 0 = v'(1) = (\gamma - 1)(\alpha - \beta).
\]
This leads to \( \alpha = \beta = -\gamma \).

Next, we prove that the inequality
\[
\Gamma(x^{-\gamma} + y^{-\gamma}) \leq \Gamma(\Gamma(x) + \Gamma(y))
\]
is valid for all \( x, y > 0 \). We distinguish two cases.

Case 1. \( x_0 \leq x^{-\gamma} + y^{-\gamma} \).

Applying Lemma 1 gives
\[
x_0 \leq x^{-\gamma} + y^{-\gamma} \leq \Gamma(x) + \Gamma(y).
\]
Since \( \Gamma \) is strictly increasing on \([x_0, \infty)\), we conclude from (3.3) that (3.2) is valid. Moreover, Lemma 1 reveals that equality holds in (3.2) only if \( x = y = 1 \).

Case 2. \( x^{-\gamma} + y^{-\gamma} < x_0 \).

We may assume that \( 0 < x \leq y \). Then we obtain
\[
x_0 > 2y^{-\gamma} \quad \text{and} \quad y > c = (x_0/2)^{-1/\gamma} = 1.721 \ldots > x_0.
\]
It follows that
\[
\Gamma(x) + \Gamma(y) > \Gamma(x) + \Gamma(c) \geq \Gamma(x_0) + \Gamma(c) = 1.798 \ldots > x_0
\]
and
\[
\Gamma(x) + \Gamma(y) \geq \Gamma(x_0) + \Gamma(y) > \Gamma(x_0) + \Gamma(c) > x_0.
\]
This yields
\[ \Gamma \left( \Gamma(x) + \Gamma(y) \right) > \Gamma \left( \Gamma(x) + \Gamma(c) \right) \tag{3.6} \]
and
\[ \Gamma \left( \Gamma(x) + \Gamma(y) \right) \geq \Gamma \left( \Gamma(x_0) + \Gamma(y) \right). \tag{3.7} \]

We consider two subcases.

Case 2.1. \( x \geq 3 \).
We have
\[ 0 < x^{-\gamma} < x^{-\gamma} + y^{-\gamma} < x_0. \]
This leads to
\[ \Gamma(x^{-\gamma}) > \Gamma \left( x^{-\gamma} + y^{-\gamma} \right). \tag{3.8} \]

Using (3.6), (3.8), and Lemma 2 gives
\[ \Gamma \left( x^{-\gamma} + y^{-\gamma} \right) < \Gamma \left( x^{-\gamma} \right) < \Gamma \left( \Gamma(x) + \Gamma(c) \right) < \Gamma \left( \Gamma(x) + \Gamma(y) \right). \]

Case 2.2. \( 0 < x < 3 \).
Since
\[ 0 < 3^{-\gamma} + y^{-\gamma} < x^{-\gamma} + y^{-\gamma} < x_0, \]
we obtain
\[ \Gamma \left( 3^{-\gamma} + y^{-\gamma} \right) > \Gamma \left( x^{-\gamma} + y^{-\gamma} \right). \tag{3.9} \]

Combining (3.7), (3.9), and Lemma 3 yields
\[ \Gamma(x^{-\gamma} + y^{-\gamma}) < \Gamma(3^{-\gamma} + y^{-\gamma}) < \Gamma \left( \Gamma(x_0) + \Gamma(y) \right) \leq \Gamma \left( \Gamma(x) + \Gamma(y) \right). \]

This completes the proof of the theorem. \( \square \)

**Remarks.** (i) The proof of the theorem reveals that the sign of equality holds in
\[ \Gamma(x^{-\gamma} + y^{-\gamma}) \leq \Gamma \left( \Gamma(x) + \Gamma(y) \right) \quad (x, y > 0) \]
if and only if \( x = y = 1 \).

(ii) In view of inequality (3.1) it is natural to ask whether there exist real parameters \( a \) and \( b \) such that the converse of (3.1), that is,
\[ \Gamma(\Gamma(x) + \Gamma(y)) \leq \Gamma(x^a + y^b) \tag{3.10} \]
is valid for all \( x, y > 0 \). The answer to this question is “no” as we conclude from the following relations. We have
\[ \lim_{x \to \infty} \Gamma \left( \Gamma(x) + \Gamma(y) \right) = \lim_{x \to 0} \Gamma \left( \Gamma(x) + \Gamma(y) \right) = \infty, \]
whereas
\[ \lim_{x \to \infty} \Gamma(x^a + y^b) = \Gamma(y^b), \quad \text{if} \ a < 0, \]
\[ \Gamma(x^a + y^b)|_{a=0} = \Gamma(1+y^b), \quad \text{and} \quad \lim_{x \to 0} \Gamma(x^a + y^b) = \Gamma(y^b), \quad \text{if} \ a > 0. \]

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References